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CONTINUA WHICH ADMIT NO MEAN

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A symmetric, idempotent, continuous binary operation on a space is called a mean. In this paper, we provide a criterion for the non-existence of mean on a certain class of continua which includes tree-like continua. This generalizes a result of Bell and Watson. We also prove that any hereditarily indecomposable circle-like continuum admits no mean.

1. Introduction and preliminaries. A *continuum* is a compact connected metric space. A continuous map $m : X \times X \to X$ satisfying the following conditions is called a *mean*:

(M1) for each $x \in X$, $m(x,x) = x$,

(M2) $m(x,y) = m(y,x)$ for each $x, y \in X$.

The properties of continua which admit or which do not admit means have been investigated by several authors. The following are some of the known facts.

(1) Every absolute retract admits a mean.

(2) ([S]) If a continuum X admits a mean, then $\widetilde{H}^*(X;\mathbb{Z}_2) = 0$. In particular, X is unicoherent (see below for the definition), and if it is 1-dimensional, then it is hereditarily unicoherent.

(3) ([Ba]) The $\sin(1/x)$-curve admits no mean, while the dyadic solenoid admits a mean.

(4) ([BeW]) There are contractible dendroids and non-contractible dendroids which admit means.

In particular, Bell and Watson [BeW] obtained criteria for the existence and non-existence of means on given continua. A result of the present paper generalizes their non-existence criterion. This generalized criterion does not apply to hereditarily indecomposable continua. Here we also prove that

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there is no mean on the pseudo-arc nor on any hereditarily indecomposable circle-like continuum. The scheme of both of these proofs is very similar to the one of Bell and Watson.

First, we give some preliminary notions and facts used in the proof.

Definition 1.1. (1) A continuum X is said to be *arc-like* (*circle-like* resp.) if, for each $\varepsilon > 0$, there is a finite open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of X, called an *ε-chain cover* (*ε-circular chain cover* resp.), with $\operatorname{mesh} \mathcal{U} < \varepsilon$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$ ($|i - j| \bmod n \leq 1$ resp.). Two points a and b of an arc-like continuum X are called *opposite end points* if, for each $\varepsilon > 0$, an ε-chain cover can be chosen so that $a \in U_1$ and $b \in U_n$. In this case, X is clearly irreducible between a and b (that is, there is no proper subcontinuum of X containing a and b). The converse does not hold in general.

(2) A continuum X is said to be *unicoherent* (*indecomposable* resp.) if $X = A \cup B$, where A and B are subcontinua of X, implies that $A \cap B$ is connected ($A \subset B$ or $A \supset B$ resp.). If every subcontinuum of X is unicoherent (indecomposable resp.), then X is said to be *hereditarily unicoherent* (*hereditarily indecomposable* resp.).

(3) The topologically unique hereditarily indecomposable arc-like continuum is called the *pseudo-arc* and denoted by P ([Bi$_1$]). Two points a and b of the pseudo-arc P are opposite end points if and only if P is irreducible between a and b ([Bi$_2$]).

Definition 1.2. A continuous map $f : X \to Y$ is said to be *weakly confluent* if, for each subcontinuum B of Y, there is a subcontinuum A of X such that $f(A) = B$.

Theorem 1.3 ([O]). *Let Y_i be arc-like continua $(i = 1, 2)$. Then for each pair of surjective maps $f_i : X_i \to Y_i$, $i = 1, 2$, the product map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is weakly confluent.*

Theorem 1.4 ([M]). *Suppose that $f, g : [0, 1] \to [0, 1]$ are PL maps of the interval. Then there are PL maps $a, b : [0, 1] \to [0, 1]$ such that $f \circ a = g \circ b$. If, in addition, $f^{-1}(0) = g^{-1}(0) = \{0\}$ and $f^{-1}(1) = g^{-1}(1) = \{1\}$, then a and b can be chosen so that $a^{-1}(0) = b^{-1}(0) = \{0\}$ and $a^{-1}(1) = b^{-1}(1) = \{1\}$.*

Lemma 1.5. *Let X be a subcontinuum of an absolute retract M. Each mean $m : X \times X \to X$ on X can be extended to a mean $m^* : M \times M \to M$ on M.*

Proof. Let $\Sigma(M) = M \times M/(x, y) \sim (y, x)$ be the symmetric product of M and let $\Delta(M)$ be the diagonal set which is naturally contained in $\Sigma(M)$. Notice that $\Sigma(M)$ is metrizable. The mean m defines a retraction of $\Sigma(X)$ onto $\Delta(X)$ which extends to a retraction of $\Sigma(M)$ onto $\Delta(M)$. This defines a mean m^* on M.

