COLLOQUIUM MATHEMATICUM

VOL. LXX 1996 FASC. 2

HAHN'S EMBEDDING THEOREM FOR ORDERS AND HARMONIC ANALYSIS ON GROUPS WITH ORDERED DUALS

BY

NAKHLÉ H. ASMAR AND
STEPHEN J. MONTGOMERY-SMITH (COLUMBIA, MISSOURI)

Let G be a locally compact abelian group whose dual group Γ contains a Haar measurable order P. Using the order P we define the conjugate function operator on $L^p(G)$, $1 \le p < \infty$, as was done by Helson [7]. We will show how to use Hahn's Embedding Theorem for orders and the ergodic Hilbert transform to study the conjugate function. Our approach enables us to define a filtration of the Borel σ-algebra on G, which in turn will allow us to introduce tools from martingale theory into the analysis on groups with ordered duals. We illustrate our methods by describing a concrete way to construct the conjugate function in $L^p(G)$. This construction is in terms of an unconditionally convergent conjugate series whose individual terms are constructed from specific ergodic Hilbert transforms. We also present a study of the square function associated with the conjugate series.

1. Introduction. Throughout this paper, G will denote a locally compact abelian group with dual group Γ. A Haar measure on G will be symbolized by μ. For $1 \le p < \infty$, we denote by $L^p(G)$ the Banach space of Haar measurable functions f on G such that $|f|^p$ is integrable. The space of essentially bounded Haar measurable functions on G will be denoted by $L^\infty(G)$.

An *order* on Γ is a subset P such that $P + P \subseteq P$; $P \cap (-P) = \{0\}$; $P \cup (-P) = \Gamma$. Given such a set P we will write $\alpha > \beta$ to mean that $\alpha - \beta \in P \backslash \{0\}$. For $f \in L^2(G)$ we use the Fourier transform to define the conjugate function $\widetilde{f}$ of f with respect to the order P by the Fourier multiplier relation

$$\widehat{\widetilde{f}}(\chi) = -i \operatorname{sgn}_P(\chi) \widehat{f}(\chi) \tag{1}$$

for almost all $\chi \in \Gamma$, where $\operatorname{sgn}_P(\chi) = -1, 0$, or 1, according as $\chi \in$

1991 *Mathematics Subject Classification*: Primary 43A17.

$(-P)\backslash\{0\}$, $\chi = 0$, or $\chi \in P\backslash\{0\}$. When G is compact, these definitions are due to Helson [7] and [8].

In [6], Garling observed that the conjugate function on $\mathbb{T}^N$, defined using a lexicographic order on $\mathbb{Z}^N$, is connected in a natural way to martingale theory. Using this connection, Garling gave simple proofs of basic properties of the conjugate function in this setting. Our goal in this paper is to show how certain notions related to Hahn's Embedding Theorem for orders [5, Chapter IV]) can be used to introduce similar tools from probability theory in the study of conjugate functions on groups with an arbitrary measurable order on the dual group. We present the material related to Hahn's Theorem in Section 2. In particular, we will state a structure theorem (Theorem 2.4 below) which describes an order in terms of a unique chain of convex subgroups of Γ. When Γ is not necessarily discrete, the chain of subgroups may contain elements that are not Haar measurable. In Section 3, we will show how to obtain a structure theorem, similar to the one in Section 2, while avoiding nonmeasurable subgroups of Γ. This study is based on the work of Hewitt and Koshi [9] concerning measurable orders. In Section 4, we use the structure theorems for orders to define the conjugate function as a martingale difference series whose individual terms are constructed by using different ergodic Hilbert transforms. In Section 5, we will show that the conjugate series of $f \in L^p(G)$ is unconditionally convergent in $L^p(G)$ when $1 < p < \infty$, and is unconditionally convergent in $L^{1,\infty}(G)$ when $p = 1$. These results yield a concrete way for constructing the conjugate function on abstract groups. We end the paper with a study of the square function associated with the conjugate series.

2. Orders on discrete groups. We start by collecting facts leading to a structure theorem for orders on discrete groups (Theorem 2.4 below). This requisite material is taken from [5, Chapter IV], where it is presented as a background for the proof of Hahn's Embedding Theorem for orders [5, Theorem 16, p. 59]. Our presentation is simplified by the fact that the groups are all abelian.

Definitions and basic properties. Let Γ denote an infinite torsion-free abelian group. The topology on Γ will play no role in this section. An order on Γ will be denoted by $P(\Gamma)$ or simply P. Because sometimes we will be dealing with more than one order on a given group, it will be convenient to write (Γ, P) or $(\Gamma, P(\Gamma))$ to denote the group and the given order on it. A subset $J \subset \Gamma$ is called *convex* if whenever $a, b \in J$, $c \in \Gamma$ and $a \leq c \leq b$, then $c \in J$. As we will see, this notion plays a prominent role in the theory of orders. For the reader's convenience, we list a few properties of convexity that will be used in the sequels (see [5, pp. 18–19, and Chapter IV]).

(a) A subgroup $C \subset \Gamma$ is convex if and only if $P(\Gamma) \cap C$ is convex in $P(\Gamma)$.

(b) If $B \subset C \subset \Gamma$, and if B is convex in C and C is convex in Γ, then B is convex in Γ.

(c) The intersection of convex subgroups is again a convex subgroup. Thus, if A is a subset of Γ, there is a smallest convex subgroup containing A. We will denote this subgroup by $\{A\}_{\Box}$. If A is a subgroup, then $\{A\}_{\Box} = (A+P) \cap (A-P)$.

(d) A subgroup $C \subset \Gamma$ is called *principal* if $C = \{c\}_{\Box}$ for some $c \in \Gamma$.

(e) Let Γ and Γ' be two ordered groups. A homomorphism $\phi : (\Gamma, P(\Gamma)) \to (\Gamma', P(\Gamma'))$ is called an *order homomorphism* if $\phi(P(\Gamma)) \subset P(\Gamma')$. It is clear that if ϕ is an order homomorphism, then $\ker \phi$ is a convex subgroup of Γ. Conversely, if H is a convex subgroup of $(\Gamma, P(\Gamma))$, then we can define an order on the quotient group Γ/H by $a + H \in P(\Gamma/H) \Leftrightarrow a \in P$. To verify this claim, suppose that $a + H = b + H$, and, say, $a \in P$ and $b \in -P$. Then $0 \le a \le a - b \in H$. Since H is convex, it follows that $a \in H$, and so $a + H = b + H = 0 + H$, which shows that $P(\Gamma/H)$ is indeed an order on Γ/H. It is also clear that the natural homomorphism $\pi : (\Gamma, P(\Gamma)) \to (\Gamma/H, P(\Gamma/H))$ is an order homomorphism. We have thus the following useful theorem (see [5, Theorem 7, p. 21]).

THEOREM 2.1. *Suppose that Γ is an ordered group and H is a subgroup of Γ. If H is convex then the natural homomorphism $\pi : (\Gamma, P) \to (\Gamma/H, \pi(P))$ is an order homomorphism. Conversely, suppose that $\phi : (\Gamma, P) \to (\Gamma', P')$ is an order homomorphism with* $\ker \phi = H$. *Then H is a convex subgroup of Γ.*

DEFINITION. An order P on Γ is called *Archimedean* if, given $a, b \in P \backslash \{0\}$, there is a positive integer n such that $na > b$.

