CR-SUBMANIFOLDS OF LOCALLY CONFORMAL KAHLER MANIFOLDS AND RIEMANNIAN SUBMERSIONS

BY

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We consider a Riemannian submersion \( \pi: M \to N \), where \( M \) is a CR-submanifold of a locally conformal Kaehler manifold \( L \) with the Lee form \( \omega \) which is strongly non-Kaehler and \( N \) is an almost Hermitian manifold. First, we study some geometric structures of \( N \) and the relation between the holomorphic sectional curvatures of \( L \) and \( N \). Next, we consider the leaves \( M \) of the foliation given by \( \omega = 0 \) and give a necessary and sufficient condition for \( M \) to be a Sasakian manifold.

1. Introduction. Let \( L \) be an almost Hermitian manifold with almost complex structure \( J \). Let \( M \) be a real submanifold of \( L \) and \( TM \) its tangent bundle. We set \( T^h M = TM \cap J(TM) \). Then we have

(a) \( JT^h_p M = T^h_p M \) for each \( p \in M \).

Let \( M \) be a CR-submanifold of an almost Hermitian manifold \( L \) such that the differentiable distribution \( T^h M : p \to T^h_p M \subset T_p M \) on \( M \) satisfies the following conditions:

(b) \( JT^\nu_p M \subset T^\nu_p M^\perp \) for each \( p \in M \), where \( T^\nu M \) is the complementary orthogonal distribution of \( T^h M \) in \( TM \);

(c) \( J \) interchanges \( T^\nu M \) and \( TM^\perp \);

(d) there is a Riemannian submersion \( \pi: M \to N \) of \( M \) onto an almost Hermitian manifold \( N \) such that (i) \( T^\nu M \) is the kernel of \( \pi_* \) and (ii) \( \pi_* : T^h_p M \to T^h_{\pi(p)} N \) is a complex isometry for every \( p \in M \).

This set up is similar to the set up of symplectic geometry. Indeed, one has the following analogue (due to S. Kobayashi) of the symplectic reduction theorem of Marsden–Weinstein.

Theorem 1 ([7]). Let \( L \) be a Kaehler manifold. Under the assumptions stated above, \( N \) is a Kaehler manifold. If \( H^L \) and \( H^N \) denote the holomorphic sectional curvatures of \( L \) and \( N \), then, for any horizontal unit vector

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\(X \in T^h M\), we have
\[
H^L(X) = H^N(\pi_* X) - 4|\sigma(X, X)|^2,
\]
where \(\sigma\) denotes the second fundamental form of \(M\) in \(L\).

In the above theorem, \(L\) is a Kaehler manifold. In this paper, we consider the case where \(L\) is a locally conformal Kaehler manifold which is strongly non-Kaehler. Then \(T^v M\) is integrable [3]. Let \(B^h\) and \(B^v\) be the horizontal part, the vertical part and the normal part of the Lee vector field \(B\) respectively. First, we show the following theorem:

**Theorem 2.** Under the assumptions (a)–(d), assume further that \(L\) is a locally conformal Kaehler manifold. Then the Lee vector field \(B \in T^h M \oplus TM^\perp\) and for any horizontal unit vector \(X \in T^h M\), we have
\[
H^L(X) = H^N(\pi_* X) - 3|A X J X|^2 - |\sigma(X, X)|^2,
\]
where \(\sigma\) is the second fundamental form of \(M\) in \(L\) and \(A\) is the integrability tensor with respect to \(\pi\). Moreover, if we assume in addition that the horizontal component \(B^h\) of the Lee vector field \(B\) is basic and \(\dim N \geq 4\) then \(N\) is also a locally conformal Kaehler manifold. In particular, if \(L\) is a generalized Hopf manifold and if the Lee vector field \(B\) is basic and horizontal then \(N\) is also a generalized Hopf manifold.

Next, we consider the case where the Lee vector field \(B \in TM^\perp\).

**Theorem 3.** Under the assumptions (a)–(d), if \(L\) is a locally conformal Kaehler manifold and \(B \in TM^\perp\), then \(N\) is a Kaehler manifold.

**Theorem 4.** Under the assumptions (a)–(d), if \(L\) is a \(P_0 K\)-manifold and \(M\) is a totally umbilical submanifold whose mean curvature vector is parallel and \(B \in TM^\perp\), then \(N\) is a locally symmetric Kaehler manifold and the holomorphic sectional curvature \(H^N\) of \(N\) is \(H^N(\tilde{X}) > 0\), where \(\tilde{X}\) is any unit tangent vector.