Definition 1.6. (1) For a subset A of $X \times X$, A^- denotes the set $\{(y,x) \mid (x,y) \in A\}$. The subset A is said to be *symmetric* if $A = A^-$.

(2) The ε-neighbourhood of a subset S of a metric space X is denoted by $N(S,\varepsilon)$. The Hausdorff metric induced by a metric d on X is denoted by d_{H}. If $\{A_n\}$ is a sequence of compacta in X, then $\operatorname{Lim} A_n$ means the limit of $\{A_n\}$ in the Hausdorff metric.

2. Results. Theorem 3.5 of [BeW] asserts that, if a continuum X contains an arc A and two sequences of arcs that are "folded in opposite directions with respect to A", and further, if these sequences converge to A in a "regular way" (essentially, the 0-regularity is assumed), then X admits no mean. Theorem 2.2 below shows that we can remove the hypothesis of the 0-regular convergence. The key tool is the Uniformization Theorem (Theorem 1.4).

Definition 2.1. Let X be a continuum and A be an arc-like subcontinuum of X which has a and b as its opposite end points. A sequence $\{A_n \mid n \in \mathbb{N}\}$ of subcontinua of X is called a *folding sequence with respect to the point* a if it satisfies the following conditions: for each $n \in \mathbb{N}$, there are two subcontinua E_n and F_n of A_n such that

(1) $A_n = E_n \cup F_n$, and $\operatorname{Lim}(E_n \cap F_n) = \{a\}$,
(2) $\operatorname{Lim} E_n = \operatorname{Lim} F_n = A$.

Theorem 2.2. *Let X be a hereditarily unicoherent continuum which has an arc-like subcontinuum A with the following properties*:

(1) *A has a and b as its opposite end points*, *and*
(2) *there exist folding sequences $\{A_n\}$ and $\{B_n\}$ with respect to a and b respectively.*

Then X admits no mean.

The strategy of the proof of the above theorem is the same as that of Theorem 3.5 of [BeW]. We prove the following two lemmas, which are analogues of Lemmas 3.2 and 3.4 of [BeW] respectively.

Lemma 2.3. *Let m be a mean on a continuum X and let A be an arc-like continuum in X which has a and b as opposite end points. Then there exists a subcontinuum K in $A \times A$ intersecting the diagonal ΔA such that one of the following conditions holds*:

(1) $K \cap m^{-1}(a) = \emptyset$ *and* $K \cap A \times \{a\} \neq \emptyset$, *or*
(2) $K \cap m^{-1}(b) = \emptyset$ *and* $K \cap A \times \{b\} \neq \emptyset$.

Proof. Since a and b are opposite end points of A, there is an inverse limit representation

$$A = \varprojlim(I_j, f_j : I_{j+1} \to I_j)$$

such that each $I_j = [0_j, 1_j]$ is the interval with end points 0_j and 1_j, and

(3) $f_j^{-1}(0_j) = \{0_{j+1}\}$, $f_j^{-1}(1_j) = \{1_{j+1}\}$, and $f_{j\infty}^{-1}(0_j) = \{a\}$, $f_j^{-1}(1_j) = \{b\}$, where $f_{j\infty} : A \to I_j$ denotes the projection to I_j.

Let $E_j = f_{j\infty} \times f_{j\infty}(m^{-1}(a) \cap A \times A)$ and $F_j = f_{j\infty} \times f_{j\infty}(m^{-1}(b) \cap A \times A)$ (notice that they are not empty). Taking a subsequence if necessary, we may assume that E_j and F_j are disjoint. By (3), E_j contains $(0_j, 0_j)$, and F_j contains $(1_j, 1_j)$. By Lemma 3.1 of [BeW], there exists a symmetric subcontinuum K_1 of $I_1 \times I_1$ intersecting the diagonal ΔI_1 such that either

(4.1) $K_1 \cap I_1 \times \{0_1\} \neq \emptyset$ and $K_1 \cap E_1 = \emptyset$, or
(4.2) $K_1 \cap I_1 \times \{1_1\} \neq \emptyset$ and $K_1 \cap F_1 = \emptyset$.