Archimedean orders have a simple characterization in terms of real-valued homomorphisms, due to O. Hölder ([5, Theorem 1, p. 45]).

THEOREM 2.2. *An order P on Γ is Archimedean if and only if Γ is isomorphic to a subgroup of $\mathbb{R}$.*

There is another useful characterization of Archimedean orders in terms of convex subgroups ([5, Corollary 5, p. 47]).

THEOREM 2.3. *Suppose that Γ is an ordered group*; *then Γ is Archimedean ordered if and only if the only convex subgroups of Γ are $\{0\}$ and Γ.*

Following [5, Chapter IV, Section 3], we let Σ denote the system of all convex subgroups in Γ. This system is in fact a chain containing $\{0\}$ and Γ. Hence if C and D are in Σ, then either $C \subset D$ or $D \subset C$. Also, whenever $\{C_\lambda\}_{\lambda \in \Lambda}$ is a collection from Σ, then $\bigcap_{\lambda \in \Lambda} C_\lambda$ and $\bigcup_{\lambda \in \Lambda} C_\lambda$ are again in Σ.

By a *jump* in Σ we mean a pair of subgroups C and D such that $D \subset C$, $D \neq C$, and Σ contains no subgroups between C and D. A jump will be denoted by $D \prec C$.

It is a fact that a subgroup $C \in \Sigma$ is the greater member of a jump (i.e. $D \prec C$) if and only if C is a principal convex subgroup. That is,

$$C = \{a\}_\square$$

for some $a \in \Gamma$ (see p. 54 of [5]).

Let Σ_0 denote the system of principal convex subgroups, and let Π be an indexing set for Σ_0. Order Π as follows: for $\varrho, \pi \in \Pi$, set $\pi \leq \varrho$ if and only if $C_\varrho \subset C_\pi$. With this order, Π has a maximal element α_0 corresponding to $\{0\} \in \Sigma_0$. Thus $C_{\alpha_0} = \{0\}$. For notational convenience, let $D_{\alpha_0} = \emptyset$. Let $D_\pi \prec C_\pi$ denote a jump in Σ, with $\pi < \alpha_0$. The quotient group C_π/D_π has no nontrivial convex subgroups. By Theorem 2.3, C_π/D_π is Archimedean ordered. Let

$$\psi_\pi : C_\pi/D_\pi \to \mathbb{R} \tag{2}$$

denote the order isomorphism mapping C_π/D_π into a subgroup of $\mathbb{R}$, and let

$$L_\pi : C_\pi \to \mathbb{R} \tag{3}$$

denote the composition of ψ_π with the natural homomorphism of C_π onto C_π/D_π. Then L_π is an order homomorphism of C_π into $\mathbb{R}$ with $\ker L_\pi = D_\pi$. Since $\mathbb{R}$ is a divisible group, the homomorphism L_π can be extended to a homomorphism of the entire group Γ into $\mathbb{R}$ (see [10, Theorem A.7, p. 441]). We keep the same notation for this extension. The next theorem summarizes this discussion. It is a basic result of this section and will be used in defining our construction of the conjugate function.

THEOREM 2.4. *Let Γ be an infinite discrete* (*torsion-free*) *ordered group with order P. Let Σ denote the chain of convex subgroups of Γ and Σ_0 the subcollection of principal convex subgroups indexed by the ordered set Π. There is a collection of real-valued homomorphisms $\{L_\pi : \pi \in \Pi\}$ of Γ into $\mathbb{R}$ such that*, *for every jump $D_\pi \prec C_\pi$*, *we have*

(i) $L_\pi(D_\pi) = \{0\}$; *and*
(ii) $\mathrm{sgn}(L_\pi(\chi)) = \mathrm{sgn}_P(\chi)$ *for all* $\chi \in C_\pi \backslash D_\pi$.

Observe that, in the notation of the previous theorem, we have

$$\Gamma = \bigcup_{\alpha \in \Pi} C_\alpha \backslash D_\alpha. \tag{4}$$

In fact, given $x \in \Gamma$, we have $\{x\}_\square = C_\alpha$ for some $\alpha \in \Pi$. Since D_α is strictly contained in C_α, it follows that x belongs to $C_\alpha \backslash D_\alpha$, which proves (4).

The following example will illustrate many of the results of this section.

Example. Suppose that $(T, <)$ is an ordered set and that γ is a limit ordinal. Let $\mathbb{Z}^{(T,\gamma)}$ be the group of sequences $(z_t)_{t\in T}$ such that the set $\{t : z_t \neq 0\}$ is reverse well ordered (i.e., every subset has a largest element) with order type less than γ. Define an order as follows: If $z = (z_t)_{t\in T} \in \mathbb{Z}^{(T,\gamma)}$ and $t_0 = \max\{t : z_t \neq 0\}$, then $z \in P$ if and only if $z_{t_0} \geq 0$.

In this example we see that Σ_0 is order isomorphic to $-T$, that $C_\alpha = \{(z_t) : z_t = 0, \text{ for } t \geq \alpha\}$, that $D_\alpha = \{(z_t) : z_t = 0, \text{ for } t > \alpha\}$, and that $L_\alpha((z_t)) = z_\alpha$.

We can also construct $\mathbb{R}^{(T,\gamma)}$ similarly.

3. Orders on locally compact abelian groups. In this section we prove a general version of Theorem 2.4 for measurable orders. The difficulty here is due to the fact that, in general, the chain of convex subgroups in Theorem 2.4 may contain nonmeasurable subgroups, or jumps of the form $D_\alpha \prec C_\alpha$ with $C_\alpha \backslash D_\alpha$ having measure zero. To overcome these measure theoretic problems, we will find a smallest open principal convex subgroup of Γ which will determine when to stop the chain while still being able to separate with continuous real-valued homomorphisms as in Theorem 2.4. This section is based on the study of orders of Hewitt and Koshi [9]. Indeed, Theorem 3.10 below is a combination of results from [9] and the material from the previous section. Throughout the present section Γ will denote an infinite locally compact torsion-free abelian group. The following basic properties of measurable orders will be needed.

Theorem 3.1. (a) *If P is a measurable order, then P has nonvoid interior* ([9, Theorem 3.1]). *Consequently, if P is a measurable order, then $-P$ has nonvoid interior.*

(b) *If Γ is an infinite compact torsion-free group, then every order on Γ is dense and has void interior* ([9, Theorem 3.2]). *Consequently, every order on a compact infinite group is nonmeasurable.*

Remark 3.2. Suppose that Γ is a locally compact abelian group, and that P is a measurable order on Γ. Use the structure theorem for locally compact abelian groups to write Γ as $\Gamma = \mathbb{R}^a \times \Omega$, where a is a nonnegative integer and Ω contains a compact open subgroup Ω_0 ([10, Theorem 24.30]). The fact that P is measurable automatically implies that either Γ is discrete or $a > 0$. In fact, if $a = 0$, then $\Gamma = \Omega$ and so Ω_0 is a compact open subgroup of Γ. The restriction of P to Ω_0 is a measurable order in Ω_0. But since Ω_0 is compact, it follows from Theorem 3.1(b) that $\Omega_0 = \{0\}$, and so Γ is discrete.

