STRONGLY CHAOTIC DENDRITES

BY

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The concept of a strongly chaotic space is introduced, and its relations to chaotic, rigid and strongly rigid spaces are studied. Some sufficient as well as necessary conditions are shown for a dendrite to be strongly chaotic.

1. Introduction. A nondegenerate topological Hausdorff space X is said to be:

(a) *chaotic* if for any two distinct points p and q of X there exists an open neighbourhood U of p and an open neighbourhood V of q such that no open subset of U is homeomorphic to any open subset of V;

(b) *rigid* if it has a trivial autohomeomorphism group, i.e., if the only homeomorphism of X onto X is the identity;

(c) *strongly rigid* if the only homeomorphism of X into X is the identity of X onto itself.

These three concepts were extensively studied in many papers. In [1], a comprehensive list of references is produced and a number of results are presented or recalled, especially those related to curves (i.e., one-dimensional metric continua). In a discussion with the authors, Dr. Krzysztof Omiljanowski and Dr. Janusz R. Prajs have proposed to define a narrower class of spaces than chaotic ones, called here strongly chaotic. In this paper we investigate this new class of spaces. First, we establish some inclusions of the new class in the previous ones. Next, we study strongly chaotic dendrites and show that no other inclusions hold, even for dendrites.

2. Preliminaries. All spaces considered in this paper are assumed to be Hausdorff and all mappings are continuous. Given a subset A of a space X, we denote by cl A its closure in X. A *continuum* means a compact connected metric space. A *dendrite* means a locally connected continuum

containing no simple closed curve. Given two points p and q of a dendrite X, we denote by pq the unique arc from p to q in X.

We shall use the notion of *order of a point* in the sense of Menger–Urysohn (see e.g. [5], §51, I, p. 274), and we denote by $\operatorname{ord}(p, X)$ the order of the space X at a point $p \in X$. It is well known (see e.g. [5], §51, pp. 274–307) that the function ord takes values from the set

$$S = \{0, 1, 2, \ldots, \omega, \aleph_0, 2^{\aleph_0}\}.$$

Points of order 1 in X are called *end points* of X; the set of all end points of X is denoted by $E(X)$. Points of order 2 are called *ordinary points*. It is known that the set of all ordinary points of a dendrite is a dense subset. For each $n \in \{3, 4, \ldots, \omega, \aleph_0, 2^{\aleph_0}\}$ points of order n are called *ramification points*; the set of all ramification points of X is denoted by $R(X)$. It is known that for each dendrite X the set $R(X)$ is at most countable, and that points of order $\aleph_0$ and $2^{\aleph_0}$ do not occur in any dendrite.

Given a dendrite X we decompose it into disjoint subsets of points of a fixed order. Namely for each $n \in \{1, 2, \ldots, \omega\}$ we put

$$R_n(X) = \{p \in X : \operatorname{ord}(p, X) = n\}.$$

3. General properties. We start with a proposition that will lead to the definition of a strongly chaotic space.

3.1. PROPOSITION. *The following conditions are equivalent for a topological Hausdorff space X:*

(3.2)   *for any two distinct points p and q of X there exists an open neighbourhood U of p and an open neighbourhood V of q such that no open subset of U is homeomorphic to any subset of V;*

(3.3)   *for any two disjoint subsets U and V of X with U being open there is no homeomorphism from U onto V;*

(3.4)   *for any two distinct subsets U and V of X with U being open there is no homeomorphism from U onto V;*

(3.5)   *for every open subset U of X every homeomorphism $h : U \to h(U) \subset X$ is the identity on U;*

(3.6)   *for any two distinct points p and q of X there exists an open neighbourhood U of p and an open neighbourhood V of q such that no subset of U is homeomorphic to any open subset of V;*

(3.7)   *for any two disjoint subsets U and V of X with V being open there is no homeomorphism from U onto V;*

(3.8)   *for any two distinct subsets U and V of X with V being open there is no homeomorphism from U onto V;*

(3.9)   *for every subset U of X and for every homeomorphism $h : U \to h(U) \subset X$ with $h(U)$ open, h is the identity on U.*