Without loss of generality, we assume the first case. Since $f_1 \times f_1$ is weakly confluent by Theorem 1.3, there is a subcontinuum K_2 of $I_2 \times I_2$ such that $f_1 \times f_1(K_2) = K_1$. By the assumption (4.1) and the condition (3), we have $K_2 \cap I_2 \times \{0_2\} \neq \emptyset \neq K_2 \cap \{0_2\} \times I_2$ and $K_2 \cap E_2 = \emptyset$. Clearly K_2 intersects the diagonal ΔI_2.

Continuing this process, we obtain an inverse sequence

$$K_1 \xleftarrow{f_1 \times f_1 | K_2} K_2 \xleftarrow{f_2 \times f_2 | K_3} K_3 \leftarrow \ldots$$

whose limit K, being naturally identified with a subcontinuum of $A \times A$, satisfies the desired conditions.

LEMMA 2.4. *Let m be a mean on a hereditarily unicoherent continuum X and suppose that A is an arc-like subcontinuum of X which has a and b as its opposite end points. If $\{A_n\}$ is a folding sequence with respect to a, then for each subcontinuum K of $A \times A$ such that $K \cap \Delta A \neq \emptyset \neq K \cap \{a\} \times A$, we have $K \cap m^{-1}(a) \neq \emptyset$.*

Proof. We may assume that X is a subset of the Hilbert cube I^∞. The mean m extends to a mean m^* on I^∞ by Lemma 1.5. Suppose that there is a continuum K in $A \times A$ intersecting ΔA and $\{a\} \times A$ which is disjoint from $m^{-1}(a)$. By the symmetric property of m, we may assume that K is symmetric. Take points $(p, p), (a, q) \in K$ and let $0 < 4\varepsilon < d(a, m^*(K))$. There is a $\delta > 0$ such that $\delta < \varepsilon/4$ and if $d((x, y), (x', y')) < \delta$, then $d(m^*(x, y), m^*(x', y')) < \varepsilon/4$.

Take a map $\varphi : A \to J = a_J b_J$ onto an arc J in I^∞ which is $\delta/2$-close to id and such that $\varphi^{-1}(a_J) = \{a\}$ and $\varphi^{-1}(b_J) = \{b\}$. Let $\{E_n\}$ and $\{F_n\}$ be the sequences as in the definition of the folding sequence with respect to A and the point a. Take a large N so that $d_{\mathrm{H}}(A_N, A) < \delta/4$, $d_{\mathrm{H}}(E_N \cap F_N, a) < \delta/4$ and choose sequences $\{P_j\}$ and $\{Q_j\}$ of arcs in I^∞

satisfying the following conditions:

(1) $\operatorname{Lim} P_j = E_N$ and $\operatorname{Lim} Q_j = F_N$. The arc P_j has the end points s_j and t_j, the arc Q_j has the end points s_j and u_j.
(2) $P_j \cap Q_j = \{s_j\}$, where $\lim s_j \in E_N \cap F_N$.
(3) There exists PL maps $h_j : P_j \to J$ and $k_j : Q_j \to J$ which are $\delta/2$-close to id.
(4) $h_j^{-1}(a_J) = \{s_j\} = k_j^{-1}(a_J)$, $h_j^{-1}(b_J) = \{t_j\}$, and $k_j^{-1}(a_J) = \{u_j\}$.

Applying Theorem 1.4 to h_j and k_j, there exist maps $\alpha_j : [0,1] \to P_j$ and $\beta_j : [0,1] \to Q_j$ such that

(5) $h_j \circ \alpha_j = k_j \circ \beta_j$ and
$\alpha_j^{-1}(s_j) = \beta_j^{-1}(s_j) = \{0\}$ and $\alpha_j^{-1}(t_j) = \beta_j^{-1}(u_j) = \{1\}$.

Let $\lambda_j = h_j \circ \alpha_j = k_j \circ \beta_j$; then by (4), (5) and Theorem 1.3, we have

(6) $\lambda_j^{-1}(a_J) = \{0\}$ and $\lambda_j^{-1}(b_J) = \{1\}$, and $\lambda_j \times \lambda_j$ is weakly confluent.

