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ON THREE PROBLEMS FROM THE SCOTTISH BOOK CONNECTED WITH ORTHOGONAL SYSTEMS

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Introduction. In this paper we consider some questions connected with the following problems from [5]:

1. Problem of Mazur ([5, Problem 154]): Let (φ_n) be an orthogonal system consisting of continuous functions and closed in C.

(a) If $f(t) \sim a_1\varphi_1(t) + a_2\varphi_2(t) + \ldots$ is the development of a given continuous function $f(t)$ and $n_1, n_2, \ldots$ denote the successive indices for which $a_{n_1} \neq 0, \ldots$, can one approximate $f(t)$ uniformly by linear combinations of the functions $\varphi_{n_1}(t), \varphi_{n_2}(t), \ldots$?

(b) Does there exist a linear summation method M such that the development of every continuous function $f(t)$ in the system $(\varphi_n(t))$ is uniformly summable by the method M to $f(t)$?

In [6] A. M. Olevskiĭ has given negative answers to both questions.

2. Problem of Banach ([5, Problem 86]): Given a sequence of functions $(\varphi_n(t))$ which is orthogonal, normed, measurable, and uniformly bounded, can one always complete it, using functions with the same bound, to a sequence which is orthogonal, normed, and complete? Consider the case when infinitely many functions are necessary for completion.

This problem was first solved by S. Kaczmarz in [2]. Various solutions of this problem were found by B. S. Kashin, A. M. Olevskiĭ, S. V. Bochkarev and K. S. Kazarian [3, 4].

3. Problem of Mazur ([5, Problem 51]):

a) Is every set of functions, measurable in $[0, 1]$ with the property that any two functions of the set are orthogonal, at most countable? (the functions are not assumed to be square-integrable!)

b) An analogous question for sequences: Is every set of sequences with the property that any two sequences (ε_n), (η_n) of this set are orthogonal, that is, $\sum_{n=1}^{\infty} \varepsilon_n\eta_n = 0$, at most countable?

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It is stated in [5] that this problem was solved by Mazurkiewicz but there is no such remark in the xerox copy of the original manuscript we have.

Let X be a separable Banach space and let X^* be its dual. A system x_n, f_n, $x_n \in X$, $f_n \in X^*$, $n = 1, 2, \ldots, \infty$ is called *biorthogonal* if $f_m(x_n) = \delta_{mn}$ (Kronecker delta). A biorthogonal system is called *fundamental* (or *complete*) if its closed linear span $[x_n]_{n=1}^{\infty}$ is equal to X, and *total* if for any non-zero element $x \in X$ there is an index n such that $f_n(x) \neq 0$. A fundamental and total biorthogonal system is called a *Markushevich basis* (an *M-basis*). A biorthogonal system is called a *strong M-basis* if $x \in [f_n(x)x_n]_{n=1}^{\infty}$ for every x in X. A system (x_n) is called a *T-basis* if there exists a regular summation method such that for every element x in X there exists a unique series $\sum_{n=1}^{\infty} b_n x_n$ which is summable to x by this method.

We say that a Banach space X is *densely embedded* in a Banach space Y if X is a dense linear subspace of Y, it does not coincide with Y and there exists a positive constant C such that $\|x\|_Y \leq C\|x\|_X$ for $x \in X$.

1. An answer to the first part of Mazur's question [5, Problem 154] follows from the following general proposition which is an improvement of results of Gurariĭ and Johnson [1, 10].

PROPOSITION 1. *Let X be a separable Banach space which is densely embedded in a Hilbert space H. There exists a non-strong M-basis in X which is an orthogonal system in H.*

For the proof we need three lemmas.