Henceforth, to avoid the cases treated in the previous section, we will assume that $\Gamma = \mathbb{R}^a \times \Omega$ with $a > 0$. For use in the sequel, we need the following result due to Hewitt and Koshi [9, Theorem 3.12].

THEOREM 3.3. *Let P be an order on $\mathbb{R}^a \times B$, where $a > 0$ and B is an infinite torsion-free locally compact abelian group that is the union of its compact open subgroups. Suppose that P has nonempty interior. Then there is a continuous real-valued homomorphism $L : \mathbb{R}^a \to \mathbb{R}$ such that*

$$L^{-1}(]0,\infty[) \times B \subset \mathrm{int}(P) \subset P \subset L^{-1}([0,\infty[) \times B. \tag{5}$$

The order $P \cap (L^{-1}(\{0\}) \times B)$ is arbitrary.

Remark 3.4. (a) It follows from Theorems 3.1 and 3.3 that an order P on $\mathbb{R}^a$ is measurable if and only if it is not dense in $\mathbb{R}^a$.

(b) Let P be a measurable order on $\mathbb{R}^a \times \Omega$, and let B denote the union of all the compact open subgroups of Ω. The restriction of P to $\mathbb{R}^a \times B$ is a nondense order, and so by Theorem 3.3 there is a continuous real-valued homomorphism $L :\ \mathbb{R}^a \to \mathbb{R}$ such that

$$L^{-1}(]0,\infty]) \times B \subset P.$$

We can now describe our candidate for a smallest open convex principal subgroup of Γ. Let

$$H = \{y \in \Omega : \mathbb{R}^a \times \{y\} \text{ has nonvoid intersections with } P \text{ and } -P\}. \tag{6}$$

PROPOSITION 3.5. *Let P be a measurable order on $\mathbb{R}^a \times \Omega$, and let H be as in* (6). *Then H is a subgroup of Ω that contains all the compact open subgroups of Ω; and $\mathbb{R}^a \times H$ is an open convex subgroup of Γ.*

Proof. The fact that H contains all the compact open subgroups of Ω follows from Theorem 3.3. Also, the fact that H is a subgroup is easily verified. Since $\mathbb{R}^a \times H$ is a subgroup with nonvoid interior it follows immediately that the subgroup is open. To establish the convexity of $\mathbb{R}^a \times H$, suppose that

$$0 < (t,y) < (t',y')$$

with $y' \in H$, $t,t' \in \mathbb{R}^a$. To show that $(t,y) \in \mathbb{R}^a \times H$, it is enough to find $x \in \mathbb{R}^a$ with $(x,y) \in -P$. Since $y' \in H$, we can find $x \in \mathbb{R}^a$ such that $(x,y') \in -P$. Hence $(t-t', y-y') + (x,y') \in -P$, or, $(t-t'+x, y) \in -P$.

LEMMA 3.6. *Let P be a measurable order in Γ and let H be as in* (6). *For $y \in H$, let $A_y = (\mathbb{R}^a \times \{y\}) \cap P$ and $B_y = (\mathbb{R}^a \times \{y\}) \cap -P$. Then A_y and B_y are nondense in $\mathbb{R}^a \times \{y\}$.*

Proof. It is enough to deal with the set A_y with $y \neq 0$. Assume that $(\mathbb{R}^a \times \{y\}) \cap P$ is dense in $\mathbb{R}^a \times \{y\}$. Let (s,y) be any element of $\mathbb{R}^a \times \{y\}$, and let $(s_0,y) \in (\mathbb{R}^a \times \{y\}) \cap P$ be such that $L(s_0) > L(s)$. We have $L(s_0 - s) > 0$, and so from Remark 3.5(b) we see that $(s_0 - s, 0) \in P$. Hence $(s_0,y) + (s - s_0, 0) = (s,y) \in P$, implying that $\mathbb{R}^a \times \{y\} \subset P$, which contradicts the fact that $y \in H$. Thus $P \cap (\mathbb{R}^a \times \{y\})$ is nondense in $\mathbb{R}^a \times \{y\}$.

We can now prove a special separation theorem for orders (compare with [9, Theorem 3.7]).

THEOREM 3.7. *Let P be a measurable order on Γ, let H be as in* (6), *and let L be as in Remarks* 3.4(b). *For every $y \in H$, there is a real number $\alpha(y)$ such that*

(i) $L^{-1}(]-\infty, \alpha(y)[) \times \{y\} \subset -P$,

(ii) $L^{-1}(]\alpha(y), \infty[) \times \{y\} \subset P$.

Moreover, the mapping $y \mapsto \alpha(y)$ is a continuous real-valued homomorphism from H.

Proof. We will write the elements of $\Gamma = \mathbb{R}^a \times \Omega$ as (x, y), where $x \in \mathbb{R}^a$, and $y \in \Omega$. Suppose that $(x_1, y) \in P$ and $L(x_2) > L(x_1)$. Then $L(x_2 - x_1) > 0$, and so from Remark 3.4(b) we have $(x_2 - x_1, 0) \in P\backslash\{0\}$, and consequently, $(x_2, y) \in P\backslash\{0\}$. Similarly, if $(x_1, y) \in -P$, and $L(x_2) < L(x_1)$, then $(x_2, y) \in (-P)\backslash\{0\}$. From these observations and the definition of H, we see that the following holds for every $y \in H$:

$$\begin{aligned} -\infty &< \sup\{L(x) : x \in \mathbb{R}^a,\ (x, y) \in (-P)\} \\ &= \inf\{L(x) : x \in \mathbb{R}^a,\ (x, y) \in P\} < \infty. \end{aligned} \tag{7}$$

For $y \in H$, let $\alpha(y)$ be defined by either the inf or the sup in (7). That α is a continuous homomorphism follows exactly as in the proof of [9, Theorem 3.7]. We omit the details.

We restate Theorem 3.7 using separating homomorphisms that reflect the order on $\mathbb{R}^a \times H$.

THEOREM 3.8. *Let P be a measurable order on Γ, and let H, L, and α be as in Theorem* 3.7. *Define the homomorphism τ on $\mathbb{R}^a \times H$ by*

$$\tau(x, y) = L(x) - \alpha(y).$$

Then

(i) *τ is continuous on $\mathbb{R}^a \times H$.*

(ii) $\tau^{-1}(]-\infty, 0[) \subset (-P)$.

(iii) $\tau^{-1}(]0, \infty[) \subset P$.

(iv) *The kernel of τ is locally null* (*i.e., if K is any compact subset of $\tau^{-1}(\{0\})$, then $\mu_\Gamma(K) = 0$*).