Next, let \(L\) be a locally conformal Kaehler manifold which is strongly non-Kaehler, \(\omega\) the Lee form and \(\mathcal{M}\) the distribution defined by \(\omega = 0\). Since \(d\omega = 0\), \(\mathcal{M}\) is integrable. Let \(M\) be a maximal connected integral submanifold of \(\mathcal{M}\), that is, \(M\) is an orientable hypersurface of \(L\). Then \(M\) is a CR-submanifold satisfying (a)–(c) such that \(TM^\perp = \{B\}\) and \(T^v M = \{J B\}\). In the case where \(L\) is a \(P_0 K\)-manifold, we get the following theorem.

**Theorem 5.** Let \(L\) be a complete \(P_0 K\)-manifold and \(M\) a maximal connected integral submanifold of \(\mathcal{M}\). Let \(N\) be an almost Hermitian manifold and \(\pi: M \to N\) be a Riemannian submersion satisfying the condition (d). Then \(N\) is isometric to the complex projective space \(P_m(\mathbb{C})\).
It is known that every orientable hypersurface of an almost Hermitian manifold has an almost contact metric structure \((\phi, V, \eta, g)\) (see [2], [17]). We show the following theorem:

**Theorem 6.** Let \(L\) be a locally conformal Kaehler manifold and \(M\) a maximal connected integral submanifold of \(\mathcal{M}\). Then \((M, \phi, V, \eta, g)\) is a Sasakian manifold if and only if

\[
k = \left(\frac{1}{2} \sqrt{\omega(B)} - 1\right) g + \alpha \eta \otimes \eta,
\]

where \(k\) is the second fundamental form of \(M\) and \(\alpha\) is a function.

**Remark 1.** (I) In [17], I. Vaisman proved that if \(L\) is a locally conformal Kaehler manifold with parallel Lee form, then a maximal connected integral submanifold \(M\) of \(\mathcal{M}\) is a totally geodesic submanifold of \(L\) and \(M\) is a Sasakian manifold. In Theorem 6, we obtain a necessary and sufficient condition for \(M\) to be a Sasakian manifold without the assumption that the Lee form is parallel.

(II) It is known that if \(M\) is an orientable hypersurface of a Kaehler manifold \(L\), then the induced almost contact metric structure \((\phi, V, \eta, g)\) is Sasakian if and only if \(k = -g + \alpha \eta \otimes \eta\), where \(k\) is the second fundamental form of \(M\) and \(\alpha\) is a function [14]. When \(L\) is a locally conformal Kaehler manifold, from Theorem 6 we obtain a similar result.

**2. Preliminaries.** Let \(L\) be an almost Hermitian manifold with metric \(g\), complex structure \(J\) and fundamental 2-form \(\Omega\). The manifold \(L\) will be called a locally conformal Kaehler manifold if every \(x \in L\) has an open neighborhood \(U\) with a differentiable function \(\gamma : U \to \mathbb{R}\) such that \(g'_U = e^{-\gamma} g|_U\) is a Kaehler metric on \(U\). The locally conformal Kaehler manifold \(L\) is characterized by

\[
d\Omega = \omega \wedge \Omega, \quad d\omega = 0,
\]

where \(\omega\) is a globally defined 1-form on \(L\). We call \(\omega\) the Lee form. Since for \(\text{dim } L = 2\) we have \(d\Omega = 0\), we may suppose \(\text{dim } L \geq 4\). Next we define the Lee vector field \(B\) by

\[
g(X, B) = \omega(X).
\]

The Weyl connection \(\text{W} \nabla\) is the linear connection defined by

\[
\text{W} \nabla_X Y := \nabla_X Y - \frac{1}{2} \omega(X) Y - \frac{1}{2} \omega(Y) X + \frac{1}{2} g(X, Y) B,
\]

where \(\nabla\) is the Levi-Civita connection of \(g\). It is shown in [15] that an almost Hermitian manifold \(L\) is a locally conformal Kaehler if and only if there is a closed 1-form \(\omega\) on \(L\) such that

\[
\text{W} \nabla_X J = 0.
\]
The equation (4) is equivalent to

\[ \nabla_X JY - \frac{1}{2} \omega(JY)X + \frac{1}{2}g(X, JY)B = J \nabla_X Y - \frac{1}{2} \omega(Y)JX + \frac{1}{2}g(X, Y)JB, \]

where \( X \) and \( Y \) are vector fields on \( L \).

