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LIOUVILLIAN FIRST INTEGRALS OF HOMOGENEOUS POLYNOMIAL 3-DIMENSIONAL VECTOR FIELDS

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Given a 3-dimensional vector field V with coordinates V_x, V_y and V_z that are homogeneous polynomials in the ring $k[x,y,z]$, we give a necessary and sufficient condition for the existence of a Liouvillian first integral of V which is homogeneous of degree 0. This condition is the existence of some 1-forms with coordinates in the ring $k[x,y,z]$ enjoying precise properties; in particular, they have to be integrable in the sense of Pfaff and orthogonal to the vector field V. Thus, our theorem links the existence of an object that belongs to some level of an extension tower with the existence of objects defined by means of the base differential ring $k[x,y,z]$. A self-contained proof of this result is given in the language of differential algebra.

This method of finding first integrals in a given class of functions is an extension of the compatibility method introduced by J.-M. Strelcyn and S. Wojciechowski; and an old method of Darboux is a special case of it.

We discuss all these relations and argue for the practical interest of our characterization despite an old open algorithmic problem.

1. Introduction—Compatibility analysis. Consider some vector field $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$ defined on $\mathbb{R}^3$, $\mathbb{C}^3$ or on some open subset U of one of these spaces.

A function f defined on U is said to be a *first integral* of V if it is regular enough, not constant on any non-empty open subset of U and if the inner product of the exterior derivative df of f by the field V is 0:

$$i_V(df) = V_x\partial_x f + V_y\partial_y f + V_z\partial_z f = 0. \tag{1}$$

This means that f is invariant under the action of the local semigroup generated by field V.

First integrals always exist locally around any regular point of the vector field by classical results (see, for instance, [2]). Their global existence is a difficult topological question [1].

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We will adopt a more algebraic approach; in fact, we are interested in the effective search of first integrals for explicitly given vector fields. Precisely, we consider a vector field V defined on the whole space whose coordinates V_x, V_y and V_z in the canonical basis $(\partial_x, \partial_y, \partial_z)$ are homogeneous polynomials of the same degree with respect to the space variables x, y and z and may depend on some parameters. We would like to know whether this vector field admits a first integral on the whole space, or at least on some significant subset of it, and also if this integral can be defined "in finite terms", i.e. explicitly by means of some elementary constructions.

In this search, it will be of interest to know the existence of a first integral of the given field V which is also a first integral of some other vector field.

Therefore, two vector fields V_1 and V_2 are said to be *compatible* if the following identity holds everywhere on their common domain of definition:

$$\det(V_1, V_2, [V_1, V_2]) = 0, \tag{2}$$

where $[V_1, V_2]$ stands for the Lie bracket of the two fields and det is the determinant of the 3×3 matrix. This condition (2) does not imply that the three fields V_1, V_2 and $[V_1, V_2]$ are linearly dependent.

Let Ω be the volume form $dx\,dy\,dz$ and let ω be the 1-form $i_{V_1}(i_{V_2}(\Omega))$; an easy but tedious computation shows that the vector fields V_1 and V_2 are compatible if and only if the form ω satisfies the *integrability condition* $\omega \wedge d\omega = 0$.

A theorem of Frobenius' shows that compatibility is a sufficient condition for the existence of a local common first integral of two vector fields in K^3 ($K = \mathbb{R}$ or $\mathbb{C}$).

The *compatibility method* of Strelcyn and Wojciechowski [18] consists in looking for first integrals of a given vector field V that are also first integrals of linear vector fields.

Indeed, first integrals of linear vector fields are well known; precisely, it is always possible to find two functionally independent first integrals of any linear vector field.

Let us remark that every homogeneous vector field V (i.e. a vector field whose coordinates are homogeneous functions of the same degree) is compatible with the special *radial* (or *Euler's*) vector field $E = x\partial_x + y\partial_y + z\partial_z$, according to the well-known Euler relation. It follows that such a field has, at least locally, a first integral which is homogeneous of degree 0 (i.e. this first integral is also a first integral of the field E).

In a parametric situation, finding the values of the parameters for which the studied polynomial vector field is compatible with some non-zero linear vector field is an algebraic problem whose solution consists in solving a system of polynomial equations. Thus, whenever a compatible linear vector field is found, the discovery of a common first integral is a matter of skill.

This has been achieved with a surprising success in the particular example of the Lotka–Volterra system in a joint work of ours with B. Grammaticos, A. Ramani, J.-M. Strelcyn and S. Wojciechowski [7]. Using a computer algebra program, we used this method to find many cases, i.e. many values of the parameters A, B and C, for which a first integral (in finite terms) exists for the system

$$\begin{cases} L_x = x\,(Cy + z), \\ L_y = y\,(Az + x), \\ L_z = z\,(Bz + y). \end{cases}$$

In fact, we need not be skilful in finding the common first integral if we change our point of view. Let us now describe the facts without details: we shall be much more precise in the next section.

If V is a non-zero homogeneous polynomial vector field in K^3 and L a linear vector field compatible with V which is not collinear with the radial field E, an integrating factor can be found for the homogeneous 1-form $\omega = i_L i_V \Omega$, where Ω is the volume form $dx\,dy\,dz$. Simply choose $1/P$, where P is the non-zero homogeneous polynomial $i_E(\omega)$, for this integrating factor. Any primitive f of ω/P is then a first integral of V.

When L is a multiple of E, the polynomial P is 0 and nothing more can be done. But, in our study of the Lotka–Volterra system, we were not able to find a first integral only by using compatibility with the field E.

Moreover, it can be thought that the true problem, due to the homogeneity of the data, consists precisely in finding a first integral which is homogeneous of degree 0, i.e. a common first integral of V and E.

This problem can be solved in a second step in the case where the divergence of the field V is zero. Recall that the divergence of a vector field V is the function $\mathrm{div}(V)$ such that $d(i_V(\Omega)) = \mathrm{div}(V)\Omega$, and denote by s the degree of V. This assumption on the divergence is not a restriction, because, if we look for a common first integral of V and E, we can replace V by a new vector field V' with zero divergence:

$$V' = V - \frac{1}{s+2}\,\mathrm{div}(V)E$$

without changing the homogeneous first integrals of degree 0.

The found first integral f of V is then the logarithm of a function which is homogeneous of degree 1 and, under the previous assumption, $\exp(-(s+2)f)$ is an integrating factor g of the 1-form $\omega_0 = i_E(i_V(\Omega))$ and any primitive of $g\omega_0$ is a first integral of V and E.

It is now clear that linear vector fields do not play any special role in the problem and that a less restrictive method can be proposed for finding first integrals of homogeneous polynomial vector fields in K^3 that are homogeneous functions of degree 0.

Let us call this method *extended compatibility*. It consists in finding some homogeneous 1-forms ω that are integrable ($\omega \wedge d\omega = 0$), orthogonal to the given vector field V ($i_V(\omega) = 0$) and non-projective ($i_E(\omega) \neq 0$). Moreover, the useful forms have to be irreducible (no non-trivial common factor of the coordinates).