Proof. Assertions (i)–(iii) follow from the definitions of the homomorphisms τ and α. Now suppose that K is a compact subset of $\tau^{-1}(\{0\})$ and $\mu_\Gamma(K) > 0$, where μ_Γ is a Haar measure on Γ. Then $K - K$ contains an open neighborhood of the identity in Γ, and hence $\tau^{-1}(\{0\})$ is an open subgroup of Γ, which implies that $\mathbb{R}^a \times \{0\} \subset \tau^{-1}(\{0\})$. This is plainly a contradiction, and so (iv) holds.

THEOREM 3.9. *Let H be as in* (6). *Then $\mathbb{R}^a \times H$ is a principal convex open subgroup of Γ.*

Proof. Because of Proposition 3.5, all we need to show is that $\mathbb{R}^a \times H$ is a principal subgroup. For this purpose, let z be any element in $\mathbb{R}^a \times H$ such that $\tau(z) > 0$. We claim that $\mathbb{R}^a \times H = \{z\}_\square$. Since by definition $\{z\}_\square$ is the smallest convex subgroup containing z, and since $\mathbb{R}^a \times H$ is convex, it is enough to show that $\{z\}_\square \supset \mathbb{R}^a \times H$. Let x be any element of $P \cap (\mathbb{R}^a \times H)$. From Theorem 3.8, $\tau(x) \geq 0$. Choose a positive integer n such that $\tau(nz) > \tau(x) \geq 0$. Again from Theorem 3.8 we see that $0 \leq x \leq nz$, and since $\{z\}_\square$ is convex, it follows that $x \in \{z\}_\square$. Hence $\{z\}_\square \supset P \cap (\mathbb{R}^a \times H)$, and consequently $\{z\}_\square \supset \mathbb{R}^a \times H$.

We are now ready to state the main result of this section. We write the group Γ as $\mathbb{R}^a \times \Omega$, where $a > 0$. Let $\Sigma(H)$ denote the chain of convex subgroups of Γ containing $\mathbb{R}^a \times H$, and let $\Sigma_0(H)$ denote the chain of principal convex subgroups containing $\mathbb{R}^a \times H$. Let $\Pi(H)$ be an indexing set for $\Sigma_0(H)$. Order $\Pi(H)$ as we did in the discrete case: for $\varrho, \pi \in \Pi(H)$, set $\pi \leq \varrho$ if and only if $C_\varrho \subset C_\pi$. With this order, $\Pi(H)$ has a maximal element α_0 corresponding to $\mathbb{R}^a \times H \in \Sigma_0(H)$. Hence $C_{\alpha_0} = \mathbb{R}^a \times H$. As in the discrete case, we denote a jump in $\Sigma(H)$ by $D_\pi \prec C_\pi$, where C_π is a principal convex subgroup of Γ containing $\mathbb{R}^a \times H$. Note that for $C_{\alpha_0} = \mathbb{R}^a \times H$ the jump occurs with an element outside of $\Sigma_0(H)$. We set, by definition, $D_{\alpha_0} = \ker \tau$, where τ is the homomorphism of Theorem 3.8. Hence D_{α_0} is locally null by Theorem 3.8. Note that

$$\Gamma = D_{\alpha_0} \cup \bigcup_{\alpha \in \Pi(H)} C_\alpha \backslash D_\alpha. \tag{8}$$

With the exception of D_{α_0}, each set on the right side of (8) is open.

THEOREM 3.10. *With the above notation, for every $\alpha \in \Pi(H)$, $\alpha \neq \alpha_0$, there is a continuous real-valued homomorphism L_α on Γ such that*

(i) $L_\alpha(D_\alpha) = \{0\}$;
(ii) $\mathrm{sgn}_P(\chi) = \mathrm{sgn}(L_\alpha(\chi))$ *for all* $\chi \in C_\alpha \backslash D_\alpha$.

When $\alpha = \alpha_0$, there is a real-valued homomorphism L_{α_0} on Γ such that

(iii) $L_{\alpha_0}(D_{\alpha_0}) = \{0\}$;
(iv) $\mathrm{sgn}_P(\chi) = \mathrm{sgn}(L_{\alpha_0}(\chi))$ *for all* $\chi \in C_{\alpha_0} \backslash D_{\alpha_0}$.

(*Since D_{α_0} is locally null*, (iv) *holds for locally almost all $\chi \in C_{\alpha_0}$.*)

Proof. We treat first the case $\alpha = \alpha_0$. Consider the homomorphism τ provided by Theorem 3.8. Since τ maps into $\mathbb{R}$ and $\mathbb{R}$ is a divisible group, τ can be extended to a homomorphism on all of Γ ([10, Theorem A.7]). We denote the extended homomorphism by L_{α_0}. By the properties of τ from

Theorem 3.8, it is clear that (iii) and (iv) hold. Now since L_{α_0} is continuous on an open subgroup of Γ, it follows by linearity that L_{α_0} is continuous on all of Γ. This proves the theorem in this case.

We now treat the remaining cases. Since $\mathbb{R}^a \times H$ is convex and open, the group $\Gamma/(\mathbb{R}^a \times H)$ is discrete and can be ordered as in Theorem 2.1. Let Φ denote the natural homomorphism of Γ onto $\Gamma/(\mathbb{R}^a \times H)$. It is easy to see that C is a convex subgroup of Γ if and only if $\Phi(C)$ is a convex subgroup of $\Gamma/(\mathbb{R}^a \times H)$. Consequently, if $D_\alpha \prec C_\alpha$ is a jump in Γ, with $\alpha \neq \alpha_0$, then $\Phi(D_\alpha) \prec \Phi(C_\alpha)$ is a jump in $\Gamma/(\mathbb{R}^a \times H)$. The theorem follows now by composing Φ with the homomorphisms provided by Theorem 2.4 for the discrete ordered group $\Gamma/(\mathbb{R}^a \times H)$.

With Theorem 3.10 in hand we can give a simple proof of a separation theorem for measurable orders [1, Theorem 5.14]. The statement here slightly improves on [1].

THEOREM 3.11. *Let P be a measurable order on Γ and let K be an arbitrary compact subset of Γ. Let $N = \emptyset$ if Γ is discrete and $N = D_{\alpha_0}$ if Γ is not discrete*, *where D_{α_0} is as in Theorem* 3.10. *Then there is a continuous real-valued homomorphism ψ of Γ such that*

$$\operatorname{sgn}_P(\chi) = \operatorname{sgn}(\psi(\chi))$$

for all $\chi \in K\backslash D_{\alpha_0}$.