The Riemannian curvature tensor field \( R^L \) of \( L \) is given by

\[ R^L(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \]

We set

\[ R^L(W, Z, X, Y) = g(R^L(X, Y)Z, W). \]

Let \( W^R \) be the curvature tensor field of the Weyl connection \( W \). Then

\[ W^R(X, Y)Z = R^L(X, Y)Z - \frac{1}{2} \left\{ \left[ (\nabla_X \omega)Z + \frac{1}{2} \omega(X)\omega(Z) \right] Y \right. \]

\[ - \left. \left[ (\nabla_Y \omega)Z + \frac{1}{2} \omega(Y)\omega(Z) \right] X - g(Y, Z)(\nabla_X B + \frac{1}{2} \omega(X)B) \right. \]

\[ + g(X, Z)(\nabla_Y B + \frac{1}{2} \omega(Y)B) \} - \frac{1}{4} |\omega|^2 (g(Y, Z)X - g(X, Z)Y), \]

where \( X, Y \) and \( Z \) are any vector fields on \( L \) [18].

A locally conformal Kaehler manifold \((L, J, g)\) is said to be a **generalized Hopf manifold** if the Lee form is parallel, that is, \( \nabla \omega = 0 \) (\( \omega \neq 0 \)). A generalized Hopf manifold is called a \( P_0K \)-manifold if the Weyl curvature tensor is zero, that is, \( W^R(X, Y) = 0 \). In this paper, we consider the case where \( L \) is a locally conformal Kaehler manifold which is strongly non-Kaehler in the sense that \( d\Omega \neq 0 \) (and so \( \omega \neq 0 \)) at every point of \( L \).

The Hopf manifolds are defined as \( H_n^\lambda = (\mathbb{C}^n - \{0\})/\Delta_\lambda, n > 1, \) where \( \mathbb{C} \) is the complex plane, \( \lambda \in \mathbb{C}, |\lambda| \neq 0, 1 \) and \( \Delta_\lambda \) is the group generated by the transformation \( z \mapsto \lambda z, z \in \mathbb{C}^n - \{0\} \) (see [15]). On the manifold \( H_n^\lambda \), we consider the Hermitian metric

\[ ds^2 = \sum_{k=1}^n \frac{1}{z^k \bar{z}^k} \sum_{j=1}^n dz^j \otimes d\bar{z}^j, \]

where \( z^j (j = 1, \ldots, n) \) are complex Cartesian coordinates on \( \mathbb{C}^n \). The Hopf manifold \( H_n^\lambda \) is an example of a \( P_0K \)-manifold which is strongly non-Kaehler.

Let \( M \) be a submanifold of a Riemannian manifold \( L \). We denote by the same \( g \) the Riemannian metric tensor field induced on \( M \) from that of \( L \). Let \( \nabla^M \) denote covariant differentiation of \( M \). Then the Gauss formula for \( M \) is written as

\[ \nabla_X Y = \nabla^M_X Y + \sigma(X, Y) \]

for any vector fields \( X, Y \) tangent to \( M \), where \( \sigma \) denotes the second fundamental form of \( M \) in \( L \). Let \( M \) be an \( n \)-dimensional submanifold of \( L \).
The mean curvature vector $g$ of $M$ is defined by $g = \frac{1}{n} \text{trace}(\sigma)$. A submanifold $M$ is called totally umbilical if the second fundamental form $\sigma$ satisfies $\sigma(X, Y) = g(X, Y)g$. A submanifold $M$ is called totally geodesic if the second fundamental form vanishes identically, that is, $\sigma = 0$.

Let $R^M$ be the Riemannian curvature tensor field of $M$. Then we have the equation of Gauss

$$R^L(W, Z, X, Y) = R^M(W, Z, X, Y) + g(\sigma(X, Z), \sigma(Y, W)) - g(\sigma(Y, Z), \sigma(X, W)).$$

Let $N$ be an almost Hermitian manifold with almost complex structure $J'$ and $\pi: M \to N$ a Riemannian submersion such that $TM \cap J(TM)$ is the horizontal part of $TM$ and, at each point $p \in M$, $\pi_*$ is a complex isometry of $T^p M = T^p M \cap J(T^p M)$ onto $T_{\pi(p)} N$. Let $X$ denote a tangent vector at $p \in M$. Then $X$ decomposes as $\mathcal{V}X + \mathcal{H}X$, where $\mathcal{V}X$ is tangent to the fiber through $p$ and $\mathcal{H}X$ is perpendicular to it. We define tensors $T$ and $A$ associated with the submersion by

$$T_X Y := \nabla_{\mathcal{V}X}^M \mathcal{H}Y + \mathcal{H} \nabla_{\mathcal{V}X}^M \mathcal{V}Y,$$
$$A_X Y := \nabla_{\mathcal{H}X}^M \mathcal{H}Y + \mathcal{H} \nabla_{\mathcal{H}X}^M \mathcal{V}Y,$$

for any vector fields $X, Y$ on $M$. Then $T$ and $A$ have the following properties [11].