LEMMA 1. *Let X be a Banach space which is densely embedded in a Banach space Y and let E be a finite-codimensional closed subspace of Y. Then $X \cap E$ is densely embedded in E.*

Proof. Let Z be a finite-dimensional complement to $X \cap E$ in X. Then for every $e \in E$ there exists a sequence $x_n + z_n \to e$ in Y-norm with $x_n \in X \cap E$ and $z_n \in Z$. Since $Z \cap E = 0$, Z and E are closed in Y and Z is finite-dimensional, we have $x_n \to x$ and $z_n \to z$ as $n \to \infty$, with $x \in E$, $z \in Z$ and $e = x + z$. Thus $z = 0$ and $x_n \to e$ as $n \to \infty$, i.e. $X \cap E$ is densely embedded in E. If $X \cap E = E$, then $X = (X \cap E) + Z = Y$. Therefore $X \cap E \neq E$.

LEMMA 2. *Let X be a Banach space which is densely embedded in a Hilbert space H. For any $\varepsilon > 0$ there exist x and x' in X such that $\|x\|_X = \|x'\|_X = 1$, $\|x - x'\|_X < \varepsilon$ and $x \perp x'$ in H.*

Proof. Let $\| \ \|$ be the norm in X and $\| \ \|_H$ be the norm in H. Without loss of generality we may suppose that there exists u in X such that $\|u\| = \|u\|_H = 1$. Let E be the orthogonal complement of u in H. Then $\operatorname{codim} E$

$= 1$. It follows from Lemma 1 that $X \cap E$ is dense in E and this embedding is not an isomorphism. Hence we may choose v in $X \cap E$ such that $\|v\|_H = 1$ and $a := \|v\|$ is sufficiently large. Put $\overline{x} = v + u$, $\overline{x}' = v - u$, $x = \overline{x}/\|\overline{x}\|$ and $x' = \overline{x}'/\|\overline{x}'\|$. Then $(\overline{x}, \overline{x}') = (v+u, v-u) = \|v\|_H - \|u\|_H = 0$, hence $x \perp x'$ in H. It is easy to see that $a - 1 < \|\overline{x}\|, \|\overline{x}'\| < a + 1$. This implies that $|\,\|\overline{x}\| - \|\overline{x}'\|\,| \le 2$ and $\|\overline{x}\| \cdot \|\overline{x}'\| \ge (a-1)^2$. Then

$$\|x - x'\| = \frac{\|\,\|\overline{x}'\|\overline{x} - \|\overline{x}\|\overline{x}'\|}{\|x\| \cdot \|x'\|} \le \frac{\|(\|\overline{x}'\| - \|\overline{x}\|)v + (\|\overline{x}'\| + \|\overline{x}\|)u\|}{(a-1)^2}$$
$$\le \frac{2\|v\| + 2(a+1)\|u\|}{(a-1)^2} \le \frac{2a + 2(a+1)}{(a-1)^2},$$

i.e. choosing a sufficiently large we may obtain $\|x - x'\|$ less than any preassigned ε. ∎

LEMMA 3. *Let X be a Banach space which is densely embedded in a Hilbert space H. Let (φ_n) be a system which is fundamental in X and orthogonal in H. Then (φ_n) is an M-basis in X.*

Proof. Since (φ_n) is orthogonal in H, there exist functionals $(\varphi_n^*) \subset H^*$ biorthogonal to (φ_n). Since X is densely embedded in H, H^* is embedded in X^* and dense in the weak* topology, hence (φ_n^*) is a total system on X, and therefore (φ_n) is an M-basis in X. ∎