So, there exists a subcontinuum $M_j \subset [0,1] \times [0,1]$ such that

$$\lambda_j \times \lambda_j(M_j) = \varphi \times \varphi(K) \supset \{(\varphi(p), \varphi(p)), (a_J, \varphi(q)), (\varphi(q), a_J)\}.$$

By (6), we have $M_j \cap ([0,1] \times \{0\}) \neq \emptyset \neq M_j \cap (\{0\} \times [0,1])$. Therefore M_j intersects the diagonal $\Delta[0,1]$. Replacing M_j by $M_j \cup M_j^-$, we may assume that M_j is symmetric. Take points $(x_j, x_j), (y_j, 0), (0, y_j) \in M_j$. Define a continuum V_j in $(P_j \cup Q_j) \times (P_j \cup Q_j)$ as follows:

$$V_j = \alpha_j \times \alpha_j(M_j) \cup \beta_j \times \alpha_j(M_j) \cup \beta_j \times \beta_j(M_j).$$

To see that V_j is indeed a continuum, observe that

$$\begin{aligned} &(\alpha_j(x_j), \alpha_j(x_j)), (\alpha_j(0) = s_j, \alpha_j(y_j)) \in \alpha_j \times \alpha_j(M_j),\\ &(\beta_j(0) = s_j, \alpha_j(y_j)), (\beta_j(y_j), \alpha_j(0) = s_j) \in \beta_j \times \alpha_j(M_j),\\ &(\beta_j(y_j), s_j = \beta_j(0_j)), (\beta_j(x_j), \beta_j(x_j)) \in \beta_j \times \beta_j(M_j). \end{aligned} \tag{7}$$

We may assume that the sequence $\{V_j\}$ converges to a continuum V in $A_N \times A_N$, and $\alpha_j(x_j)$ and $\beta_j(x_j)$ converge to u and v respectively, as $j \to \infty$. By the continuity of m^*, we have $\operatorname{Lim} m^*(V_j) = m(V)$.

By (7), we have $(u,u), (v,v) \in V$ and the condition (M1) implies that $u, v \in m(V)$. Since X is hereditarily unicoherent, $m(V) \cap A_N$ is a subcontinuum of A_N. From the construction, we see that $u \in E_N$ and $v \in F_N$, hence $m(V) \cap A_N$ intersects $E_N \cap F_N$. Take a point $z \in m(V) \cap E_N \cap F_N$; then it is $\varepsilon/4$-close to the point a by the choice of N and δ. A contradiction is derived by proving the following claim.

Claim. $d(m(V), a) > \varepsilon$.

Proof of Claim. First recall that $d(m^*(K), a) > 4\varepsilon$. Since φ is δ-close to id, we have $d(m^*(\varphi \times \varphi(K)), a) > 4\varepsilon - \varepsilon/2 > 3\varepsilon$. Take a point

$(m, n) \in M_j$. Since $h_j \times h_j(\alpha_j(m), \alpha_j(n)) = \lambda_j \times \lambda_j(m, n) \in \varphi \times \varphi(K)$ and h_j is $\delta/2$-close to id, we see that $\alpha_j \times \alpha_j(m, n) \in N(\varphi \times \varphi(K), \delta)$. By the same argument, we can prove that $V_j \subset N(\varphi \times \varphi(K), \delta)$. Hence $\alpha_j \times \alpha_j(M_j) \subset N(\varphi \times \varphi(K), \delta)$. Therefore, $d(m^*(V_j), a) > 3\varepsilon - \varepsilon = 2\varepsilon$. Taking the limit, we see that $d(m(V), a) > \varepsilon$.

This completes the proof of Lemma 2.4.

Theorem 2.2 follows immediately from the above two lemmas. However, it does not apply to hereditarily indecomposable continua, because it assumes the existence of decomposable subcontinua. The non-existence of means on the pseudo-arc is proved from the following result.

THEOREM 2.5. *Let X be a hereditarily unicoherent continuum which contains a pseudo-arc. Then X admits no mean.*

COROLLARY 2.6. *The pseudo-arc and each hereditarily indecomposable circle-like continuum admit no mean.*

For the proof, we need the following result, which is obtained by the proof of [L], Theorem 1.

