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A NEW PROOF OF A THEOREM OF BALCERZYK, BIAŁYNICKI-BIRULA AND ŁOŚ

BY

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Let $\mathbb{Z}^\kappa$ be the direct product of κ copies of the integers $\mathbb{Z}$, where κ is less than the least measurable cardinal number. In [1] the authors cited in the title showed, among other things, that, if A is a direct summand of the abelian group $\mathbb{Z}^\kappa$, then $A \cong \mathbb{Z}^\alpha$ for some cardinal number α. A key ingredient in their proof was showing that $\mathbb{Z}^\kappa$ as well as $\mathbb{Z}^{(\kappa)}$, the direct sum of κ copies of Z, were $\mathbb{Z}$-dual. An abelian group G is $\mathbb{Z}$-*dual* (in modern terminology $\mathbb{Z}$-*reflexive*) if G is isomorphic to $\mathrm{Hom}(\mathrm{Hom}(G, Z), Z)$. About the same time R. L. Nunke in [3] obtained the same result using similar arguments. We here present a proof of their result which avoids the use of duality and Hom and seems to us to be more natural.

Unless otherwise indicated our notation and terminology is the same as in [2]. We rely heavily on the following version of a famous Theorem of Łoś (see Theorem 94.4 in [2]): if P and S are the direct product and direct sum respectively of torsion-free abelian groups G_i, $i \in \kappa$, where κ is less than the least measurable cardinal number, and $\eta : P \to \mathbb{Z}$ is a homomorphism, then

(a) $\eta(G_i) = 0$ for almost all i, and
(b) if $\eta(S) = 0$, then $\eta(P) = 0$.

THEOREM 1 (see the Corollary to Theorem 1 in [1]; also, Theorem 5 in [3]). *Let* $\mathbb{Z}^\kappa = \prod_{i\in\kappa} \langle e_i \rangle = A \oplus B$, *where* κ *is a set of cardinality less than the least measurable cardinal number. Then* A (*similarly* B) *is a direct product of copies of* $\mathbb{Z}$.

We begin with two lemmas whereby in Proposition 4 we reduce Theorem 1 to the countable case. This case is treated in Proposition 5. A formal proof of Theorem 1 follows. We conclude with some generalizations of Theorem 1.

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LEMMA 2. *The set κ can be ordered as an ordinal such that, if P_s equals $\prod_{i\geq s}\langle e_i\rangle$ for each $s\in\kappa$, then, for $s=0$ or a limit ordinal, $P_s=A_s\oplus B_s$ with $A_s\subseteq A$ and $B_s\subseteq B$.*

Proof. Let $\pi:\mathbb{Z}^\kappa\to A$ and $\pi_i:\mathbb{Z}^\kappa\to A\to\langle e_i\rangle$ be the natural projections. By Łoś' Theorem for each $i\in\kappa$, $\pi_i(e_j)=0$ for almost all $j\in\kappa$. We totally order κ as follows. Assume for ordinal j that we have chosen a subset $I=\{i:i<j\}$ from κ and that $\kappa-I$ is nonempty. Choose j from $\kappa-I$ such that $\pi_i(e_j)\neq 0$ for the least possible $i\in I$; if no such i exists let j be arbitrary. Now let s be 0 or a limit ordinal in κ. By our ordering of κ we have $\pi_i(P_s)=0$ for all $i<s$ by Łoś' Theorem. Let $A_s=\pi(P_s)$ and let B_s be the projection of P_s to B. Since A_s is contained in P_s, so is B_s. Therefore $P_s=A_s\oplus B_s$.

Let S be the set consisting of 0 and all limit ordinals in κ. For each s in S define $P(s)=\prod_{s\leq i<s+\omega}\langle e_i\rangle$ where $\omega=\{0,1,2,\ldots\}$. In what follows, P_s, A_s and B_s are as in Lemma 2. However, if s happens to be maximal in κ, then $s+\omega$ is not in S, so we set $P_{s+\omega}$, $\mathrm{A}_{s+\omega}$ and $B_{s+\omega}$ all equal to 0. We also note that, if s is maximal in κ, then $P(s)$ could have finite rank.

LEMMA 3. *For each s in S,*

(a) $A_s=A(s)\oplus A_{s+\omega}$, *where* $A(s)=A_s\cap(P(s)\oplus B_{s+\omega})$,
(b) $B_s=B(s)\oplus B_{s+\omega}$, *where* $B(s)=B_s\cap(P(s)\oplus A_{s+\omega})$, *and*
(c) $P(s)\cong A(s)\oplus B(s)\cong\mathbb{Z}^m$ *for some* $m\leq\omega$.

Proof. By Lemma 2, $P_s=A_s\oplus B_s$ and $P_{s+\omega}=A_{s+\omega}\oplus B_{s+\omega}$. Also, P_s equals $P(s)\oplus A_{s+\omega}\oplus B_{s+\omega}$. Thus, since $A_s\supseteq A_{s+\omega}$, (a) is true; similarly, (b) is true. For (c) note that we now have $A(s)\oplus B(s)\cong P_s//P_{s+\omega}\cong\mathbb{Z}^m$ for some $m\leq\omega$.