(i) $T_X$ and $A_X$ are skew symmetric linear operators on the tangent space of $M$, and interchange the horizontal and vertical parts.

(ii) $T_X = T_{\mathcal{V}X}$ while $A_X = A_{\mathcal{H}X}$.

(iii) For $V, W$ vertical, $T_V W$ is symmetric, that is, $T_V W = T_W V$. For $X, Y$ horizontal, $A_X Y$ is skew symmetric, that is, $A_X Y = -A_Y X$.

A vector field $X$ on $M$ is said to be basic if $X$ is horizontal and $\pi$-related to a vector field $\tilde{X}$ on $N$. Every vector field $\tilde{X}$ on $N$ has a unique horizontal lift $X$ to $M$, and $X$ is basic. We denote it by $X = \text{h.l.}(\tilde{X})$.

**Lemma 1** ([11]). Let $X$ and $Y$ be any basic vector fields on $M$. Then

(i) $g(X, Y) = \bar{g}(\tilde{X}, \tilde{Y}) \circ \pi$;

(ii) $\mathcal{H}[X, Y]$ is the basic vector field corresponding to $[\tilde{X}, \tilde{Y}]$;

(iii) $\mathcal{H}\nabla_X^M Y$ is the basic vector field corresponding to $\nabla_{\tilde{X}} \tilde{Y}$, where $\bar{g}$ is the metric of $N$ and $\nabla^N$ is the covariant differentiation on $N$.

Let $R^N$ denote the curvature tensor field of $N$. The horizontal lift of the curvature tensor $R^N$ of $N$ will also be denoted by $R^N$. We recall the following curvature identity which will be needed in the sequel:

$$R^M(W, Z, X, Y) = R^N(\tilde{W}, \tilde{Z}, \tilde{X}, \tilde{Y}) - g(A_Y Z, A_X W) + g(A_X Z, A_Y W) + 2g(A_X Y, A_Z W),$$
where $X, Y, Z, W$ are any basic vector fields on $M$. As before, this result is proven in [11].

Let $X$ and $Y$ be any basic vector fields on $M$. We define the operator $\nabla^N$ by

$$\nabla^N_X Y := \mathcal{H}\nabla^M_X Y.$$  (14)

Then, by Lemma 1(iii), $\nabla^N_X Y$ is a basic vector field and

$$\pi_*(\nabla^N_X Y) = \nabla^N_{\tilde{X}} \tilde{Y}.$$  (15)

Next, we give the definition of a Sasakian manifold. A Riemannian manifold $(M, g)$ is said to be a Sasakian manifold if there exist a tensor field $\phi$ of type (1, 1), a unit vector field $V$ and a 1-form $\eta$ such that

$$\phi V = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)V,$$  (16)

$$\nabla^M_X \phi = g(X, Y)V - \eta(Y)X,$$  (17)

for any vector fields $X, Y$ on $M$ [2].

3. Proof of Theorem 2. We put $B = B^h + B^v + B^\perp$, where $B^h, B^v$ and $B^\perp$ are the horizontal part, the vertical part and the normal part of the Lee vector field $B$ respectively.

From (9) and (12), for any horizontal vector fields $X$ and $Y$, we have

$$\nabla_X Y = \mathcal{H}\nabla^M_X Y + A_X Y + \sigma(X, Y).$$  (17)

Since $M$ is a CR-submanifold of $L$, using (5) and (17), we obtain

$$\mathcal{H}\nabla^M_X JY = \frac{1}{2}\omega(JY)X + \frac{1}{2}g(X, JY)B^h = J\mathcal{H}\nabla^M_X Y - \frac{1}{2}\omega(Y)JX + \frac{1}{2}g(X, JY)JB^h \in T^h M,$$  (18)

$$A_X JY + \frac{1}{2}g(X, JY)B^v = J\sigma(X, Y) + \frac{1}{2}g(X, JY)JB^\perp \in T^v M,$$  (19)

$$\sigma(X, JY) + \frac{1}{2}g(X, JY)B^\perp = JA_X Y + \frac{1}{2}g(X, Y)JB^v \in TM^\perp,$$  (20)

where $X$ and $Y$ are any horizontal vector fields on $M$.

From (19) and (20), for any horizontal vector fields $X$ and $Y$, we obtain $\sigma(JX, JY) = \sigma(X, Y) + g(JX, Y)JB^v$, $A_{JX} JY = A_X Y - g(X, Y)B^v$, because $A_X Y$ is skew symmetric. In the last equation, we set $X = Y$; then we have $A_{JX} JX = A_X X - g(X, X)B^v$. Since $A_X X = 0$, we obtain $B^v = 0$.