Proof of Proposition 1. Let (y_n) be some M-basis in X and let (ε_n) be a sequence of positive scalars such that $\lim_n \varepsilon_n = 0$. We proceed by induction. In the first step we put $z_1 = y_1$ and choose x_1 and x_1' in $X \cap y_1^{\perp}$ which satisfy the conclusion of Lemma 2 with $\varepsilon = \varepsilon_1$. In the nth step we put $Y_{n-1} = (y_i, x_i, x_i')_{i=1}^{n-1}$, take $z_n \in \operatorname{lin}(Y_{n-1}, y_n)$ with $z_n \perp Y_{n-1}$ and choose x_n and x_n' in $X \cap (Y_{n-1} \cup \{y_n\})^{\perp}$ which satisfy Lemma 2 with $\varepsilon = \varepsilon_n$. Then the subspaces $X_1 = [x_n, z_n]_{n=1}^{\infty}$ and $X_2 = [x_n']_{n=1}^{\infty}$ are quasi-complementary but not complementary in X and orthogonal in H. It is known (see [8] for example) that we can choose a subspace X_1^0 of X_1 such that $\dim X_1/X_1^0 = 1$ and so that X_1^0 and X_2 remain quasi-complementary in X. Take a system (u_n) which is complete in X_1^0 and orthogonalize it in H. We get a system $(v_n) \subset X_1^0$ for which all conditions of Lemma 3 are valid, hence (v_n) is an M-basis in X_1^0, orthogonal in H. Put $\varphi_{2n-1} = v_n$ and $\varphi_{2n} = x_n$ for $n = 1, 2, \ldots$ Then (φ_n) is an M-basis in X, it is orthogonal in H by Lemma 3, but it is not a strong M-basis because $[\varphi_{2n-1}]_{n=1}^{\infty} \subset X_1^0$ and $([\varphi_{2n}]_{n=1}^{\infty})^{\perp} \supset X_1$. ∎

Remark. Since every T-basis (summation basis) is a strong M-basis (see [11, p. 357]), there exists an M-basis in X, orthogonal in H, which is not a T-basis in X. In the case when X has a conditional basis which is orthogonal in H, a negative answer to the second part of Mazur's question [5, Problem 154] can be obtained significantly simpler than in the article of

A. M. Olevskiĭ [6]. Such bases exist in L_p, $p > 2$ (trigonometric system), and in C (Franklin system). We will show that such bases exist in some symmetric function spaces which are embedded in L_2.

PROPOSITION 2. *Let X be a Banach space densely embedded in a Hilbert space H and suppose that X has a conditional basis orthogonal in H. Then there exists a strong M-basis in X, orthogonal in H, which is not a T-basis in X.*

Proof. This easily follows from the fact that every conditional basis has a permutation which is not a T-basis (see [11, p. 357]). It is clear that the rearranged system remains an M-basis and orthogonal in H. ∎

The following statement is well known (see [9, p. 31], for example).

LEMMA 4. *No orthonormal basis $(x_n(t))_{n=1}^{\infty}$ in $L_2(0,1)$ with $|x_n(t)| \equiv 1$ for all n can be an unconditional basis of a symmetric space E on $(0,1)$ different from L_2.*

Let E be a symmetric function space, let p_E and q_E be its Boyd indices (see e.g. [9, p. 27] for definition). It is known that the Walsh system is a basis in L_p, $1 < p < \infty$. If $1 \le p_E \le q_E < \infty$, then E is an interpolation space between L_{p_E} and L_{q_E} ([9, p. 27]). The above observations imply that the Walsh system is a conditional basis in E when $2 < p_E \le q_E < \infty$.

2. The following proposition gives, in particular, a negative answer to Banach's question [5, Problem 86].

PROPOSITION 3. *Let X be a Banach space which is densely embedded in a Hilbert space H and this embedding is not compact. Then there exists a sequence (φ_n) such that*

(i) (φ_n) *is bounded in X;*
(ii) (φ_n) *is orthogonal in H;*
(iii) (φ_n) *admits no extension to a fundamental and orthogonal sequence in H, using elements from X;*
(iv) *the closed linear span of (φ_n) in H has an infinite codimension in H.*

We need two lemmas for the proof.

LEMMA 5. *Let X be a Banach space which is densely embedded in a Hilbert space H and the embedding is not compact. Then there exists a positive scalar a such that for any finite-codimensional closed subspace $E \subset H$ there exists $x \in X \cap E^{\perp}$ such that $\|x\|_X \le a\|x\|_X$.*

Proof. Suppose the converse. Then for every a there exists a finite-codimensional subspace $E \subset H$ with $\|x\|_X \ge a\|x\|_H$ for every $x \in X \cap E$. We will show that this implies the compactness of the embedding of X in H.