This will be the first and easier part of our main theorem: in the above situation, a homogeneous first integral of degree 0 can be built from an integral and an exponential. Thus, this first integral is Liouvillian, according to the definition given by Michael Singer. The second and more difficult part of the theorem will be the converse: the existence of some 1-form with coordinates in $k[x, y, z]$ with the above properties is a necessary condition for the existence of a Liouvillian first integral homogeneous of degree 0.

All that will be described in the completely formal framework of differential algebra; the class of Liouvillian elements over a given differential field is indeed defined in this context.

The algorithmic status of the problem of finding good 1-forms with coordinates in $k[x, y, z]$ is not yet known. Nevertheless, our theorem is not so poor: in a wide range of situations, the non-existence of such forms can be proven. In particular, a powerful method that can be found in the book of J.-P. Jouanolou [8] can be used. A comprehensive study of this method, together with new examples and remarks in more than 3 variables, will soon be available [9].

Moreover, for a homogeneous vector field V, the existence of a (non-homogeneous) Liouvillian first integral implies the existence of a homogeneous one; and this homogeneous first integral can be obtained by the extended compatibility method. That will be the content of our second theorem. This is the best to be expected: we then give a counterexample in which some homogeneous vector field has a homogeneous Liouvillian first integral but has no Liouvillian first integral which is homogeneous of degree 0.

An old method due to Gaston Darboux [6] is a special case of our extended compatibility method. It relies on the search of particular algebraic integrals (we shall call them *Darboux curves*) of a polynomial vector field V. A Darboux curve is an irreducible homogeneous polynomial f in $k[x, y, z]$ such that the inner product $i_V(df) = V_x\partial_x f + V_y\partial_y f + V_z\partial_z f$ is a multiple mf of f by some homogeneous polynomial "eigenvalue" m. If sufficiently many such Darboux curves can be found, a first integral can be built which is moreover homogeneous of degree 0. It is easy to show that Darboux's method is formally less powerful than extended compatibility. We give an example showing that our method is really stronger; this example is a special case of Lotka–Volterra's system.

Darboux curves have another relation with our method so that the search of all such curves for a given vector field is of great interest to us.

From an algorithmic point of view, the following question remains open: find an effective upper bound for the degree of Darboux curves of a given polynomial vector field. Michael Singer asks this question [14, 17], and Henri Poincaré was already interested in this problem, which he first considered easy [11–13]. A theorem of Jean-Pierre Jouanolou's [8] gives the non-effective existence of such an upper bound. But the method quoted before can sometimes be used to decide that a given vector field has no Darboux curve, and then that no good 1-form exists for it.

We conclude this paper with miscellaneous remarks.

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2. Main theorem

2.1. *The frame.* In the present section we are interested in a three-dimensional vector field

$$V = V_x\partial_x + V_y\partial_y + V_z\partial_z$$

whose coordinates are homogeneous polynomials of the same degree n in the polynomial ring $k[x, y, z]$; field k is an extension of the field $\mathbb{Q}$ of all rational numbers.

Denote by K the differential field $K = k(x, y, z)$ of all rational functions in three variables with coefficients in k. The derivations of K are of course the three commuting derivations ∂_x, ∂_y and ∂_z with respect to the space variables. We shall look for solutions to some problems in abstract differential extensions of K.

An element f of K is *homogeneous* of degree n, i.e. is the quotient of two homogeneous polynomials with coefficients in k whose degrees differ by n, if and only if $i_E(df) = nf$ (*Euler's identity*). In differential extensions of K, this identity will be the definition of homogeneity.

The *exterior derivative* of an element f on some differential extension of k is the formal three-dimensional object $df = \partial_x f\, dx + \partial_y f\, dy + \partial_z f\, dz$. More generally, we shall use notations of exterior calculus as compact ways to deal with the three derivations ∂_x, ∂_y and ∂_z at the same time.

The so-called Liouvillian extensions of K [17] will be of special interest to us. A *simple Liouvillian extension* is a pair (L, L') of differential fields, where L' is a differential extension of L which is generated by one element ($L' = L(t)$), and where this element has one of the following three properties.

• t is algebraic over L,

• t is transcendental over L and the derivatives of t belong to L (*integral case*),

• t is transcendental over L and the quotients by t of the derivatives of t belong to L (*exponential-integral case*)

A pair (K, L) of differential fields is said to be a *Liouvillian extension* if there exists a finite tower of simple Liouvillian extensions $(L_0, L_1), (L_1, L_2), \ldots, (L_{n-1}, L_n)$, where $L_0 = K$ and $L_n = L$.

2.2. *The result*

THEOREM 1. *Let V be a homogeneous polynomial vector field of degree n in three variables with coefficients in field k. Suppose that the divergence of V is* 0 *and that V is not proportional to Euler's field E, i.e. the* 1-*form $\omega_0 = i_E(i_V(\Omega))$ is not* 0, *Ω being the volume form $dx\,dy\,dz$. Then there exists a Liouvillian first integral of V which is homogeneous of degree* 0 *if and only if there exists some homogeneous polynomial* 1-*form ω with coordinates in the ground ring $k[x, y, z]$, in which the coordinates of V lie, which satisfies the following conditions*:

• *ω is integrable*: $\omega \wedge dw = 0$,

• *ω is orthogonal to V*: $i_V(\omega) = 0$,

• *ω is not projective*: $i_E(\omega) \neq 0$.

Remark 1. This means that extended compatibility is the way to get Liouvillian first integrals, i.e. is a well-suited method to solve "in finite terms" the problem of the existence of a first integral of a homogeneous polynomial vector field.

2.3. *The proof*. In one direction, this proof consists in giving all details, in the chosen formal frame, of what we described in the previous section.

In the other direction, our proof, although inspired by some results of Michael Singer [17], is probably simpler and more natural. Singer is interested in non-homogeneous two-variable polynomial vector fields: this difference is only a matter of style. As Michael is a great connoisseur of the subject, his proof refers to classical results by Rosenlicht [16] and Risch [15] while ours is self-contained. The stability of the subfield of constants plays a role in their arguments; in our opinion, this seems irrelevant here.

The reader can now follow us climbing up and down Liouvillian towers.

Several lemmas will be used in the proof; assumptions on V are the same for all of them and we shall speak of four *problems* in which R stands for some differential ring which is an extension of the polynomial ring $k[x, y, z]$:

Problem 1. Find an element f of R such that $i_V(df) = 0$, $i_E(df) = 0$ and $df \neq 0$.

Problem 2. Find a 1-form ω whose coordinates are elements of R such that $i_V(\omega) = 0$, $i_E(\omega) = 0$, $\omega \neq 0$ and $d\omega = 0$.

Problem 3. Find a 1-form ω whose coordinates are elements of R such that $i_V(\omega) = 0$, $i_E(\omega) = 1$ and $d\omega = 0$.