THEOREM 2.7. *Let P be a pseudo-arc in a metric space M which is irreducible between x and y. For each $\varepsilon > 0$, there exists a $\delta > 0$ such that for each pseudo-arc Q in M which is irreducible between s and t and such that $d_{\mathrm{H}}(P, Q) < \delta$, $d(x, s) < \delta$ and $d(y, t) < \delta$, there is a homeomorphism $h : (P, x, y) \to (Q, s, t)$ such that $d(h, \mathrm{id}) < \varepsilon$.*

For any homogeneous continuum K and for each $\varepsilon > 0$, there exists a $\delta > 0$, called *the Effros number for ε*, such that for each pair x, y of points of K with $d(x, y) < \delta$, there exists a homeomorphism $h : K \to K$ such that $d(h, \mathrm{id}) < \varepsilon$ and $h(x) = y$.

Proof of Theorem 2.5. Let Q_0 be a pseudo-arc in X and let $P \subset Q_0$ be a proper sub-pseudo-arc of Q_0. The strategy of the proof is exactly the same as that of Theorem 2.2. We derive a contradiction by combining Lemma 2.3 with the following assertion:

ASSERTION. *Suppose that the pseudo-arc P above is irreducible between a and b and suppose that m is a mean on X. For any subcontinuum K of $P \times P$ with $K \cap \Delta P \neq \emptyset \neq K \cap \{a\} \times P$, we have $K \cap m^{-1}(a) \neq \emptyset$.*

Proof of Assertion. Let K be a subcontinuum of $P \times P$ which intersects ΔP and $\{a\} \times P$, and suppose that K is disjoint from $m^{-1}(a)$. We may assume that K is symmetric.

Let $\varepsilon > 0$ be a positive number such that $d(a, m(K)) > 4\varepsilon$. Let $\pi_j : P \times P \to P$ be the projection to the jth factor $(j = 1, 2)$. Observe that $\pi_1(K) = \pi_2(K) =: R$. Notice that R is also a pseudo-arc.

First we prove the following:

(1) There exists a continuum $L \subset R \times R$ such that

(1.1) $\pi_j(L) = R,\ j = 1, 2,$
(1.2) $L \cap \Delta P \neq \emptyset \neq L \cap \{a\} \times P,$
(1.3) $d(m(L), a) > 4\varepsilon,$
(1.4) there is a point $r \in R$ such that $(r, r) \in L$ and R is irreducible between a and r.

Indeed, let $(p, p), (a, q) \in K$. Notice that the points a and p need not belong to different composants of R. Take a $\beta > 0$ such that for any subcontinuum S of $P \times P$ with $d_{\mathrm{H}}(S, K) < \beta$, we have $d(m(S), a) > 4\varepsilon$.

Let $\gamma > 0$ be the Effros number for β and R. There exists a point $r \in R$ which belongs to a different composant of R than the point a and is such that $d(r, p) < \gamma$. By the choice of γ, there is a homeomorphism $u : R \to R$ such that $d(u, \mathrm{id}_R) < \beta$ and $u(p) = r$.

Define $L = u \times u(K) \subset R \times R$. Clearly $d_{\mathrm{H}}(L, K) < \beta$, so $d(m(L), a) > 4\varepsilon$. Let $c \in R$ be a point such that $u(c) = a$. Then $(c, d) \in K$ for some $d \in R$, thus $(a, u(d)) \in L$. Therefore L is the desired continuum, and (1) is proved.

In the rest of the proof, we assume that $K = L$ and $p = r$. For convenience, let us summarize the properties of K.

(2.1) $\pi_1(K) = \pi_2(K) =: R \subset P.$
(2.2) $(p, p), (a, q) \in K$ and $d(a, m(K)) > 4\varepsilon.$
(2.3) The points a and p belong to different composants in R.

Take δ and η with $0 < \delta < \varepsilon/2$, $0 < \eta < \delta$ as follows:

(3.1) If $d((x, y), (x', y')) < \delta$, then $d(m(x, y), m(x', y')) < \varepsilon/4$.
(3.2) If a pseudo-arc T in X satisfies $d_{\mathrm{H}}(T, R) < \eta$, T is irreducible between s and t, and $d(a, s) < \eta$ and $d(p, t) < \eta$, then there exists a homeomorphism $h : (R, a, p) \to (T, s, t)$ such that $d(h, \mathrm{id}) < \delta/2$.

(Such an η exists by Theorem 2.7.)