P r o o f. The implications $(3.2){\Rightarrow}(3.3)$, $(3.3){\Rightarrow}(3.2)$, $(3.4){\Rightarrow}(3.3)$ and $(3.5){\Rightarrow}(3.4)$ are obvious. We show $(3.3){\Rightarrow}(3.5)$. Assume (3.3) and suppose on the contrary that there an open set U and a homeomorphism $h : U \to h(U) \subset X$ which is not the identity on U. Then there is a point $x \in U$ with $h(x) \neq x$. Since X is Hausdorff, there exist disjoint open sets U_1 and V_1 such that $x \in U_1$ and $h(x) \in V_1$. Put $h_1 = h|U_1$. Then $U_2 = h_1^{-1}(V_1 \cap h(U))$ is an open subset of U_1. The homeomorphism $h_2 = h_1|U_2 = h|U_2$ sends the set U_2 into V_1, so $h(U_2)$ is disjoint from $U_2 \subset U_1$. This contradicts (3.3). Thus the equivalence of the conditions (3.2) through (3.5) is established.

Interchanging the roles of U and V and considering the homeomorphism h^{-1} in place of h we get the conditions (3.6)–(3.9) from (3.2)–(3.5). The proof is complete.

A nondegenerate topological space X is said to be *strongly chaotic* if it satisfies any of the conditions (3.2)–(3.9) listed in Proposition 3.1. Putting $U = X$ in (3.5) we get the following.

3.10. OBSERVATION. *Each strongly chaotic space is strongly rigid.*

3.11. COROLLARY. *If a chaotic space is not strongly rigid, then it is not strongly chaotic.*

3.12. R e m a r k. Chaotic and not strongly rigid spaces are known: see e.g. [1], Statements 7 and 8, pp. 226 and 227. A chaotic and not strongly rigid (thus not strongly chaotic) dendrite is constructed in Statement 10 of [1], p. 229.

3.13. PROPOSITION. *For every topological space X we have the following four implications and none of them can be reversed, even if X is a dendrite.*

$$(X \text{ is strongly chaotic}) \implies (X \text{ is strongly rigid})$$
$$\Downarrow \qquad\qquad\qquad \Downarrow$$
$$(X \text{ is chaotic}) \implies (X \text{ is rigid})$$

P r o o f. The two vertical implications are obvious. The upper horizontal one is just Observation 3.10, and the lower horizontal one is Proposition 6 of [1], p. 221. To see that the two vertical implications cannot be reversed for dendrites, one can take the example of a chaotic (thus rigid) dendrite which is not strongly rigid (thus not strongly chaotic), presented in Statement 10 of [1], p. 229. It is shown in Example 33 of [4] that there exists a strongly rigid and not chaotic dendrite. Thus the two horizontal implications cannot be reversed. The argument is complete.

4. Universal dendrites. A dendrite is said to be *universal* if it contains a homeomorphic image of any other dendrite. Similarly, if the order of each point of a dendrite X is bounded by a number $n \in \{3, 4, \dots, \omega\}$, and X contains homeomorphic copies of other dendrites whose points have

orders not greater than n, then X is called a *universal dendrite of order n*. Thus, since no dendrite contains points of order exceeding ω ([5], §51, VI, Theorem 4, p. 301), a universal dendrite of order ω is universal according to the former definition.

Observe that if a dendrite X contains a universal dendrite Y, then X is universal itself. The same holds for universal dendrites of order n. Hence, to avoid any confusion with other universal dendrites, we shall consider, for each $n \in \{3, 4, \ldots, \omega\}$, some special universal dendrite D_n of order n, which will be called the *standard universal dendrite of order n*. It is well known that D_n is characterized by the following two conditions (see e.g. [3], Theorem 3.1, p. 169):

(4.1)   each ramification point of D_n is of order n,

and

(4.2)   for every arc A contained in D_n the set of all ramification points of D_n which belong to A is a dense subset of A.

Assuming (4.1), condition (4.2) is equivalent to the following:

(4.3)   for every arc A contained in D_n we have $A \cap R_n(X) \neq \emptyset$.