Problem 4. Find a 1-form ω whose coordinates are elements of R such that $i_V(\omega) = 0$, $i_E(\omega) \neq 0$ and $\omega \wedge d\omega = 0$.

Theorem 1 can be expressed in the following form: Problem 1 can be solved in some Liouvillian extension L of K if and only if Problem 4 has a homogeneous solution whose coordinates lie in the polynomial ring $k[x, y, z]$ itself.

Lemma 1. *Problem 4 has a homogeneous solution whose coordinates belong to $k[x, y, z]$ if and only if Problem 3 has a solution whose coordinates belong to the field $K = k(x, y, z)$.*

Proof. Let ω be a 1-form whose coordinates ω_x, ω_y and ω_z are homogeneous elements of K of the same degree and suppose that $i_V(\omega) = 0$, $i_E(\omega) = 1$ and $d\omega = 0$. Let Q be the least common multiple of the denominators of the coordinates of ω. Multiplying ω by Q, we get a new 1-form ω' whose coordinates are homogeneous polynomials of the same degree with coefficients in the field k and ω' is a solution to Problem 4.

In the other direction, consider a 1-form ω whose coordinates belong to $k[x, y, z]$ such that $i_V(\omega) = 0$, $i_E(\omega) \neq 0$ and $\omega \wedge d\omega = 0$. Denote by P the non-zero polynomial $i_E(\omega)$ and divide ω by P to get a new 1-form ω' whose coordinates belong to K.

The linear properties $i_V(\omega') = 0$ and $i_E(\omega') \neq 0$ still hold for ω'. The property $d\omega' = 0$ remains to be deduced from the integrability condition $\omega \wedge d\omega = 0$ on ω.

Calculate therefore the exterior derivative of $\omega' = \omega/P$:

$$d\omega' = d(\omega/P) = (1/P^2)(Pd\omega - dP \wedge \omega).$$

To prove that the numerator is the zero 2-form, it suffices to obtain the following two equalities:

$$\omega \wedge i_E(d\omega) = Pd\omega, \tag{3}$$

$$\omega \wedge i_E(d\omega) = dP \wedge \omega. \tag{4}$$

Taking the inner product by Euler's field E of the identity $\omega \wedge d\omega = 0$ yields equality (3). As the coordinates of ω are homogeneous of some degree m, the generalized Euler formula (for 1-forms) yields

$$di_E(\omega) + i_E(d\omega) = (1 + m)\omega$$

and then the exterior product by ω gives (4). ∎

LEMMA 2. *If Problem 1 has a solution in a differential ring R which is an extension of $k[x,y,z]$ then Problem 2 can be solved in the same ring R.*

Proof. If f is a solution to Problem 1, then its exterior derivative df is a solution to Problem 2. ∎

LEMMA 3. *If Problem 2 can be solved in a differential field L which is an extension of $K = k(x,y,z)$, then there exists an integrating factor for the 1-form $\omega_0 = i_E(i_V(\Omega))$ in L, i.e. a non-zero element ϕ of L such that $d(\phi\omega_0) = 0$.*

Proof. Let ω be a solution to Problem 2 and calculate the exterior product of ω by $\omega_0 = i_E(i_V(\Omega))$, where Ω is the volume form $dx\,dy\,dz$:

$$\omega \wedge i_E(i_V(\Omega)) = i_E(\omega \wedge i_V(\Omega))$$

as $i_E(\omega)$ is equal to 0. Similarly, $\omega \wedge i_V(\Omega) = -i_V(\omega \wedge \Omega)$ as $i_V(\omega)$ is equal to 0. But a 4-form like $\omega \wedge \Omega$ is 0, so that $\omega \wedge \omega_0$ is 0. This means that ω and ω_0 are collinear, and precisely, as ω_0 is not 0, that there exists a non-zero element ϕ of L such that $\omega = \phi\omega_0$. ∎

LEMMA 4. *Let ϕ be a non-zero element of some differential field L which is an extension of $K = k(x,y,z)$. This ϕ is an integrating factor for $\omega_0 = i_E(i_V(\Omega))$ if and only if ϕ is a homogeneous first integral of V of degree $-(n+2)$, n being the degree of V.*

Proof. As $\omega = \phi\omega_0$ is supposed to be closed,

$$d\omega = 0 = d(\phi\omega_0) = d\phi \wedge \omega_0 + \phi\, d\omega_0. \tag{5}$$

The inner product of (5) by Euler's field yields

$$0 = i_E(d\phi)\omega_0 - 0 + \phi i_E(d\omega_0)$$

and thus

$$i_E(d\phi/\phi)\omega_0 = -i_E(d\omega_0).$$

On the other hand, as ω_0 is a homogeneous 1-form of degree $n+1$, the generalized Euler formula (for 1-forms) yields

$$(n+2)\omega_0 = i_E(d\omega_0) + di_E(\omega_0) = i_E(d\omega_0).$$

All that proves the identity

$$i_E(d\phi) = -(n+2)\phi.$$

The inner product of (5) by V yields

$$0 = i_V(d\phi)\omega_0 - 0 + \phi i_V(d\omega_0)$$

and so

$$i_V(d\phi)\omega_0 = -\phi i_V(d\omega_0). \tag{6}$$

But $i_V(\Omega)$ is a homogeneous 2-form of degree n and we get (Euler's formula for 2-forms)

$$(n+2)i_V(\Omega) = i_E(di_V(\Omega)) + di_E(i_V(\Omega)).$$

As the divergence of V is 0, $d\omega_0 = (n+2)i_V(\Omega)$. And a second inner product by V yields 0, which proves $i_V(d\omega_0) = 0$ and, together with (6), the identity $i_V(d\phi) = 0$.

Thus, assuming that $\omega = \phi\omega_0$ is closed leads to the fact that ϕ is a first integral of V and is homogeneous of degree $-(n+2)$.

In the other direction, suppose that

$$i_V(d\phi) = 0, \qquad i_E(d\phi) = -(n+2)\phi.$$

Then calculate the exterior derivative of $\omega = \phi\omega_0$:

$$d\omega = d(\phi\omega_0) = d\phi \wedge \omega_0 + \phi d\omega_0. \tag{7}$$

To prove that the right hand side is 0, it suffices to start from the trivial fact that the 4-form $d\phi \wedge \Omega$ is 0; taking the inner product by V gives $d\phi \wedge i_V(\Omega) = 0$ as ϕ is a first integral of V.