We apply Theorem 2.7 (or [L], Theorem 1) to obtain a $\xi > 0$ such that for each pseudo-arc P' in X with $d_{\mathrm{H}}(P, P') < \xi$, there is a homeomorphism $g : P \to P'$ such that $d(g, \mathrm{id}) < \eta/2$.

Since P is a subcontinuum of Q_0, there is a pseudo-arc Q in Q_0 such that $d_{\mathrm{H}}(P, Q) < \xi$. By the choice of ξ above, there exists a homeomorphism $h : P \to Q$ such that

(4) $d(h, \mathrm{id}) < \eta/2.$

Let $c = h(a)$ and $d = h(b)$.

We will define sequences $\{Q_j^1\}$ and $\{Q_j^2\}$ of subcontinua of Q and sequences of homeomorphisms $\{h_j^1 : R \to Q_j^1\}$ and $\{h_j^2 : R \to Q_j^2\}$ as follows:

First let $S = h(R)$ and observe that $\{c, h(p)\} \subset S$. By the choice of h, $d_{\mathrm{H}}(S, R) < \eta/2$. Fix two points p^1 and p^2 in S such that

(5) $d(p^t, h(p)) < \eta/2$, $t = 1, 2$, and S is irreducible between any pair of points from $\{c, p^1, p^2\}$.

For $t = 1, 2$, let $\{Q_j^t\}_{j=1}^{\infty}$ be a sequence of proper subcontinua of S such that

(6) $p^t \in Q_j^t$ for each $j \geq 1$ and $t = 1, 2$ (in particular, Q_i^1 and Q_j^2 belong to different composants of S for each i, j), and $\mathrm{Lim}_j\, Q_j^t = S$ $(t = 1, 2)$.

By taking a subsequence if necessary, we may assume that

(7) $d_{\mathrm{H}}(Q_j^t, R) < \eta/2$ for each $j \geq 1$ and $t = 1, 2$.

For $t = 1, 2$, take a sequence $\{c_j^t\}_{j=1}^{\infty}$ of points of Q_j^t such that

(8) $\lim_j c_j^t = c$, $t = 1, 2$, and c_j^t and p^t belong to different composants of Q_j^t for each $j \geq 1$.

Note that, by (4) and (5),

(9) $d(p^t, p) < \eta$,

and we may assume (by (4) and by taking a subsequence if necessary) that

(10) $d(c_j^t, a) < \eta$.

By the choice of η and (7) and (8), there is a homeomorphism $h_j^t : R \to Q_j^t$ such that

(11) $d(h_j^t, \mathrm{id}) < \delta/2$, $h_j^t(a) = c_j^t$, $h_j^t(p) = p^t$ for $t = 1, 2$ and for each $j \geq 1$.

Define a closed set $V_j \subset S \times S$ as follows:

$$V_j = h_j^1 \times h_j^1(K) \cup h_j^1 \times h_j^2(K) \cup h_j^2 \times h_j^2(K).$$

Recalling that K is symmetric and $(p, p), (a, q), (q, a) \in K$, we have

$$\begin{aligned} &(p^1, p^1), (c_j^1, h_j^1(q)), (h_j^1(q), c_j^1) \in h_j^1 \times h_j^1(K), \\ &(p^1, p^2), (c_j^1, h_j^2(q)), (h_j^1(q), c_j^2) \in h_j^1 \times h_j^2(K), \\ &(p^2, p^2), (c_j^2, h_j^2(q)), (h_j^2(q), c_j^2) \in h_j^2 \times h_j^2(K). \end{aligned} \tag{12}$$

We may assume that $\{V_j\}$ converges to a compactum V. Although V_j is not connected, the conditions (8) and (12) imply that V is a continuum. Since $\{p^1, p^2\} \subset m(V_j)$ for each $j \geq 1$, we have $m(V) \supset \{p^1, p^2\}$.

Since X is hereditarily unicoherent, $m(V) \cap S$ is a subcontinuum of S intersecting different composants by (6). Thus, $m(V) \cap S = S$, that is, $c \in S \subset m(V)$. Therefore, $d(m(V), a) \leq d(c, a) < \eta < \delta < \varepsilon/4$.

On the other hand, a computation similar to Lemma 2.4 shows that $d(m(V), a) > \varepsilon$, which contradicts the above.

This completes the proof of the Assertion and hence also of Theorem 2.5.

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