The construction of D_n is known from Ważewski's doctoral dissertation (see [7], Chapter K, p. 137). It has been simplified by K. Menger in [6], Chapter X, §6, p. 318, and is recalled in [3], Chapter 3, p. 167. Another description of these continua for finite n, which uses limits of inverse sequences of finite dendrites (i.e. dendrites having a finite number of end points only) with monotone onto bonding mappings, is given in [2], p. 491.

The following result generalizes Proposition 3.2 of [3], p. 169.

4.4. THEOREM. *Let a dendrite X be given, and let $n \in \{3, 4, \ldots, \omega\}$ be fixed. If*

(4.5) $A \cap R_n(X) \neq \emptyset$ *for every arc A in X*,

then X contains a homeomorphic copy of the standard universal dendrite D_n of order n.

P r o o f. Fix an arbitrary arc $A \subset X$ and, for every point $p \in A \cap R_n(X)$, consider n arcs emanating from p and pairwise disjoint apart from p. On each of them repeat this construction. According to the characterization (4.1) and (4.2) of D_n the closure of the union of all these arcs is homeomorphic to D_n.

5. Strongly chaotic dendrites. As an application of Theorem 4.4 we have the following result.

5.1. PROPOSITION. *Let a dendrite X be given, and let $n \in \{3, 4, \ldots, \omega\}$ be fixed. If*

$$(5.2) \qquad \operatorname{ord}(p, X) \le n \quad \text{for each } p \in X,$$

and if

$$(4.5) \qquad A \cap R_n(X) \neq \emptyset \quad \text{for every arc } A \text{ in } X,$$

then the dendrite X is not strongly chaotic.

P r o o f. We use condition (3.5) of Proposition 3.1. Let U be an open subset of X. Take an open connected subset of X whose closure V is disjoint from U. As a subcontinuum of the dendrite X, the set V is also a dendrite ([5], §51, VI, a corollary to Theorem 4, p. 301). Then condition (4.5) implies a similar condition for V:

$$(5.3) \qquad A \cap R_n(V) \neq \emptyset \quad \text{for every arc } A \subset V.$$

Thus we infer from Theorem 4.4 that V contains a homeomorphic copy Y of the standard universal dendrite D_n of order n. Now condition (5.2) and the universality of Y imply that Y contains a homeomorphic copy X_0 of X. Let U_0 be a homeomorphic copy of U contained in X_0. So, we have $U_0 \subset X_0 \subset Y \subset V$, and we see that V contains a homeomorphic copy of U. Since $U \cap V = \emptyset$, it follows that the homeomorphism is not the identity, and therefore X is not strongly chaotic. The proof is complete.

5.4. R e m a r k. It can be observed from the construction that the chaotic dendrite D of Statement 13 of [1], p. 231, satisfies conditions (5.2) and (4.5) of Proposition 5.1 for $n = 4$; analogously, the chaotic dendrites $X(m, n)$ of Theorem 27 of [4] also satisfy these conditions. Consequently, no one of them is strongly chaotic. Moreover, each $X(m, n)$ is strongly rigid. Thus not only can no other implication be added to the diagram of Proposition 3.13, but also the condition of being strongly rigid *and* chaotic does not imply being strongly chaotic, even for dendrites.

However, a similar construction can be applied to obtain examples of strongly chaotic dendrites. In fact, it is enough to change the role of the numbers m and n in the definition of the dendrites $X(m, n)$ of Theorem 27 of [4] and to modify the construction a little to get some extra properties. For clarity, however, we repeat the whole construction.