The inner product by E then yields

$$i_E(d\phi) \wedge i_V(\Omega) = d\phi \wedge \omega_0.$$

As previously seen, the zero divergence of V gives the value $(n+2)i_V(\Omega)$ for the exterior derivative of ω_0. As ϕ is homogeneous of degree $-(n+2)$, the two terms of (7) cancel, which proves the result. ∎

LEMMA 5. *If Problem* 2 *can be solved in a differential field L which is an extension of $K = k(x,y,z)$ then Problem* 3 *can be solved in the same field L.*

P r o o f. According to the previous two lemmas, an integrating factor ϕ of ω_0 can be found in L. This integrating factor is a non-trivial first integral of V and is homogeneous of degree $-(n+2)$. As the third property

$$d\left[-\frac{1}{n+2}\frac{d\phi}{\phi}\right] = 0$$

is clear, $-\frac{1}{n+2}\frac{d\phi}{\phi}$ is a solution to Problem 3. ∎

LEMMA 6. *If Problem* 3 *has a solution ω whose coordinates belong to some differential field L which is an extension of K, then a simple Liouvillian extension $L(t)$ can be built in which Problem* 2 *has a solution.*

P r o o f. Let $L(t)$ be the field of rational fractions with coefficients in the differential field L. To extend derivations from L to $L(t)$, it suffices to define their values for t; let $t\omega_x$, $t\omega_y$ and $t\omega_z$ be the values of the three derivations for t. This determines completely, and in a unique way, the extension of the derivations to $L(t)$.

As ω is closed, these three derivations, which commute on L, also commute for the generator t so that they are commuting derivations of $L(t)$. Then $\omega/(t^{n+2})$ is a solution to Problem 2 and its coordinates belong to $L(t)$, which is a simple Liouvillian extension of L of exponential-integral type. ∎

LEMMA 7. *If Problem* 2 *has a solution* ω *whose coordinates belong to some differential field* L *which is an extension of* K, *then a simple Liouvillian extension* $L(t)$ *can be built in which Problem* 1 *has a solution.*

Proof. Let $L(t)$ be the field of rational fractions with coefficients in the differential field L. Let ω_x, ω_y and ω_z be the values of the three derivations for t. This determines completely, and in a unique way, the extension of the derivations to $L(t)$.

As ω is closed, these three derivations, which commute on L, also commute for the generator t so that they are commuting derivations of $L(t)$. And t, whose derivative is ω, is a solution to Problem 1 and belongs to $L(t)$, which is a simple Liouvillian extension of L of integral type. ∎

We can now summarize the proof of the theorem in one direction, i.e. we can build a solution to Problem 1 in a two-level Liouvillian extension of $K = k(x, y, z)$ from a solution to Problem 4 in the ring $k[x, y, z]$. Indeed, Lemma 1 gives a solution to Problem 3 in K; then Lemma 6 gives a solution to Problem 2 in a simple Liouvillian extension of K of exponential-integral type; applying Lemma 7 then yields a solution to Problem 1 in a simple Liouvillian extension of integral type of the preceding field.

Climbing down Liouvillian towers allows us to get a proof in the other direction; this proof relies on the next three lemmas.

LEMMA 8. *If Problem* 3 *has a solution whose coordinates belong to* $L(t)$, *where* t *is algebraic over* L, *then a solution to Problem* 3 *can be found with coordinates in* L.

Proof. Let $\overline{L}$ be the splitting field of the minimal polynomial of t over L. The Galois group of algebraic automorphisms of $\overline{L}$ over L commutes with derivations, because of the well-known uniqueness of the extension of a derivation in the algebraic case.

The defining properties of Problem 3 are affine, and replacing every coordinate of ω by the mean value of its conjugates gives a new solution to Problem 3, but with coordinates in the ground field L. ∎

LEMMA 9. *If Problem* 3 *has a solution whose coordinates belong to* $L(t)$, *where* t *is transcendental over* L *and of exponential-integral type*, *then a solution to Problem* 3 *can be found with coordinates in* L.

Proof. Let ω be a solution to Problem 3 with coordinates ω_x, ω_y and ω_z belonging to $L(t)$. These coordinates can be written as

$$\omega_x = \frac{P(t)}{S(t)}, \quad \omega_y = \frac{Q(t)}{S(t)}, \quad \omega_z = \frac{R(t)}{S(t)},$$

where P, Q, R and S are univariate polynomials in t with coefficients in L and where S is unitary. Moreover, the derivative dt of t is equal to $t\eta_x\,dx + t\eta_y\,dy + t\eta_z\,dz$, where η_x, η_y and η_z belong to L.

The hypotheses on ω are then the following:

$$V_xP + V_yQ + V_zR = 0, \qquad xP + yQ + zR = S,$$
$$\partial_y(P/S) = \partial_x(Q/S), \quad \partial_z(Q/S) = \partial_y(R/S), \quad \partial_x(R/S) = \partial_z(P/S).$$

Consider now the Euclidean divisions of polynomials P, Q and R by the polynomial S:

$$P = P_1S + P_2, \quad Q = Q_1S + Q_2, \quad R = R_1S + R_2,$$

and denote by P_1^0, Q_1^0 and R_1^0 the constant terms (as polynomials in t) of the quotient polynomials P_1, Q_1 and R_1. These terms belong to L and satisfy the two identities

$$V_xP_1^0 + V_yQ_1^0 + V_zR_1^0 = 0 \quad \text{and} \quad xP_1^0 + yQ_1^0 + zR_1^0 = 1,$$

as can be seen by dividing the corresponding expressions for P, Q and R by S and then considering the coefficients of t^0 in the quotients.

The property of the “mixed derivatives” remains to be proven in order to show that $\omega' = P_1^0\,dx + Q_1^0\,dy + R_1^0\,dz$ is a solution to Problem 3:

$$\partial_y(P_1^0) = \partial_x(Q_1^0), \quad \partial_z(Q_1^0) = \partial_y(R_1^0), \quad \partial_x(R_1^0) = \partial_z(P_1^0).$$

We know similar properties for P/S, Q/S and R/S; for instance,

$$\partial_y(P/S) = \partial_y(P_1 + P_2/S) = \partial_x(Q/S) = \partial_x(Q_1 + Q_2/S).$$

If the polynomial S is equal to 1, the conclusion $\partial_y(P_1) = \partial_x(Q_1)$ is immediate, and if the degree of S is greater than or equal to 1, this is a consequence of the fact that the partial derivatives of P_2/S and Q_2/S, as fractions in t, have numerators with a degree strictly smaller than 2 times the degree of S.

Thus $\partial_y(P_1)$ is the sum of $\partial_y(P_1^0)$ and of a polynomial in t without constant term, because the partial derivatives of t are multiples of t by elements of L.

Considering now the constant terms (with respect to t) of the identities

$$\partial_y(P_1) = \partial_x(Q_1), \quad \partial_z(Q_1) = \partial_y(R_1) \quad \text{and} \quad \partial_x(R_1) = \partial_z(P_1)$$

leads us to the sought result:

$$\partial_y(P_1^0) = \partial_x(Q_1^0), \quad \partial_z(Q_1^0) = \partial_y(R_1^0), \quad \partial_x(R_1^0) = \partial_z(P_1^0). \ \blacksquare$$

Lemma 10. *If Problem* 3 *has a solution whose coordinates belong to* $L(t)$, *where* t *is transcendental over* L *and of integral type*, *then a solution to Problem* 3 *can be found with coordinates in* L.