Proof. We treat the descrete case first. Without loss of generality, we may assume that K is a finite subset of Γ not containing 0. We appeal to Theorem 2.4 and use its notation. Let

$$D_{\alpha_1} \subset C_{\alpha_1} \subset D_{\alpha_2} \subset C_{\alpha_2} \subset \ldots \subset D_{\alpha_n} \subset C_{\alpha_n}$$

be a finite collection in Σ such that $K \cap (C_{\alpha_j} \backslash D_{\alpha_j}) \neq \emptyset$ for all $j = 1, \ldots, n$, and

$$K \subset \bigcup_{j=1}^{n} C_{\alpha_j} \backslash D_{\alpha_j},$$

and let L_{α_j} be the real-valued homomorphism of Γ corresponding to α_j. We have, from Theorem 2.4,

$$L_{\alpha_j}(D_{\alpha_j}) = \{0\}, \tag{9}$$

$$L_{\alpha_j}(K \cap P \cap (C_{\alpha_j} \backslash D_{\alpha_j})) \subset \,]0, \infty[, \tag{10}$$

$$L_{\alpha_j}(K \cap -P \cap (C_{\alpha_j} \backslash D_{\alpha_j})) \subset \,]-\infty, 0[. \tag{11}$$

We construct the homomorphism ψ as a linear combination

$$\psi = \sum_{j=1}^{n} a_j L_{\alpha_j},$$

where the coefficients a_j are defined inductively as follows. Set $a_1 = 1$. If a_j is defined for $j = 1, \ldots, k-1$, let

$$A_k = \max_{x \in K} \sum_{j=1}^{k-1} |a_j L_{\alpha_j}(x)|, \qquad B_k = \min_{x \in K \cap (C_{\alpha_k} \setminus D_{\alpha_k})} |L_{\alpha_k}(x)|.$$

Note that B_k is positive. Choose a_k so that $a_k B_k > A_k$. Using (9)–(11), it is straightforward to check that the homomorphism ψ has the desired property.

To treat the general case, we appeal to Theorem 3.10, and borrow its notation. Let Φ denote the quotient homomorphism of Γ onto the discrete group $\Gamma/(\mathbb{R}^a \times H)$ (recall that $\mathbb{R}^a \times H$ is open). Since K is compact, $\Phi(K)$ is a finite subset of $\Gamma/(\mathbb{R}^a \times H)$. Order $\Gamma/(\mathbb{R}^a \times H)$ as in Theorem 2.1. By the case we just treated, we can find a homomorphism ψ^* of $\Gamma/(\mathbb{R}^a \times H)$ separating the set $\Phi(K)$. It is clear that the homomorphism $\psi^* \circ \Phi$ separates the set $K \setminus (\mathbb{R}^a \times H)$. If $K \cap ((\mathbb{R}^a \times H) \setminus D_{\alpha_0}) = \emptyset$ then we are done. If not, consider the homomorphism

$$\psi = L_{\alpha_0} + \frac{a}{b} \psi^* \circ \Phi,$$

where $a = \max_{x \in K} |L_{\alpha_0}(x)|$ and $b = \min_{x \in K \setminus \mathbb{R}^a \times H} |\psi^* \circ \Phi(x)|$. A simple argument that we omit shows that ψ has the desired property.

4. The conjugate function and its basic properties. In this section, we use the structure of orders that we derived earlier to define the conjugate series of a function in $L^p(G)$, $1 \le p < \infty$. For use in the following section, we also recall some basic properties of the conjugate function operator, such as generalized versions of M. Riesz's and Kolmogorov's Theorems.

Let Γ denote a locally compact abelian group containing a measurable order P, and let G denote the dual group of Γ. Recall that the conjugate function operator $f \mapsto \widetilde{f}$ is defined on $L^2(G)$ by the multiplier relation (1). It is convenient to write $\mathcal{H}_P(f)$ as an alternative notation for the conjugate function. We appeal to Theorems 2.4 and 3.10 and write the group Γ as a disjoint union of open sets

$$\Gamma = C_{\alpha_0} \cup \bigcup_{\alpha \ne \alpha_0} C_\alpha \setminus D_\alpha. \tag{12}$$

For each $\alpha \ne \alpha_0$, let L_α denote the continuous homomorphism from Γ into $\mathbb{R}$ such that

$$\operatorname{sgn}(L_\alpha(\chi)) = \operatorname{sgn}_P(\chi) \tag{12}$$

for all $\chi \in C_\alpha \setminus D_\alpha$, and let L_{α_0} denote the continuous homomorphism from Γ into $\mathbb{R}$ such that

$$\operatorname{sgn}(L_{\alpha_0}(\chi)) = \operatorname{sgn}_P(\chi) \tag{14}$$

for locally almost all $\chi \in C_{\alpha_0}$. To simplify notation, let us write Π for the indexing set in both cases of Theorems 2.4 and 3.10. For each $\alpha \in \Pi$, the subgroup C_α is open, and similarly, D_α is open for $\alpha \neq \alpha_0$. Hence the annihilators in G, $A(G, C_\alpha)$ and $A(G, D_\alpha)$, are compact. Let μ_α, respectively ν_α, denote the normalized Haar measure on $A(G, C_\alpha)$, respectively $A(G, D_\alpha)$. We have

$$\widehat{\mu}_\alpha = 1_{C_\alpha} \quad \text{and} \quad \widehat{\nu}_\alpha = 1_{D_\alpha}, \tag{15}$$

where, if A is a set, 1_A is the indicator of A. For $f \in L^p(G)$, $1 \leq p < \infty$, we have

$$\|f * \mu_\alpha\|_p \leq \|f\|_p \quad \text{and} \quad \|f * \nu_\alpha\|_p \leq \|f\|_p$$

([10, Theorem 20.12]). For $\alpha \neq \alpha_0$, we let

$$d_\alpha f = f * \mu_\alpha - f * \nu_\alpha \quad \text{and} \quad d_{\alpha_0} f = f * \mu_{\alpha_0}.$$

It is clear that if $f \in L^2(G)$, then the support of $\widehat{f}$ is σ-compact. Hence it has nonvoid intersection with only countably many of the sets appearing on the right side of (12). It follows from (15) that, except for countably many α's, $d_\alpha f$ is zero almost everywhere. By approximating with functions in $L^2(G)$, we see that the same is true for any $f \in L^p(G)$, $1 \leq p < \infty$. As a convention, when $d_\alpha f = 0$ a.e., we take it to be identically 0. With this convention, the formal difference series

$$\sum_{\alpha \in \Pi} d_\alpha f \tag{16}$$

has only countably many nonzero terms. The conjugate function will be defined by a series conjugate to (16). Central to our construction is the ergodic Hilbert transform, which we introduce next. This transform has been systematically studied by Cotlar [4], Calderón [2], and Coifman and Weiss [3].

Let L_α be as in (14) or (13), and let ϕ_α denote its adjoint homomorphism. Thus ϕ_α is a continuous homomorphism mapping $\mathbb{R}$ into G and satisfying

$$\chi \circ \phi_\alpha(r) = L_\alpha(\chi)(r) \tag{17}$$

for all $r \in \mathbb{R}$ and all $\chi \in \Gamma$ ([10, Section 24]). The *truncated Hilbert transform in the direction of* L_α is the operator defined on $L^p(G)$, $1 \leq p < \infty$, by

$$H_{L_\alpha, n} f(x) = \frac{1}{\pi} \int_{1/n \leq |t| \leq n} f(x - \phi_\alpha(t)) \frac{1}{t}\, dt. \tag{18}$$

The (*ergodic*) *Hilbert transform in the direction of* L_α is the operator defined on $L^p(G)$, $1 \leq p < \infty$, by

$$H_{L_\alpha} f(x) = \lim_{n \to \infty} H_{L_\alpha, n} f(x). \tag{19}$$

The fact that this limit exits μ-a.e. on G follows from [4] (see also [2] or [3]). In fact, several other properties of this transform follow from those of the Hilbert transform on $\mathbb{R}$ and the transference methods of [2] and [3]. For ease of reference, we state some properties that are needed in the sequel. Let L denote an arbitrary continuous nonzero homomorphism from Γ into $\mathbb{R}$, and let ϕ denote its adjoint homomorphism. The operator $H_{L,n}$ is defined as in (18).