We need to show that for every $\varepsilon > 0$ there exists a finite cover of $B(X)$ (the unit ball of X) by balls $S_1, \ldots, S_m$ in H with radius ε. It follows from the assumption that $B(X) \cap E \subseteq \varepsilon B(H)$ if $\varepsilon = 1/a$. Compactness of the embedding now easily follows from the fact that $X \cap E$ is closed and finite-codimensional. ■

LEMMA 6. *Let X be a Banach space which is densely embedded in a Hilbert space H. Let E be a finite-codimensional closed subspace of H, let $\varepsilon > 0$ and $v \in X$. Then there exists $y \in X \cap E$ such that $d(v, \mathrm{lin}(E^\perp, y)) < \varepsilon$, where d means the distance in H.*

Proof. Decompose v in H as $v = v^* + v^{**}$, where $v^* \in E$ and $v^{**} \in E^\perp$. Hence

$$\begin{aligned} d(v, \mathrm{lin}(E^\perp, y)) &= \inf\{\|v - z\|_H : z \in \mathrm{lin}(E^\perp, y)\} \\ &= \inf\{(\|v^* - \lambda y\|^2 + \|v^{**} - u\|^2)^{1/2} : \lambda \in \mathbb{R},\ u \in E^\perp\} \\ &= \inf\{\|v^* - \lambda y\|_H : \lambda \in \mathbb{R}\}. \end{aligned}$$

Since $X \cap E$ is densely embedded in E by Lemma 1, v^* can be approximated arbitrarily closely by an element y from $X \cap E$. ■

Proof of Proposition 3. The proof is a modification of arguments from [7]. The reasoning uses the orthogonal transformation of A. M. Olevskiĭ and takes into account results from [8].

Let $(v_n)_{n=1}^\infty \subset X$ be a complete sequence in H such that each element is repeated infinitely many times. Let $(\varepsilon_n)_{n=1}^\infty$ be a sequence of positive scalars such that $\lim_n \varepsilon_n = 0$. By [8] there exists a closed infinite-dimensional subspace Z in H such that $Z \cap X = 0$. We proceed by induction. Let a be the constant from Lemma 5. For elements $(z_i, x_i, y_i)_{i=1}^n \subset H$ we put $H_n = \mathrm{lin}(z_i, x_i, y_i)_{i=1}^n$. In the first step we use Lemma 5 to find $z_1 \in Z$, $z_1 \neq 0$, and $x_1 \in X$ with $x_1 \perp z_1$, $\|x_1\|_H = 1$ and $\|x_1\|_X \le a$. Next we use Lemma 6 to choose $y_1 \in X$ such that $y_1 \in (z_1, x_1)^\perp$, $\|y_1\|_H = 1$ and $d(v_1, H_1) < \varepsilon_1$. In the nth step we take $z_n \in Z \cap H_{n-1}^\perp$, $z_n \neq 0$, choose $x_n \in X \cap H_{n-1}^\perp \cap z_n^\perp$ such that $\|x_n\|_H = 1$ and $\|x_n\|_X \le a$ and choose $y_n \in X \cap H_{n-1}^\perp \cap (z_n, x_n)^\perp$ such that $\|y_1\|_H = 1$ and $d(v_n, H_n) < \varepsilon_n$.