5.5. THEOREM. *For any two integers m and n with $3 \le n < m$ there exists a dendrite X such that*

(5.6) $\operatorname{ord}(x, X) \in \{1, 2, n, m\}$ *for each $x \in X$;*
(4.5) $A \cap R_n(X) \neq \emptyset$ *for every arc A in X;*
(5.7) *if $\alpha \in \{1, 2, n, m\}$, then $\operatorname{cl} R_\alpha(X) = X$;*
(5.8) *X is strongly chaotic.*

P r o o f. First we define two auxiliary dendrites D_0 and D_1. Within a straight line segment ab ordered from a to b by $<$ we choose a sequence of points $\{a_i : i \in \mathbb{N}\}$ so that

$$a_{i+1} < a_i \quad \text{and} \quad \lim_{i \to \infty} a_i = a.$$

Within each interval $a_{i+1}a_i$ choose a sequence of points $\{a_{i,j} : j \in \mathbb{N}\}$ so that

$$a_{i,j} < a_{i,j+1} \quad \text{and} \quad \lim_{j \to \infty} a_{i,j} = a_i.$$

At each point a_i and $a_{i,j}$ erect $m - 2$ straight line segments mutually disjoint apart from these points and having only these points in common with the segment ab. Take the segments so that for any positive number ε only finitely many of them have length greater than ε. The set of points obtained in this way is called D_0. It is clear that D_0 is a dendrite.

The definition of D_1 is the same except that the points $a_{i,j}$ are taken within the intervals $a_{i+1}a_i$ so that

$$a_{i,j+1} < a_{i,j} \quad \text{and} \quad \lim_{j \to \infty} a_{i,j} = a_{i+1}.$$

So D_1 is also a dendrite. The point a is called the *origin* of either D_0 or D_1, and the straight line segments which we have erected are all referred to as *straight line segments of rank* 1.

Denoting by S^{d} the derived set of a set S in a topological space T (i.e. the set of all accumulation points of S in T) we see that

(5.9) $[ab \cap R_m(D_0)]^{\mathrm{dd}} = \{a\} = [ab \cap R_m(D_1)]^{\mathrm{dd}}.$

The defined dendrites D_0 and D_1 start an inductive construction of dendrites $D_{\gamma_1 \ldots \gamma_k}$, where $k \in \mathbb{N}$ and $\gamma_1 \ldots \gamma_k$ is a zero-one sequence. Assume now that we have defined dendrites $D_{\gamma_1 \ldots \gamma_k}$ for some $k \in \mathbb{N}$. Assume furthermore that we have defined the expressions: the origin of $D_{\gamma_1 \ldots \gamma_k}$ and the straight line segments of rank k of $D_{\gamma_1 \ldots \gamma_k}$ (whose one end point is an end point of $D_{\gamma_1 \ldots \gamma_k}$). To define $D_{\gamma_1 \ldots \gamma_k 0}$ we proceed as follows. We divide each straight line segment of rank k of $D_{\gamma_1 \ldots \gamma_k}$ into three equal parts. Next we replace the middle part by a copy of D_0 diminished so that the diameter of the copy equals the length of the middle part. The copy of D_0 is located in such a way that its origin is the closest point of the copy to that end point of the considered segment of rank k which is the end point of $D_{\gamma_1 \ldots \gamma_k}$. Furthermore, we do this, as we clearly can, so that the resulting set $D_{\gamma_1 \ldots \gamma_k 0}$ is a dendrite. By the origin of $D_{\gamma_1 \ldots \gamma_k 0}$ we mean the origin of $D_{\gamma_1 \ldots \gamma_k}$, and by the straight line segments of rank $k+1$ of $D_{\gamma_1 \ldots \gamma_k 0}$ we mean the segments of rank 1 of the sets D_0 used in obtaining $D_{\gamma_1 \ldots \gamma_k 0}$ from $D_{\gamma_1 \ldots \gamma_k}$.

The definition of $D_{\gamma_1 \ldots \gamma_k 1}$ is the same except that in obtaining $D_{\gamma_1 \ldots \gamma_k 1}$ from $D_{\gamma_1 \ldots \gamma_k}$ we use the sets D_1 instead of D_0. The inductive definition of $D_{\gamma_1 \ldots \gamma_k}$ for each $k \in \mathbb{N}$ is thus finished.