THEOREM 4.1. *Let $f \in L^p(G)$, where $1 \le p < \infty$.*

(i) *The limit*

$$H_L f(x) = \lim_{n\to\infty} H_{L,n} f(x)$$

exists μ-a.e.

(ii) *If $1 < p < \infty$, then the limit converges in $L^p(G)$, and*

$$\|H_L f\|_p \le A_p \|f\|_p,$$

where A_p is the bound of the Hilbert transform operator on $L^p(\mathbb{R})$.

(iii) *For $f \in L^1(G)$, we have*

$$\mu(\{x \in G : |H_L f(x)| > y\}) \le \frac{A}{y}\|f\|_1$$

for all $y > 0$, where A is the weak type $(1,1)$ norm of the Hilbert transform on $L^1(\mathbb{R})$.

(iv) *For $f \in L^2(G)$, we have*

$$\widehat{H_L f}(\chi) = -i\,\mathrm{sgn}(L(\chi))\widehat{f}(\chi)$$

for almost all $\chi \in \Gamma$.

The usefulness of this theorem is due in great part to the fact that all the estimates are independent of L or G. Property (iv) justifies using the terminology "the Hilbert transform in the direction of L" and shows a clear connection between the ergodic Hilbert transform and the conjugate function on groups. The proof of (iv) is straightforward, using (ii) and (17) (see [1, Theorem 6.7]). For use in the sequel, we recall the generalizations of M. Riesz's Theorem and Kolmogorov's Theorem from [1]. (These results are due to Helson [7] and [8] when G is compact.) Also, having all the necessary ingredients to prove these results, we will sketch short proofs to make the paper more self contained and to illustrate the use of the separation theorems.

THEOREM 4.2. *Let G be a locally compact abelian group with dual group Γ, and let P denote an arbitrary measurable order on Γ. For all $f \in L^p(G)$, $1 < p < \infty$, we have*

$$\|\mathcal{H}_P f\|_p \le A_p \|f\|_p,$$

where A_p is the norm of the Hilbert transform on $L^p(\mathbb{R})$.

THEOREM 4.3. *Let G be a locally compact abelian group with dual group Γ, and let P denote an arbitrary measurable order on Γ. For all $f \in L^2 \cap L^1(G)$ and all $y > 0$, we have*

$$\mu(\{x \in G : |\mathcal{H}_P f(x)| > y\}) \leq \frac{A}{y} \|f\|_1,$$

where A is the weak type $(1,1)$ norm of the Hilbert transform on $L^1(\mathbb{R})$.

Both theorems are proved in a similar way. It is enough to consider $f \in L^2(G)$ with compactly supported Fourier transform. Let $K \subset \Gamma$ denote the compact support of $\widehat{f}$. Apply Theorem 3.11 to obtain a real-valued homomorphism L of Γ such that

$$\operatorname{sgn}_P(\chi) = \operatorname{sgn}(L(\chi))$$

for almost all $\chi \in K$. Thus, from Theorem 4.1(iv) and the fact that $\widehat{f}$ is supported in K, it follows from the uniqueness of the Fourier transform that

$$\mathcal{H}_P f = H_L f$$

a.e. on G. The inequalities in Theorems 4.2 and 4.3 follow now from the corresponding ones for H_L in Theorem 4.1.

Because of Theorem 4.3, the operator $\mathcal{H}_P$ extends from $L^2 \cap L^1(G)$ to an operator on $L^1(G)$ satisfying the same weak type $(1,1)$ estimate. We keep the same notation for the extended operator.

The next theorem is our first step toward building the conjugate function. We continue with the notation leading to (16).

THEOREM 4.4. *Let $f \in L^p(G)$, where $1 \leq p < \infty$, and let $\alpha \in \Pi$. Then*

(i) $\mathcal{H}_P(d_\alpha f) = H_{L_\alpha}(d_\alpha f)$ *μ-a.e.*

If $f \in L^2 \cap L^p(G)$, then we also have

(ii) $\mathcal{H}_P(d_\alpha f) = d_\alpha(\mathcal{H}_P f)$ *and* $H_{L_\alpha}(d_\alpha f) = d_\alpha(H_{L_\alpha} f)$ *μ-a.e.*

Proof. The equalities in (ii) are clear since all operators in question are multiplier operators and so they commute. To prove (i) we note that since d_α is a bounded operator from $L^1(G)$ into $L^1(G)$, and since $\mathcal{H}_P$ and H_{L_α} are bounded from $L^1(G)$ into $L^{1,\infty}(G)$, it is enough to consider $f \in L^2(G)$. Since sgn_P and $\operatorname{sgn}(L_\alpha(\cdot))$ agree a.e. on $C_\alpha \backslash D_\alpha$, and since d_α projects the Fourier transform on $C_\alpha \backslash D_\alpha$, it is easy to see that the Fourier transforms of $\mathcal{H}_P(d_\alpha f)$ and $H_{L_\alpha}(d_\alpha f)$ agree almost everywhere on Γ, and so (i) follows.

As we argued for (16), we will agree that, for $f \in L^p(G)$ $(1 \leq p < \infty)$, the formal series

$$\sum_{\alpha \in \Pi} H_{L_\alpha}(d_\alpha f) \tag{20}$$

has only countably many terms. We will refer to (20) as the *conjugate* (*difference*) series of f.

5. Unconditional convergence of conjugate difference series. We will show that the conjugate series (20) converges unconditionally in $L^p(G)$ when $1 < p < \infty$ and unconditionally in $L^{1,\infty}(G)$ when $p = 1$. This will further justify our notation in (20) since the order of summation will become irrelevant in (20).

For use with weak type estimates, we recall a few facts about the Lorentz spaces $L^{p,\infty}(G)$. All details can be found in [11, Chapter V, Section 3]. Although the presentation in the cited reference is confined to σ-finite measure spaces, the results that we need on locally compact abelian groups follow easily by restricting a given function to its σ-compact support.

Given a measurable function f on G, let λ_f denote its distribution function, and let f^* denote the decreasing rearrangement of f. Define

$$\|f\|^*_{p,\infty} = \sup_{y>0} y(\lambda_f(y))^{1/p} = \sup_{y>0} y^{1/p} f^*(y) \tag{21}$$

and

$$\|f\|_{p,\infty} = \sup_{y>0} y^{1/p} m_f(y), \tag{22}$$

where

$$m_f(y) = \frac{1}{y}\int_0^y f^*(u)\,du.$$

Let $L^{p,\infty}(G)$ consist of all measurable functions on G such that $\|f\|^*_{p,\infty} < \infty$. It is well known that, when $1 < p < \infty$, (21) and (22) are equivalent and define a norm on $L^{p,\infty}(G)$. In fact, $\|f\|_{p,\infty}$ is a norm for all $1 \le p < \infty$, and when $1 < p < \infty$, we also have

$$\|f\|^*_{p,\infty} \le \|f\|_{p,\infty} \le \frac{p}{p-1}\|f\|^*_{p,\infty} \tag{23}$$

(see [11, Chap. V, Theorem 3.21]).