Now we rearrange the sequence (y_n) and relabel it as $(\psi_{i_m})_{m=1}^\infty$, where $(i_m)_{m=1}^\infty$ is an increasing sequence such that for every m, $i_m - i_{m-1} = 2^{s_m}$, where the positive integer s_m is chosen to satisfy $2^{-s_m/2}\|\psi_{i_m}\|_X < 2^{-m}$. We relabel $(x_n)_{n=1}^\infty$ using the remaining positive integers to get the sequence $(\psi_i : i \notin (i_m)_{m=1}^\infty)$. Let us apply for each block $(\psi_i : i_{m-1} < i \le i_m)$ the orthogonal transformation of Olevskiĭ [7]. We obtain a sequence $(\varphi_k)_{k=1}^\infty$ which is bounded in X and orthonormal in H. The closed linear span of $(\varphi_k)_{k=1}^\infty$ in H coincides with the closed linear span of $(x_n, y_n)_{n=1}^\infty$ in H. It is clear that the subspace $[z_n]_{n=1}^\infty$ is an orthogonal complement to this closed linear span. ■

3. In this section we will answer Mazur's question [5, Problem 51]. First we consider the discrete variant. We need the following known lemma ([11, p. 208]).

LEMMA 7. *Let N be a countable set. Then there exists a family $\{M_\alpha\}_{\alpha\in A}$ of subsets of N with the following properties*:

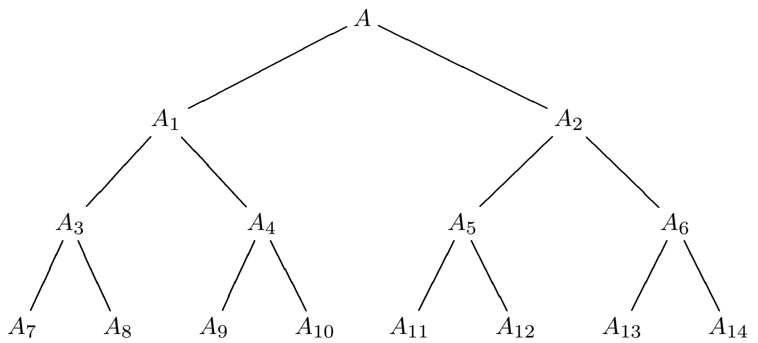
(i) *The index set A has cardinality continuum.*
(ii) *Each set M_α is infinite.*
(iii) *$M_\alpha \cap M_\beta$ is finite for $\alpha \neq \beta$.*

Proof. Let N be the set of all rational numbers in $(0,1)$, A be the set of all irrational numbers in $(0,1)$ and, for each $\alpha \in A$, let M_α be an arbitrary infinite sequence in N converging to α. ■

PROPOSITION 4. *There exist continuum many sequences $x^\alpha = (x_1^\alpha, x_2^\alpha, \ldots)$, $\alpha \in A$, such that $x_n^\alpha = 0, 1$, or -1 for every n and α, and for $\alpha \neq \beta$ the series $\sum_{n=1}^\infty x_n^\alpha x_n^\beta$ contains a finite number of non-zero terms and its sum is equal to zero.*

Proof. In a countable set N_0 choose continuum many non-empty subsets M_α, $\alpha \in A$, such that $M_\alpha \cap M_\beta$ is a finite set for $\alpha \neq \beta$, by Lemma 7. Put $x^\alpha = (x_1^\alpha, x_2^\alpha, \ldots)$, where $x_n^\alpha = 1$ if $n \in M_\alpha$, and $x_n^\alpha = 0$ if $n \notin M_\alpha$. We have constructed continuum many sequences x^α so that for every $\alpha \neq \beta$ the series $\sum_{n=1}^\infty x_n^\alpha x_n^\beta$ is a finite sum.

Now we represent A as a dyadic tree $A = A_1 \sqcup A_2$, $A_1 = A_3 \sqcup A_4$, $A_2 = A_5 \sqcup A_6, \ldots$, where $\sqcup$ denotes disjoint union, by the scheme:



We make this representation in such a way that

$(*)$ *every chain $(A_{k_i})_{i=1}^\infty$ has one-point intersection $\bigcap_{i=1}^\infty A_{k_i}$.*

We shall add to N_0 a countable number of countable sets N_i, $i = 1, 2, \ldots$, and shall complete the definition of our sequences on $\bigcup_{i=1}^\infty N_i$ by 0, 1, and -1 so that in the ith step for $\alpha \neq \beta$ the series

$$S(\alpha, \beta, i) := \sum_{n \in \bigcup_{k=0}^i N_k} x_n^\alpha x_n^\beta$$

will contain a finite number of non-zero terms; if $S(\alpha,\beta,i) = 0$ then $S(\alpha,\beta,j) = 0$ for $j > i$; and for every $\alpha \neq \beta$ there exists i such that $S(\alpha,\beta,i) = 0$.