Now we define the desired dendrite X. The construction uses the sequence of dendrites

$$D_0, \quad D_{10}, \quad D_{110}, \quad \ldots, \quad D_{11\ldots10}, \quad \ldots$$

which we re-label in the same order as

$$W_1, \quad W_2, \quad W_3, \quad \ldots, \quad W_k, \quad \ldots$$

Putting $X_1 = W_1$ we have $R_n(X_1) = \emptyset$, and, by (5.9),

$$[ab \cap R_m(X_1)]^{\mathrm{dd}} = \{a\}.$$

Further, if $y \in R_m(X_1)$, then for each arc $B \subset X_1$ ending at y we have $y \notin [B \cap R_m(X_1)]^{\mathrm{dd}}$.

Recall that a *free arc* in a dendrite D is an arc such that all its points but the ends are of order 2 in D. In particular, a *maximal free arc* in a dendrite D is an arc $st \subset D$ such that

$$st \cap (E(D) \cup R(D)) = \{s, t\}.$$

Note that each arc in X_1 contains a free subarc. Consider now an arbitrary maximal free arc in X_1. It is evident from the construction that every such arc is a straight line segment. Denote its mid point by x. We obtain, of course, a countable set of points x. With this countable set we associate, in a one-to-one way, the sets W_k of odd indices k, i.e.,

$$W_3, \quad W_5, \quad \ldots, \quad W_{2r+1}, \quad \ldots,$$

taking x as the origin of the associated set $W_{2r+1} = W(x)$ in such a way that X_1 and $W(x)$ have only the point x in common. Moreover, to the point x we attach $n - 3$ straight line segments having x as one end point and having only x in common with $W(x) \cup X_1$. All this can clearly be done in such a way that the resulting set X_2 is a dendrite. Note that

(5.10)   for every maximal free arc in the dendrite X_1 its mid point x becomes a point of order $n < m$ in the constructed dendrite X_2, and there are no other points of order n in X_2,

and that each arc in X_2 contains a free subarc. Further, observe that

(5.11)   for every $x \in R_n(X_2)$ there exists an arc $A \subset X_2$ ending at x (namely an arc contained in $W(x)$) with $[A \cap R_m(X_2)]^{\mathrm{dd}} = \{x\}$,

and that

(5.12)   if $y \in R_m(X_2)$ then $y \notin [B \cap R_m(X_2)]^{\mathrm{dd}}$ for each arc $B \subset X_2$ ending at y.

Now, X_3 is related to X_2 in the same way as X_2 is to X_1, except that we make use of the sets $W_{2(2k+1)}$ instead of W_{2k+1}. In general, X_{i+1} is related to X_i in the same way as X_i is to X_{i-1} except that we make use of

$W_{2^{i-1}(2k+1)}$ instead of $W_{2^{i-2}(2k+1)}$. It can be observed that each arc in X_i contains a free subarc, and

(5.13)   for every maximal free arc in the dendrite X_i its mid point x becomes a point of order $n < m$ in the constructed dendrite X_{i+1},

and that, analogously to (5.11) and (5.12), similar properties hold for X_i, namely

(5.14)   for every $x \in R_n(X_i)$ there exists an arc $A \subset X_i$ ending at x such that $[A \cap R_m(X_i)]^{\mathrm{dd}} = \{x\}$,

(5.15)   if $y \in R_m(X_i)$, then $y \notin [B \cap R_m(X_i)]^{\mathrm{dd}}$ for each arc $B \subset X_i$ ending at y.

It is known that this construction can be carried through so that the closure of the union of the dendrites X_i obtained is itself a dendrite. We may then assume that $X = \mathrm{cl}(\bigcup\{X_i : i \in \mathbb{N}\})$ is a dendrite.

Now we prove the desired properties of X. We notice first that any ramification point of X is either of order n or m. Thus (5.6) follows from the construction. The points of order n are the points x which arise at the successive stages of the construction. Since, for each $i \in \mathbb{N}$, in the construction of X_i we take the mid points x of all maximal free arcs in X_i, condition (4.5) follows from (5.13). Thus $R_n(X)$ is dense in X.

Furthermore, notice that (5.14) and (5.15) lead to the following two properties of the dendrite X:

(5.16)   for every $x \in R_n(X)$ there exists an arc $A \subset X$ ending at x such that $x \in [A \cap R_m(X)]^{\mathrm{dd}}$,

(5.17)   if $y \in R_m(X)$, then $y \notin [B \cap R_m(X)]^{\mathrm{dd}}$ for each arc $B \subset X$ ending at y.