Let $\varepsilon \in \{-1,1\}^{\Pi}$. We will write $\varepsilon(P)$ for the subset of Γ obtained from P by changing the sign on $C_\alpha \backslash D_\alpha$ according to $\varepsilon(\alpha)$. That is, if $x \in C_\alpha \backslash D_\alpha$ and $\alpha \ne \alpha_0$, or if $x \in C_{\alpha_0}$, then $x \in \varepsilon(P)$ if and only if $\varepsilon(\alpha)x \in P$.

It is easy to see that $\varepsilon(P)$ is an order on Γ.

Suppose that $\eta \in \{0,1\}^{\Pi}$. Define a projection operator $\mathcal{P}_\eta$ on $L^2(G)$ by

$$\widehat{\mathcal{P}_\eta(f)} = \widehat{f}\,1_{\bigcup_{\alpha\in\Pi,\ \eta(\alpha)=1} C_\alpha\backslash D_\alpha}.$$

Define the conjugate projection operator $\widetilde{\mathcal{P}_{\eta,P}}$ by

$$\widetilde{\mathcal{P}_{\eta,P}} f = \mathcal{H}_P(\mathcal{P}_\eta f). \tag{24}$$

Thus,

$$\widetilde{\mathcal{P}_{\eta,P}f}(\chi) = \begin{cases} 0 & \text{if } \chi \notin \bigcup_{\alpha,\eta(\alpha)=1} C_\alpha \backslash D_\alpha, \\ -i\,\mathrm{sgn}_P(\chi)\widehat{f}(\chi) & \text{otherwise.} \end{cases} \tag{25}$$

To establish the unconditional convergence of the conjugate series, the following result is fundamental. It is a simple consequence of Theorems 4.2 and 4.3.

THEOREM 5.1. *Let η be any element of $\{0,1\}^\Pi$, and let P be an arbitrary order on Γ.*

(i) *The operator $\widetilde{\mathcal{P}_{\eta,P}}$ is bounded on $L^p(G)$ for $1 < p < \infty$ with norm $\le A_p$, where A_p is as in Theorem 4.2.*

(ii) *The operator $\widetilde{\mathcal{P}_{\eta,P}}$ is of weak type $(1,1)$ on $L^2 \cap L^1(G)$ with norm $\le 2A$, where A is the weak type norm in Theorem 4.3.*

Proof. Define $\varepsilon \in \{-1,1\}^\Pi$ by

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if } \eta(\pi) = 1, \\ -1 & \text{if } \eta(\pi) = 0. \end{cases}$$

It is easy to check using the Fourier transform that for all $f \in L^2(G)$, we have

$$\widetilde{\mathcal{P}_{\eta,P}}f = \tfrac{1}{2}(\mathcal{H}_P f + \mathcal{H}_{\varepsilon(P)} f).$$

The theorem follows now from Theorems 4.2 and 4.3 applied to the operators $\mathcal{H}_P$ and $\mathcal{H}_{\varepsilon(P)}$.

As a simple consequence we have the following.

COROLLARY 5.2. *Let $\{\alpha_1, \ldots, \alpha_n\}$ be a finite subset of Π. Then the operator*

$$f \mapsto \sum_{j=1}^{n} H_{L_{\alpha_j}}(d_{\alpha_j} f)$$

is of weak type $(1,1)$ on $L^1(G)$ with norm $\le 2A$ and is bounded from $L^p(G)$ into $L^p(G)$ with norm $\le A_p$, where A and A_p are as in Theorems 4.2 and 4.3.

Proof. Define $\eta \in \{0,1\}^\Pi$ by $\eta(\alpha_j) = 1$ for $j = 1, \ldots, n$ and $\eta(\pi) = 0$ otherwise. Then

$$\sum_{j=1}^{n} H_{L_{\alpha_j}}(d_{\alpha_j} f) = \widetilde{\mathcal{P}_{\eta,P}}f. \tag{26}$$

Now apply Theorem 5.1.

We are now ready to establish the unconditional convergence of the conjugate series (20).

THEOREM 5.3. *Let $f \in L^p(G)$, $1 \le p < \infty$, and let $\{\alpha_j\} \subset \Pi$ be an arbitrary enumeration of the countable set of $\alpha \in \Pi$ such that $d_\alpha f \not\equiv 0$.*

(i) *If $p = 1$, the series $\sum_j H_{L_{\alpha_j}} d_{\alpha_j} f$ converges in $L^{1,\infty}(G)$ to $\mathcal{H}_P f$.*

(ii) *If $1 < p < \infty$, the series $\sum_j H_{L_{\alpha_j}} d_{\alpha_j} f$ converges in $L^p(G)$ to $\mathcal{H}_P f$.*

Proof. We will deal with the case $p = 1$ only. The other case is done similarly. The assertions of the theorem are clear if $\widehat{f}$ is compactly supported, since in this case only finitely many $d_\alpha f$ are nonzero. Suppose that f is an arbitrary function in $L^1(G)$, and approximate f in $L^1(G)$ by functions with compactly supported Fourier transforms, say $\{g_n\}$. Then using Corollary 5.2 and Theorem 4.3, we get

$$\begin{aligned}
\Big\| \sum_{j=1}^{N} H_{L_{\alpha_j}} d_{\alpha_j} f - \mathcal{H}_P f \Big\|^*_{1,\infty} & \\
&\le 2 \Big\| \sum_{j=1}^{N} H_{L_{\alpha_j}} d_{\alpha_j} (f - g_n) - \mathcal{H}_P (f - g_n) \Big\|^*_{1,\infty} \\
&\quad + 2 \Big\| \sum_{j=1}^{N} H_{L_{\alpha_j}} d_{\alpha_j} g_n - \mathcal{H}_P g_n \Big\|^*_{1,\infty} \\
&\le 12A \|f - g_n\|^*_{1,\infty} + 2 \Big\| \sum_{j=1}^{N} H_{L_{\alpha_j}} d_{\alpha_j} g_n - \mathcal{H}_P g_n \Big\|^*_{1,\infty}.
\end{aligned}$$

Given $\varepsilon > 0$, we can make the left side smaller than ε by first choosing n so that $\|f - g_n\|^*_{1,\infty} < \varepsilon/(12A)$ and then choosing $N = N(n)$ so that $\| \sum_{j=1}^{N} H_{L_{\alpha_j}} d_{\alpha_j} g_n - \mathcal{H}_P g_n \|^*_{1,\infty} = 0$. This completes the proof.

The conjugate square function. We end this section with a study of the square function associated with the conjugate series (20). We start with a definition. For $f \in L^p(G)$, $1 \le p < \infty$, let

$$\widetilde{S} f = \Big(\sum_{\alpha \in \Pi} |H_{L_\alpha}(d_\alpha f)|^2 \Big)^{1/2}, \tag{27}$$

where the index of summation runs over those α's for which $d_\alpha f \not\equiv 0$.