First step. Let N_1 be a copy of N_0 and $\varphi_1 : N_0 \to N_1$ be an identifying map. Put

$$x^{\alpha}_{\varphi_1(n)} = \begin{cases} x^{\alpha}_n & \text{if } \alpha \in A_1, \\ -x^{\alpha}_n & \text{if } \alpha \in A_2, \end{cases} \qquad n \in N_0.$$

Then for every α, β the series $S(\alpha,\beta,1)$ has a finite number of non-zero terms and its sum is zero for $\alpha \in A_1$, $\beta \in A_2$.

Second step. Let N_2 be a copy of $N_0 \cup N_1$ and $\varphi_2 : N_0 \cup N_1 \to N_2$ be an identifying map. Put

$$x^{\alpha}_{\varphi_2(n)} = \begin{cases} x^{\alpha}_n & \text{if } \alpha \in A_3, \\ -x^{\alpha}_n & \text{if } \alpha \in A_4, \\ 0 & \text{if } \alpha \notin A_1, \end{cases} \qquad n \in N_0 \cup N_1.$$

Then for every α, β the series $S(\alpha,\beta,2)$ has a finite number of non-zero terms, its sum is zero for $\alpha \in A_3$, $\beta \in A_4$, and also for $\alpha \in A_1$, $\beta \in A_2$, since it is then equal to $S(\alpha,\beta,1)$.

Third step. Let N_3 be a copy of $N_0 \cup N_1 \cup N_2$ and $\varphi_3 : N_0 \cup N_1 \cup N_2 \to N_3$ be an identifying map. Put

$$x^{\alpha}_{\varphi_3(n)} = \begin{cases} x^{\alpha}_n & \text{if } \alpha \in A_5, \\ -x^{\alpha}_n & \text{if } \alpha \in A_6, \\ 0 & \text{if } \alpha \notin A_2, \end{cases} \qquad n \in N_0 \cup N_1 \cup N_2.$$

Then for every α, β the series $S(\alpha,\beta,3)$ has a finite number of non-zero terms, its sum is zero for $\alpha \in A_5$, $\beta \in A_6$, for $\alpha \in A_1$, $\beta \in A_2$ (being equal to $S(\alpha,\beta,1)$) and for $\alpha \in A_3$, $\beta \in A_4$ (being equal to $S(\alpha,\beta,2)$).

We have constructed our sequences so that in the ith step for $\alpha \neq \beta$ the series $S(\alpha,\beta,i)$ has a finite number of non-zero terms and if $S(\alpha,\beta,i) = 0$ then $S(\alpha,\beta,j) = S(\alpha,\beta,i) = 0$ for $j > i$. Condition $(*)$ ensures that for any distinct α, β there exists i such that $S(\alpha,\beta,i) = 0$. ∎

An uncountable orthogonal system on an interval can be obtained as a result of the following transformation. We decompose $(0,1)$ into a countable union of disjoint sets $(\Delta_n)_{n=1}^{\infty}$ of positive measure, and for every sequence $x = (x_n)$ we define a function $f_x(t) = x_n/\sqrt{\mu(\Delta_n)}$ for $t \in \Delta_n$. It is easy to see that if $x^{\alpha}, \alpha \in A$, are the sequences from Proposition 4, then the set of functions $f_{x^{\alpha}}$, $\alpha \in A$, has the property desired in [5, Problem 51].

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