Proof. The proof is similar to the preceding one. Consider a solution to Problem 3 with coordinates in $L(t)$:

$$\omega_x = \frac{P(t)}{S(t)}, \quad \omega_y = \frac{Q(t)}{S(t)}, \quad \omega_z = \frac{R(t)}{S(t)},$$

where P, Q, R and S are univariate polynomials in t with coefficients in L and where S is unitary. Euclidean division by S also gives a solution to Problem 3 in which the coordinates are the quotient polynomials P_1, Q_1 and R_1.

Let now m be the maximum degree of P_1, Q_1 and R_1. The corresponding coefficients P_1^m, Q_1^m and R_1^m are the coordinates of some 1-form. If m is equal to 0, this form is a solution to Problem 3, and if it is not 0, we get a solution to Problem 2. Indeed, derivation with respect to t lowers the degrees, so that the property of mixed derivatives holds for these leading coefficients; as V and E have coordinates which are polynomials of degree 0 (with respect to t of course), the identities involving inner products are proved.

In the first case, the proof is finished; in the second case, Lemma 5 allows us to get a solution to Problem 3 with coordinates in the same field as a given solution to Problem 2, which completes the proof. ∎

We can now conclude the proof of Theorem 1.

Consider the following tower of simple Liouvillian extensions:

$$K = L_0 = k(x, y, z) \subset L_1 = L_0(t_1) \subset \ldots \subset L_n = L_{n-1}(t_n) = L$$

and suppose that Problem 1 has a solution in L.

According to Lemmas 2 and 5, Problem 3 has a solution in the biggest field L. By the last three lemmas, for every type of Liouvillian simple extension $(L_i, L_i(t))$, if there exists a solution to Problem 3 in $L_i(t)$, then a solution to the same problem can be found in L_i; so that, by induction, there exists a solution to Problem 3 with coordinates in the first field K. By Lemma 1, Problem 4 can be solved with coordinates in the ring $k[x, y, z]$.

2.4. *A complementary result*. The existence of a non-homogeneous Liouvillian first integral of a homogeneous vector field V is already interesting; indeed, it implies that the extended compatibility method can be applied with success to V, as follows from the next theorem.

Theorem 2. *Let* V *be a homogeneous polynomial vector field of degree* n *in three variables with coordinates in the ring* $k[x, y, z]$. *Suppose that* V *is not proportional to Euler's field* E, *i.e. the* 1*-form* $\omega_0 = i_E(i_V(\Omega))$ *is not*

0, *Ω being the volume form $dx\,dy\,dz$. Let D be the divergence of V, i.e. the element of $k[x,y,z]$ given by $d(i_V(\Omega)) = D\Omega$. Let $\overline{V}$ be the translate $\overline{V} = V - (D/(n+2))E$ of V; the divergence of $\overline{V}$ is 0. Then, if V has some Liouvillian first integral, there exists a homogeneous polynomial 1-form ω with coordinates in the ground ring $k[x,y,z]$, in which the coordinates of V lie, such that ω satisfies the following conditions*:

- *ω is integrable*: $\omega \wedge dw = 0$,
- *ω is either orthogonal to V or to $\overline{V}$* : $i_V(\omega) = 0$ *or* $i_{\overline{V}}(\omega) = 0$,
- *ω is not projective*: $i_E(\omega) \neq 0$.

Like the proof of the preceding theorem, this one consists in finding a solution to some problem in a differential field L from a solution to the same or another problem in a simple Liouvillian extension $L(t)$ of L. We shall therefore need several lemmas.

LEMMA 11. *Let V be a homogeneous vector field with coordinates in the ring $R = k[x,y,z]$; if a closed 1-form ω has coordinates in some differential extension L of R and satisfies $i_V(\omega) = 0$ then the identity $i_V(df) = 0$ holds for the element $f = i_E(\omega)$ of L.*

Proof. The proof is an easy computation; simply write the inner product $i_V(df)$, use the fact that ω is closed, and the three derivatives of the identity $i_V(\omega) = 0$ to get

$$i_V(df) = -\omega_x(i_E(V_x)) - \omega_y(i_E(V_y)) - \omega_z(i_E(V_z)).$$

But, as the coordinates of V are homogeneous of the same degree, the right hand side is a multiple of $i_V(\omega)$ and thus it is equal to 0. ∎

LEMMA 12. *Let L be a differential field that is an extension of $k(x,y,z)$ and let $L(t)$ be an algebraic extension of L. Suppose that there exists an element f of $L(t)$ which is a first integral of V, i.e. such that $i_V(df) = 0$ and $df \neq 0$. Then there exists a first integral of V in L.*

Proof. Let P be the minimal unitary polynomial of f with coefficients in L:

$$P(f) = \sum_{i=0}^{m} P_i f^i = 0, \qquad P_m = 1.$$

Differentiating $P(f)$ with respect to x, y and z, and then taking the inner product by V yields

$$\sum_{i=0}^{m-1} f^i i_V(dP_i) = 0$$

and all inner products $i_V(dP_i)$ are 0. One of the dP_i is not 0, as f is not a constant; and the corresponding P_i is the sought first integral of V that belongs to L. ∎

LEMMA 13. *Let L be a differential field that is an extension of $k(x,y,z)$ and let $L(t)$ be a transcendental extension of L. Suppose that $L(t)$ is a simple Liouvillian extension of L of exponential-integral type, i.e. dt/t is a closed 1-form η with coordinates in L. If there exists an element f of $L(t)$ which is a first integral of V, then there exists a non-zero closed 1-form ω with coordinates in L such that $i_V(\omega)=0$.*

Proof. The element f can be written as $f=P(t)/Q(t)$, where P and Q are relatively prime polynomials with coefficients in L and where Q is unitary. As f is a first integral of V, the 1-form $PdQ-QdP$ is not 0 but its inner product by V is 0.

The derivatives of P and Q are

$$dQ=\sum_{i=0}^{n} t^i dQ_i+\eta\sum_{i=0}^{n} it^iQ_i,\qquad dP=\sum_{i=0}^{m} t^i dP_i+\eta\sum_{i=0}^{m} it^iP_i,$$

and the inner product by V of $PdQ-QdP$ gives the following equality of polynomials:

$$P\sum_{i=0}^{n} t^i(i_V(dQ_i)+iQ_i i_V(\eta))=Q\sum_{i=0}^{m} t^i(i_V(dP_i)+iP_i i_V(\eta)).$$

As P and Q are relatively prime, there exists an element α of L such that, for every i,

$$i_V(dQ_i)+iQ_i i_V(\eta)=\alpha Q_i,\qquad i_V(dP_i)+iP_i i_V(\eta)=\alpha P_i.$$

Moreover, α is easily seen to be equal to $ni_V(\eta)$ because $Q_n=1$.