Consequently, by (5.16), each open neighbourhood of a point $y \in R_n(X)$ contains points of order m in X and, hence, the density of $R_m(X)$ in X follows from the density of $R_n(X)$ in X. The set $R_2(X)$ is always dense in a dendrite X ([6], p. 309; cf. [5], §51, VI, Theorem 8, p. 302). Finally, the density of $R_1(X) = E(X)$ is equivalent to the density of $R(X)$ according to Theorem 2.4 of [3], p. 167. Thus (5.7) is shown.

Now we prove that X is strongly chaotic using condition (3.3) of Proposition 3.1. Let U and V be disjoint subsets of X with U open. Suppose, on the contrary, that there is a homeomorphism h of U onto V. First, we notice that $R_n(X) \cap U \neq \emptyset \neq R_m(X) \cap U$, the sets $R_n(X)$ and $R_m(X)$ being dense in X. Further, observe that h must carry each point of $R_n(X) \cap U$ to a point of either $R_n(X) \cap V$ or $R_m(X) \cap V$ since no point of X is of order greater than m.

Take $u \in R_n(X) \cap U$ and put $v = h(u) \in (R_n(X) \cup R_m(X)) \cap V$. According to (5.16) there exists an arc $A \subset U \subset X$ ending at u and such

that

$$[A \cap R_m(X)]^{\mathrm{dd}} = \{u\}.$$

The existence of such an arc is preserved by a homeomorphism, so v has the same property, i.e., there is an arc $B \subset V$ (viz. $B = h(A)$) such that v is an end point of B and that

$$[B \cap R_m(X)]^{\mathrm{dd}} = \{v\}.$$

Thus v cannot be in $R_m(X)$ by (5.17), and consequently $v \in R_n(X) \cap V$. Assume for definiteness (the argument is similar in the opposite case) that the set W_i with origin u has index lower than the one with origin v. Consider now an arc $ub_u \subset U$ that contains a sequence of ramification points of order m converging to u. It is clear that there is a subarc ub'_u of ub_u such that ub'_u and $h(ub'_u)$ are straight line segments.

If $W(u) = W_1$, we can easily reach a contradiction. Really, $W_1 = D_0$ and $W(v) = W_i = D_{111\ldots10}$, which means that ub'_u contains ramification points of order m which are limit points from the left of ramification points of order m, while $h(ub'_u)$ contains no such points.

If $W(u) = W_2$, consider a fixed ramification point s_u of order m in X which is interior to ub'_u. Consider a straight line segment standing upright to ub'_u at s_u whose end point e_u is in $E(X) \cap U$. Order the arc $e_u s_u$ from e_u to s_u. Then there is a point on the arc $e_u s_u$ which is the origin of the inserted copy of D_0, i.e., it is the limit of a sequence of ramification points of order m which are limit points from the left of ramification points of order m. The image $h(e_u s_u)$ in V contains no such points since $W(v) = W_i = D_{111\ldots10}$. The argument exemplified above can be extended to apply to the general case where $W(u) = W_i$ and $W(v) = W_j$ for $i < j$ or $j < i$, respectively. This contradiction completes the proof.

REFERENCES

[1] J. J. Charatonik, *On chaotic curves*, Colloq. Math. 41 (1979), 219–236.

[2] —, *Open mappings of universal dendrites*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 28 (1980), 489–494.

[3] —, *Monotone mappings of universal dendrites*, Topology Appl. 38 (1991), 163–187.

[4] —, *On chaotic dendrites*, preprint.

[5] K. Kuratowski, *Topology*, Vol. 2, Academic Press and PWN, 1968.

[6] K. Menger, *Kurventheorie*, Teubner, 1932.

[7]   T. W a ż e w s k i, *Sur les courbes de Jordan ne renfermant aucune courbe simple fermée de Jordan*, Ann. Soc. Polon. Math. 2 (1923), 49–170.

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