THEOREM 5.4. (i) *Let $1 < p < \infty$. There is a constant B_p, depending only on p, such that for all $f \in L^p(G)$, we have*

$$\|\widetilde{S} f\|_p \le B_p \|f\|_p.$$

(ii) *There is an absolute constant B such that, for all $f \in L^1(G)$, and* *all* $y > 0$, *we have*

$$\mu(\{x \in G : |\widetilde{S}f(x)| > y\}) \le \frac{B}{y}\|f\|_1.$$

Proof. Part (i) is a well-known consequence of Theorem 5.3(ii). We will omit the proof. (In fact one can prove it by reproducing the argument that we present for part (ii).) To prove (ii), let p be an arbitrary but fixed number in $]0,1[$. We will need Khinchin's Inequality [12, Theorem V.8.4, p. 213], which we will cite here in a notation convenient for our proof. Let $a_1, \ldots, a_N$ be arbitrary complex numbers, and write $\mathbb{E}$ for the expected value over the probability space $\{-1,1\}^N$. Then Khinchin's Inequality asserts that there are constants α_p and β_p, depending only on p, such that

$$\alpha_p\Big\{\sum_{j=1}^N |a_j|^2\Big\}^{1/2} \le \Big\{\mathbb{E}\Big|\sum_{j=1}^N a_j\varepsilon_j\Big|^p\Big\}^{1/p} \le \beta_p\Big\{\sum_{j=1}^N |a_j|^2\Big\}^{1/2}. \tag{28}$$

Returning to the proof of (ii), we note by monotone convergence that it is enough to consider a finite sum

$$\Big(\sum_{j=1}^N |H_{L_{\alpha_j}} d_{\alpha_j} f|^2\Big)^{1/2}.$$

Applying Khinchin's Inequality, we see that, pointwise on G, we have

$$\Big(\sum_{j=1}^N |H_{L_{\alpha_j}} d_{\alpha_j} f|^2\Big)^{1/2} \le C_p\Big(\mathbb{E}\Big|\sum_{j=1}^N \varepsilon_j H_{L_{\alpha_j}} d_{\alpha_j} f\Big|^p\Big)^{1/p}.$$

We think of each $\varepsilon \in \{-1,1\}^N$ as an element of $\{-1,1\}^{\Pi}$ by setting $\varepsilon(\alpha_j) = \varepsilon(j)$ for $j = 1, \ldots, N$, and $\varepsilon(\pi) = 1$ for $\pi \notin \{\alpha_1, \ldots, \alpha_N\}$. Let $\eta = \eta(\varepsilon)$ be defined as in the proof of Corollary 5.2 (see (26)) so that

$$\sum_{j=1}^N \varepsilon_j H_{L_{\alpha_j}} d_{\alpha_j} f = \widetilde{\mathcal{P}}_{\eta(\epsilon),\varepsilon(P)} f.$$

Then

$$\Big\|\Big(\sum_{j=1}^N |H_{L_{\alpha_j}} d_{\alpha_j} f|^2\Big)^{1/2}\Big\|_{1,\infty}^* \le C_p\|(\mathbb{E}|\widetilde{\mathcal{P}}_{\eta(\varepsilon),\varepsilon(P)} f|^p)^{1/p}\|_{1,\infty}^*. \tag{29}$$

It is easy to prove from definitions that, for any $s > 0$ and for any measurable function f on G, $\||f|^s\|_{p,\infty}^* = \|f\|_{sp,\infty}^{*\,s}$. The fact that $\|\cdot\|_{1/p,\infty}^*$ is equivalent to a norm (see (23)) implies that

$$\|\mathbb{E}f\|_{1/p,\infty}^* \le \frac{1}{1-p}\mathbb{E}\|f\|_{1/p,\infty}^*.$$

We can now estimate the right side of (29)as follows:

$$C_p\|(\mathbb{E}|\widetilde{\mathcal{P}}_{\eta(\varepsilon),\varepsilon(P)}f|^p)^{1/p}\|^*_{1,\infty} = C_p(\|\mathbb{E}|\widetilde{\mathcal{P}}_{\eta(\varepsilon),\varepsilon(P)}f|^p\|^*_{1/p,\infty})^{1/p}$$
$$\le C_p\left(\frac{1}{1-p}\mathbb{E}\||\widetilde{\mathcal{P}}_{\eta(\varepsilon),\varepsilon(P)}f|^p\|^*_{1/p,\infty}\right)^{1/p} = C_p\left(\frac{1}{1-p}\mathbb{E}\|\widetilde{\mathcal{P}}_{\eta(\varepsilon),\varepsilon(P)}f\|^{*p}_{1,\infty}\right)^{1/p}$$
$$\le C_p\left(\frac{2^pA^p}{1-p}\mathbb{E}\|f\|_1^p\right)^{1/p} = 2AC_p\left(\frac{1}{1-p}\right)^{1/p}\|f\|_1.$$

The penultimate inequality follows from Corollary 5.2. This completes the proof of the theorem.

Acknowledgements. The research of the authors was supported by grants from the National Science Foundation (U.S.A.) and the Research Board of the University of Missouri. Both authors are grateful for conversations with Professors Nigel Kalton and Saleem Watson.

REFERENCES

[1] N. Asmar and E. Hewitt, *Marcel Riesz's Theorem on conjugate Fourier series and its descendants*, in: Proc. Analysis Conf., Singapore, 1986, S. T. Choy *et al.* (eds.), Elsevier, New York, 1988, 1–56.

[2] A. Calderón, *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. U.S.A. 157 (1971), 137–153.

[3] R. R. Coifman and G. Weiss, *Transference Methods in Analysis*, CBMS Regional Conf. Ser. in Math. 31, Amer. Math. Soc., Providence, R.I., 1977.

[4] M. Cotlar, *A unified theory of Hilbert transforms and ergodic theorems*, Rev. Mat. Cuyana 1 (1955), 105–167.

[5] L. Fuchs, *Partially Ordered Algebraic Systems*, Pergamon Press, 1960.

[6] D. J. H. Garling, *Hardy martingales and the unconditional convergence of martingales*, Bull. London Math. Soc. 23 (1991), 190–192.

[7] H. Helson, *Conjugate series and a theorem of Paley*, Pacific J. Math. 8 (1958), 437–446.

[8] —, *Conjugate series in several variables*, ibid. 9 (1959), 513–523.

[9] E. Hewitt and S. Koshi, *Orderings in locally compact Abelian groups and the theorem of F. and M. Riesz*, Math. Proc. Cambridge Philos. Soc. 93 (1983), 441–457.

[10] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, 2nd ed., Grundlehren Math. Wiss. 115, Springer, 1979.

[11] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Math. Ser. 32, Princeton Univ. Press, 1971.

[12] A. Zygmund, *Trigonometric Series*, 2nd ed., 2 vols., Cambridge Univ. Press, 1959.

Department of Mathematics
University of Missouri-Columbia
Columbia, Missouri 65211
U.S.A.

Reçu par la Rédaction le 13.7.1995