Thus, for every i, $i_V(dP_i)=P_i(n-i)i_V(\eta)$. As f is not a constant, some of the coefficients P_i or Q_i is different from 0 and satisfies $dP_i/P_i\neq(n-i)\eta$. If $i_V(\eta)=0$, η is the sought result. Otherwise $\omega=dP_i/P_i-(n-i)\eta$ has the desired properties. ∎

LEMMA 14. *Let L be a differential field which is an extension of $k(x,y,z)$ and let $L(t)$ be a transcendental extension of L. Suppose that $L(t)$ is a simple Liouvillian extension of L of integral type, i.e. dt is a closed 1-form η with coordinates in L. If there exists an element f of $L(t)$ which is a first integral of V, then there exists a first integral of V in L.*

Proof. The element f can be written as $f=P(t)/Q(t)$, where P and Q are relatively prime polynomials with coefficients in L and where Q is unitary. As f is a first integral of V, the 1-form $PdQ-QdP$ is not 0 but its inner product by V is 0.

The derivatives of P and Q are

$$dQ=\sum_{i=0}^{n} t^i dQ_i+\eta\sum_{i=1}^{n} it^{i-1}Q_i,\qquad dP=\sum_{i=0}^{m} t^i dP_i+\eta\sum_{i=1}^{m} it^{i-1}P_i,$$

and the inner product by V of $PdQ - QdP$ gives the following equality of polynomials:

$$P\sum_{i=0}^{n} t^i(i_V(dQ_i))+i_V(\eta)\sum_{i=1}^{n} it^{i-1}Q_i = Q\sum_{i=0}^{m} t^i(i_V(dP_i))+i_V(\eta)\sum_{i=1}^{m} it^{i-1}P_i.$$

As P and Q are relatively prime, there exists an element α of L such that, for every i,

$$i_V(dQ_i) + (i+1)Q_{i+1}i_V(\eta) = \alpha Q_i, \qquad i_V(dP_i) + (i+1)P_{i+1}i_V(\eta) = \alpha P_i.$$

Moreover, α is easily seen to be equal to 0 because $Q_n = 1$. Thus, $i_V(dP_m) = 0$ while P_m is not 0 and P_m is the sought first integral. ∎

We can now conclude the proof of Theorem 2. Consider the following tower of simple Liouvillian extensions:

$$K = L_0 = k(x,y,z) \subset L_1 = L_0(t_1) \subset \ldots \subset L_n = L_{n-1}(t_n) = L$$

and suppose that f is a first integral of V that belongs to L. Climbing down this tower, we find a first integral of V in K, unless this process stops when applying Lemma 13, in which case we get some non-zero closed 1-form ω with coordinates in an intermediate L_i such that $i_V(\omega) = 0$. According to Lemma 11, $\phi = i_E(\omega)$ is a first integral of V in L_i or a constant.

The process stops if ϕ is a constant. If this constant is 0, then ω is a solution to Problem 2 for V and also for the translated vector field $\overline{V}$ with divergence 0; the proof of Theorem 1 can then be applied to get the result. If this constant is different from 0, we can divide by it to get a solution to Problem 3 in L_i and go further.

Finally, we get either a first integral of V in $K = k(x,y,z)$ or a solution to Problem 3 for V or $\overline{V}$ with coordinates in K. If the final result of the process is a first integral ϕ of V in the field $K = k(x,y,z)$ of rational fractions, then the quotient of the homogeneous components of highest degree of the numerator and denominator of ϕ is a homogeneous first integral of V, which concludes the proof.

2.5. *A counterexample*. The previous improvement of the main theorem is the best result to be expected in that direction; indeed, here follows a special case of the Lotka–Volterra system which has no Liouvillian first integral that is homogeneous of degree 0, while some homogeneous first integral does exist. Let us postpone this example until we have defined Darboux curves because its proof shares arguments with further statements.

3. Darboux's method and Darboux curves

3.1. *Darboux's method*. Another method for finding first integrals of homogeneous polynomial vector fields in finite terms dates back to a memoir by Darboux [6]. Let us describe it.

A *particular algebraic solution* or *Darboux curve* of a given homogeneous polynomial vector field V is an irreducible homogeneous polynomial f such that

$$i_V(df) = V_x\partial_x f + V_y\partial_y f + V_z\partial_z f$$

i.e. $i_V(df)$ is a multiple of f by some polynomial m.

Looking at the base field k as a subfield of $\mathbb{R}$ or $\mathbb{C}$, the geometric meaning of this property is that the projective curve (or conic surface) $\{f = 0\}$ consists of trajectories of the vector field V or that the local semigroup generated by V preserves this set.

Suppose now that the divergence of V is 0; this assumption is not a restriction for what we are interested in. Suppose that we can find several such Darboux curves f_i ($i_V(df_i) = m_i f_i$), so that a linear combination $\sum a_i m_i$ of the "eigenvalues" m_i is 0 with some a_i different from 0. The product $f = \prod f_i^{a_i}$ is then a homogeneous first integral of V; of course, if the exponents are not rational, this integral may not be a true function, which means that we stay in the abstract differential algebra frame.

Then, either $\sum a_i \deg(f_i) = 0$, in which case f has a homogeneity degree 0 and the job is done, or some well-chosen power of f is an integrating factor of the 1-form $\omega_0 = i_E(i_V(\Omega))$ and the conclusion follows from the same computations as in the extended compatibility method.

3.2. *Extended compatibility versus Darboux's method.* Darboux's method is formally less general than extended compatibility in the sense that, if there are sufficiently many Darboux curves, the 1-form

$$\sum a_i \frac{df_i}{f_i}$$

is a solution to Problem 2 in the special case where $\sum a_i \deg(f_i) = 0$ and, in the other cases, its quotient by this coefficient $\sum a_i \deg(f_i)$ is a solution to Problem 3. Thus, Darboux's method is a way to get what we look for in the extended compatibility method more quickly than usual, in some situations.

To show that our method is really more general than Darboux's, we now give an example in which we are able to exhibit a 1-form ω solving Problem 4 which yields a Liouvillian first integral according to Theorem 1; in this example, all Darboux curves can be found, and they are not numerous enough to yield a result according to Darboux's method.

EXAMPLE 1. Consider Lotka–Volterra's field

$$\begin{cases} L_x = x(Cy + z), \\ L_y = y(Az + x), \\ L_z = z(Bz + y), \end{cases}$$

and take particular values $A = -1$, $B = 1/2$ and $C = 0$ of the parameters. Let $\bar{L}$ be the corresponding field with zero divergence and consider the

1-form ω

$$\omega = \frac{1}{2}\frac{dx}{x} - \frac{1}{2}\frac{dy}{y} + \frac{dz}{z} - dF,$$

where F is given by

$$F = \frac{(x+y)^2}{xy}.$$

Then ω satisfies $i_{\bar{L}}(\omega) = 0$ and $i_E(\omega) = 1$, and, as it is closed, this form is a solution to Problem 3, so that extended compatibility gives a result in this situation.