PROPOSITION 4. $A\cong\prod_{s\in S}A(s)$, *where $A(s)$ is as in Lemma* 3.

Proof. Let $a_s\in A(s)$ for each $s\in S$. Each a_s equals $\pi(p_s)$ for some $p_s\in P(s)$. Since $\prod_S P(s)$ is a product, for each $i\in\kappa$ we have $\pi_i(a_s)=0$ for almost all s by Łoś' Theorem. Thus, if $a_s=(a_{si})$, $i\in\kappa$, in $\mathbb{Z}^\kappa$ for each s, we can write $\sum_{s\in S}a_s$ for the element $(\sum_{s\in S}a_{si})$, $i\in\kappa$, in $\mathbb{Z}^\kappa$. Let $C=\{\sum_{s\in S}a_s:a_s\in A(s)\}$. Suppose that $\sum_{s\in S}a_s=0$ but that $a_t\neq 0$ for minimal $t=s$. Let $\phi_t:A\to P(t)$ in $\prod_{s\in S}P(s)$ be the natural projection. Then $0=\phi_t(\sum_{s\in S}a_s)=\phi_t(a_t)=p_t\neq 0$ (since $\pi(p_t)=a_t\neq 0$), a contradiction. Hence $C\cong\prod_{s\in S}A(s)$, the external direct product. We now show that $A=C$. Since $A_s\subseteq A$ for all s, $C\subseteq A$ by Łoś' Theorem. Let $a\in A$. We complete the proof by showing $a\in C$. Suppose for each $s<t$ in S we have found $a_s\in A(s)$ such that $a-\sum_{r\leq s}a_r$ is in $A_{s+\omega}$ for each s. Then the element $a-\sum_{r<t}a_r$ is in $A_{s+\omega}$ for each $s<t$ since it equals $(a-\sum_{r\leq s}a_r)-\sum_{s<r<t}a_r$ for each such s. So it is in A_t. By Lemma 3(a),

for some a_t in $A(t)$, the element $a - \sum_{r\le t} a_r$ is in $A_{t+\omega}$. Inductively then we can find $a_s \in A(s)$ for each $s \in S$ such that $a - \sum_{r\le s} a_r$ is in $A_{s+\omega}$ for each s. Since the element $a - \sum_{s\in S} a_s$ equals $(a - \sum_{r\le s} a_r) - \sum_{r>s} a_r$ for each s, it is in $\bigcap_{s\in S} A_{s+\omega}$, which is 0. Therefore $a = \sum_{s\in S} a_s$, an element in C.

Our proof of the next result is well known but, since it does not seem to be in standard references, we include it for completeness.

PROPOSITION 5. *If* $\mathbb{Z}^\omega = A \oplus B$, *then* $A \cong \mathbb{Z}^m$ *for some* $m \le \omega$.

Proof. Write $\mathbb{Z}^\omega = \{(x_n) : n \in \omega,\ x_n \in Z\}$. For each $s \in \omega$, let A_s equal $\{(x_n) \in A : x_n = 0 \text{ for } n < s\}$. Choose $a_s \in A_s$ for each s such that its s-component is the least possible positive integer; if no such element exists, let $a_s = 0$. Observe that, if $x \in A_s$, then $x - ma_s$ is in A_{s+1} for some integer m. Write $a_s = (a_{sn})$, $n \in \omega$. Since a_{sn} is 0 whenever $s > n$, we can identify $\sum_{s\in\omega} m_s a_s$ $(m_s \in Z)$ with the element $(\sum_{s\in\omega} m_s a_{sn})$, $n \in \omega$, in $\mathbb{Z}^\omega$. Let C be the set of all such elements. By our choice of each a_s it follows that $C \cong \prod_{s\in\omega}\langle a_s\rangle \cong \mathbb{Z}^m$ for some $m \le \omega$. Also, $C \subseteq A$. Let $a \in A$. We complete the proof by showing $a \in C$. By induction we can find $m_s \in Z$ for each s such that $a - \sum_{r\le s} m_r a_r$ is in A_{s+1} for each s. It follows easily then that $a = \sum_{s\in S} m_s a_s$, an element in C.

Proof of Theorem 1. By Proposition 4 and Lemma 3, A is isomorphic to $\prod_{s\in S} A(s)$, where each $A(s)$ is isomorphic to a direct summand of $\mathbb{Z}^\omega$. Proposition 5 completes the proof.

GENERALIZATIONS. (a) Suppose $P = \prod_{i\in\kappa} G_i$, where each G_i is isomorphic to a proper subgroup of the rational numbers and where κ is less than the least measurable cardinal number. In [4] we showed that any direct summand of P is a direct product of rank-one abelian groups. This generalized all the results in [1]. The proof in [4] was much more difficult than the proof of Theorem 1 above but still avoided the use of duality and Hom.

(b) In [5] by other means we showed that Theorem 1 can be extended to the case where κ is the least measurable cardinal number.

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