The proof of this fact is an easy and uninteresting computation. Moreover, this particular set of values of the parameters A, B and C does not appear in the complete list given in [7], so that ordinary compatibility with linear vector fields is not sufficient here.

Conversely, Darboux's method cannot be applied here according to the next proposition.

PROPOSITION 1. *In the above example, there are only three Darboux curves, x, y and z; the corresponding eigenvalues are linearly independent, so that no non-trivial linear combination of the logarithmic derivatives is a first integral of $\overline{L}$.*

Proof. It suffices to prove that there does not exist any homogeneous non-trivial polynomial f such that $i_L(df) = lf$ without x, y or z as a factor. The eigenvalue l is here a first degree homogeneous polynomial $\lambda x + \mu y + \nu z$ because the degree of V is 2.

Such an f would satisfy

$$x(Cy+z)\partial_x f + y(Az+x)\partial_y f + z(Bx+y)\partial_z f = (\lambda x + \mu y + \nu z) f.$$

As f is supposed not to be divisible by x, y or z, let P, Q and R be the three homogeneous non-zero two-variable polynomials obtained by setting $x=0$, $y=0$ and $z=0$ respectively in f. These three polynomials have the same degree n, which is the degree of f, and satisfy

$$\begin{cases} (\mu y + \nu z)P = yz(A\partial_y P + \partial_z P), \\ (\nu z + \lambda x)Q = zx(B\partial_z Q + \partial_x Q), \\ (\lambda x + \mu y)R = xy(C\partial_x R + \partial_y R). \end{cases}$$

It is not very difficult to prove that there exist 6 natural numbers β_1, γ_1, α_2, γ_2, α_3 and β_3 such that P, Q and R are non-zero multiples of $y^{\beta_1} z^{\gamma_1}(y - Az)^{n-\beta_1-\gamma_1}$, $z^{\gamma_2} x^{\alpha_2}(z - Bx)^{n-\gamma_2-\alpha_2}$ and $x^{\alpha_3} y^{\beta_3}(x - Cy)^{n-\alpha_3-\beta_3}$ respectively; moreover, these numbers satisfy the following equations and inequalities:

$$\begin{cases} \lambda = \beta_3 = \gamma_2 B, \\ \mu = \gamma_1 = \alpha_3 C, \\ \nu = \alpha_2 = \beta_1 A, \\ \beta_1 + \gamma_1 \le n, \\ \alpha_2 + \gamma_2 \le n, \\ \alpha_3 + \beta_3 \le n. \end{cases}$$

With the given values of the parameters A, B and C, we get

$$\begin{cases} \lambda = \beta_3 = \gamma_2/2, \\ \mu = \gamma_1 = 0, \\ \nu = \alpha_2 = -\beta_1, \end{cases}$$

and f, that can generally be written as

$$\begin{cases} f = \alpha(y^{\beta_1} z^{\gamma_1} (y - Az)^{n-\beta_1-\gamma_1}) + x? \\ \quad = \beta(z^{\gamma_2} x^{\alpha_2} (z - Bx)^{n-\gamma_2-\alpha_2}) + y? \\ \quad = \gamma(x^{\alpha_3} y^{\beta_3} (x - Cy)^{n-\alpha_3-\beta_3}) + z?, \end{cases}$$

where α, β and γ are some non-zero numbers and where the question mark stands for any uninteresting homogeneous polynomial of degree $n-1$, reads

$$\begin{cases} f = \alpha(y + z)^n + x?, \\ \quad = \beta(z^{2\lambda}(z - x/2)^{n-2\lambda}) + y?, \\ \quad = \gamma(x^{n-\lambda} y^{\lambda}) + z?, \end{cases}$$

as $\alpha_2 = \beta_1 = 0$.

Looking at the coefficients of y^n, we get $\lambda = n$, which is impossible since 2λ must be smaller than n. ■

There is another link between extended compatibility analysis and Darboux curves, which is given by the following proposition.

PROPOSITION 2. *Let V be a vector field whose coordinates are homogeneous polynomials of the same degree. Let ω be an irreducible 1-form whose coordinates are homogeneous polynomials of the same degree and let P be some homogeneous polynomial. If ω/P is closed and $i_V(\omega) = 0$, then the irreducible factors of P are Darboux curves of V.*

Proof. As ω/P is closed, ω satisfies the integrability condition $\omega \wedge d\omega = 0$. Taking the inner product of this identity by V yields

$$i_V(\omega) \wedge d\omega = \omega \wedge i_V(d\omega)$$

and ω and $i_V(d\omega)$ are collinear. But ω is irreducible, so that there exists a polynomial N such that $i_V(d\omega) = N\omega$. The condition $d(\omega/P) = 0$ reads $Pd\omega = dP \wedge \omega$; taking the inner product by V and cancelling the non-zero form ω gives

$$i_V(dP) = NP.$$

Thus P is a particular solution of V and it is not difficult to finish the proof: irreducible factors P_i of P (maybe in some extension of the field k) also satisfy the identities $i_V(dP_i) = N_i P_i$ for some polynomial eigenvalues N_i, i.e. these factors are Darboux curves. ∎

3.3. *The announced counterexample*. Consider the previously quoted Lotka–Volterra field:

$$\begin{cases} L_x = x(Cy+z), \\ L_y = y(Az+x), \\ L_z = z(Bz+y), \end{cases}$$

and take the particular values $A = 0$, $B = 0$ and $C = 0$ of the parameters. Let $\overline{L}$ be the corresponding field with zero divergence. For a suitable choice of λ, μ and ν, $\overline{L} + (\lambda x + \mu y + \nu z)E$ has the homogeneous first integral $f = xyz$. But there is no Liouvillian first integral of degree 0 for it as follows from the next proposition.

PROPOSITION 3. *Let V_0 be the vector field*

$$\begin{cases} L_x = xz, \\ L_y = yx, \\ L_z = zy. \end{cases}$$

There is no 1*-form ω whose coordinates are homogeneous polynomials of $k[x,y,z]$ such that $i_{\overline{V}_0}(\omega) = 0$, $i_E(\omega) \neq 0$ and $\omega \wedge d\omega = 0$ and, according to Theorem* 1, *V_0 has no Liouvillian first integral which is homogeneous of degree* 0.

Proof. First look for Darboux curves of V_0; of course, x, y and z are some of them. There is no non-zero homogeneous polynomial f such that $i_{V_0}(df) = (\lambda x + \mu y + \nu z)f$ without x, y or z as a factor by arguments similar to those for Proposition 2. No non-trivial linear combination ω of dx/x, dy/y and dz/z satisfies $i_{V_0}(\omega)$ and Darboux's method cannot be applied.

The extended compatibility method consists in finding a closed 1-form ω with homogeneous coordinates of degree -1 in $k(x,y,z)$ such that $i_{V_0}(\omega) = 0$. The least common multiple of the denominators of the coordinates of ω would be proportional to some monomial $x^{a'}y^{b'}z^{c'}$ as x, y and z are the only Darboux curves of V_0. Such an ω would then read

$$\omega = \frac{\omega_x dx + \omega_y dy + \omega_z dz}{x^a y^b z^c},$$

where ω_x, ω_y and ω_z are relatively prime homogeneous polynomials of the same degree in $k[x,y,z]$. Simple computations lead to the decomposition

$$\omega = \lambda \frac{dx}{x} + \mu \frac{dy}{y} + \nu \frac{dz}{z} + d\left(\frac{N}{x^a y^b z^c}\right)$$

where N is some non-zero homogeneous polynomial of degree $a+b+c \geq 1$ in $k[x,y,z]$; moreover, N is not divisible by x, y or z. And N would satisfy

$$i_{V_0}\left(d\left(\frac{N}{x^a y^b z^c}\right)\right) = -(\lambda z + \mu x + \nu y),$$

which means that $zxN_x + xyN_y + yzN_z - N(az+bx+cy)$ is a multiple of $x^a y^b z^c$. A careful and cumbersome case analysis (how many zeros among a, b and c) shows that such a polynomial N cannot exist, which concludes the proof. ∎

4. Final remarks

4.1. *Finding all Darboux curves.* In order to transform the previous methods in true algorithms, it would be of interest to find all Darboux curves of a given polynomial vector field; we need therefore an upper bound for the degree of such curves, that would reduce the problem to linear algebra. A theorem of Jouanolou's [8] gives the existence of this upper bound; but this proof is not effective. Nevertheless, it is sometimes possible to show that no Darboux curve exists for a vector field, which proves that it is not possible to find a Liouvillian first integral which is homogeneous of degree 0.

To show that such a situation is in fact generic, Jouanolou [8] proves that the following vector field has no Darboux curve:

$$V = z^m\partial_x + x^m\partial_x + y^m\partial_x, \qquad m \geq 2.$$

His very interesting proof does not seem to be a first step in the direction of an algorithm; indeed, very special arithmetic properties of this example are used. The proof that we gave in the above example of the fact that extended compatibility is more powerful than Darboux's method, also uses specific arithmetic properties to show that there are only three Darboux curves.

For us, all that gives an illustration of the true difficulty of this algorithmic question: find all Darboux curves of a given polynomial vector field. However, the class of examples [9] to which the method of Jouanolou can be applied is far from being empty. For these examples, the extremal situation can be proven: no Darboux curve and then no Liouvillian first integral of degree 0 exists for the given derivation.

4.2. *Finding some Darboux curves.* For a particular system, like Lotka–Volterra's, the search of Darboux curves with a given degree is quite feasible using computer algebra. We found many exceptional values of the parameters A, B and C for which a new fourth Darboux curve occurs (x, y and z are always Darboux curves of this factorizable system and, in the case of a second degree vector field, four different Darboux curves are enough to apply Darboux's method); we did the complete job for degrees 1 to 6.

4.3. *Are these first integrals true functions*? It can happen that the elements f lying in some Liouvillian extension of the ground field, which can be considered as first integrals of some vector fields, are not true functions from $\mathbb{R}^3$ to $\mathbb{R}$, or even from $\mathbb{R}^3$ to the circle, due to topological properties of the natural open set in which they would be defined. Nevertheless, we did not find an explicit example of such a situation: all first integrals found in the examples that we have dealt with up to now using extended compatibility are true functions. Further computations are to be done in order to understand this point.

4.4. *A formal Frobenius theorem.* We conclude this paper by giving an analogue of the theorem of Frobenius that we quoted in the first section in the frame of differential algebra.

THEOREM 3. *Let K be a field of characteristic* 0; *suppose that ∂_x, ∂_y and ∂_z are three commuting derivations of K. Let ω be a 1-form with coordinates ω_x, ω_y and ω_z in K and suppose that ω satisfies the integrability condition $\omega \wedge dw = 0$, which reads*

$$\omega_x(\omega_{yz} - \omega_{zy}) + \omega_y(\omega_{zx} - \omega_{xz}) + \omega_z(\omega_{xy} - \omega_{yx}) = 0.$$

Then the three coordinates ϕ_x, ϕ_y and ϕ_z of a closed 1-form ϕ can be found in a differential ring R that extends K, such that $\phi \wedge \omega = d\omega$. There then exists an element g such that $dg = \phi$ in a simple Liouvillian extension of the quotient field $\overline{R}$ of R; this extension is of integral type. Thus, $\exp(-g)$ *is an integrating factor of ω according to the previous computations.*

P r o o f. If $\omega = 0$, there would be no work to do, so that one of the coordinates, for instance ω_z, can be supposed to be different from 0.

Consider now the ring $R = K[\phi_1, \phi_2, \ldots]$ in infinitely many coordinates with coefficients in the field K. Setting $\partial_z \phi_i = \phi_{i+1}$ for every integer $i \geq 1$ makes R a free differential extension of K for the derivation ∂_z.

The first variable ϕ_1 can also be written ϕ_z. Denote then by ϕ_x and ϕ_y the two elements of R defined by (recall that $\omega_z \neq 0$)

$$\omega_z \phi_x = \omega_x \phi_z + \omega_{zx} - \omega_{xz}, \qquad \omega_z \phi_y = \omega_y \phi_z + \omega_{zy} - \omega_{yz}.$$

With such a choice, the 1-form ϕ clearly satisfies $\phi \wedge \omega = d\omega$; this is a simple consequence of the integrability assumption on ω.

Then extensions of derivations ∂_x and ∂_y from K to R are given by their values on the indeterminates ϕ_i; for $\phi_1 = \phi_z$, we set $\partial_x \phi_z = \partial_z \phi_x$ and $\partial_y \phi_z = \partial_z \phi_y$, and, inductively for $i > 1$, $\partial_x \phi_i = \partial_z \partial_x \phi_{i-1}$ and $\partial_y \phi_i = \partial_z \partial_y \phi_{i-1}$. The ϕ so defined is a closed form: the only identity to be proven is $\partial_x \phi_y = \partial_y \phi_x$, as the two other are consequences of the definition of ∂_x and ∂_y. The proof of this identity relies on a simple but long and cumbersome computation.

It remains to be shown that the extensions of the derivations commute. By construction, ∂_x and ∂_z commute because their bracket gives the value 0 to elements of K and to all ϕ_i, and the same argument is true for ∂_y and ∂_z. The identity $\partial_x\partial_y\phi_i = \partial_y\partial_x\phi_i$ can be proved by an easy induction on i, as soon as the result is established for $\phi_1 = \phi_z$. But $\partial_x\partial_y\phi_z = \partial_x\partial_z\phi_y = \partial_z\partial_x\phi_y$ and $\partial_y\partial_x\phi_z = \partial_y\partial_z\phi_x = \partial_z\partial_y\phi_x$; and the proof is completed according to the "cumbersome" identity $\partial_x\phi_y = \partial_y\phi_x$. ∎

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