Since $B^v = 0$, for any horizontal vector fields $X$ and $Y$, we obtain

$$\sigma(JX, JY) = \sigma(X, Y), \quad A_{JX} JY = A_X Y.$$  (21)

Next, we compare the holomorphic sectional curvatures of $L$ and $N$. We set $Z = JW$ and $Y = JX$ in (10) and (13) to obtain
Therefore, \( \pi \) field. Since basic and horizontal. Since the Lee form \( \omega \) (25) closed.

Thus, for any horizontal unit vector \( X \) on \( M \), we obtain

\[
R^L(X, JW, X, JX) = R^N(\tilde{X}, J\tilde{X}, \tilde{X}, J\tilde{X}) = 0, \quad (\text{22})
\]

where \( X \) and \( W \) are any basic vector fields on \( M \).

Setting \( X = W \) in the above equation, using (21), by \( \sigma(X, JX) = 0 \), we obtain

\[
R^L(X, JW, X, JX) = R^N(\tilde{X}, J\tilde{X}, \tilde{X}, J\tilde{X}) - 3|A_X JX|^2 - |\sigma(X, X)|^2. \quad (\text{23})
\]

Thus, for any horizontal unit vector \( X \) on \( M \), we obtain

\[
H^L(X) = H^N(\pi_* X) - 3|A_X JX|^2 - |\sigma(X, X)|^2. \quad (\text{24})
\]

Now, we assume that the horizontal component \( B^h \) of the Lee vector field \( B \) is basic and \( \dim N \geq 4 \). We put \( \tilde{B} := \pi_\ast(B^h) \). Let \( \omega' \) be the 1-form on \( M \) induced by the Lee form \( \omega \) on \( L \). For any vector field \( \tilde{X} \) on \( N \), we set \( \tilde{\omega}(\tilde{X}) := \tilde{\gamma}(\tilde{X}, \tilde{B}) \). Then \( (\pi^* \tilde{\omega})(X) = \omega'(X) \), where \( X \) is any basic vector field. Since \( \pi^* \) commutes with \( d \) and \( \pi \) is a Riemannian submersion, \( \tilde{\omega} \) is closed.

From the definition of \( \tilde{\omega} \), we obtain

\[
\tilde{\gamma}(\tilde{X}, \tilde{B}) \circ \pi = \tilde{\omega}(\tilde{X}) \circ \pi = \omega'(X) = \omega(X) = g(X, B), \quad (\text{25})
\]

where \( \tilde{X} \) is any vector field on \( N \) and \( X = h.l.(\tilde{X}) \). We define the Weyl connection \( W^N_X \) of \( N \) by

\[
W^N_X Y = \nabla^N_X Y - \frac{1}{2}\tilde{\omega}(\tilde{X})Y - \frac{1}{2}\tilde{\omega}(Y)\tilde{X} + \frac{1}{2}\tilde{\gamma}(\tilde{X}, Y)\tilde{B}. \quad (\text{26})
\]

From Lemma 1, (18), (25) and (26), for any vector fields \( \tilde{X}, \tilde{Y} \) and \( \tilde{Z} \), we obtain

\[
\tilde{\gamma}((W^N_X J')\tilde{Y}, \tilde{Z}) \circ \pi = \tilde{\gamma}(W^N_X J'\tilde{Y}, \tilde{Z}) \circ \pi - \tilde{\gamma}(J'W^N_X \tilde{Y}, \tilde{Z}) \circ \pi = g(H\nabla^M_X Y - \frac{1}{2}\omega(JY)X + \frac{1}{2}g(X, Y)B)
\]

\[
- JH\nabla^M_X Y - \frac{1}{2}\omega(JY)X - \frac{1}{2}g(X, Y)JB, Z) = 0,
\]

where \( X, Y \) and \( Z \) are the horizontal lifts of \( \tilde{X}, \tilde{Y} \) and \( \tilde{Z} \) respectively. Therefore \( W^N_X J' = 0 \), that is, \( N \) is a locally conformal Kaehler manifold.

Let \( L \) be a generalized Hopf manifold and let the Lee vector field \( B \) be basic and horizontal. Since the Lee form \( \omega \) of \( L \) is parallel, for any vector field \( X \) tangent to \( M \), we have \( \nabla_X B = 0 \). Hence, by \( \nabla_X B = \nabla^M_X B + \sigma(X, B) \),
we have $\nabla^M_X B = 0$. From Lemma 1 and (25), we obtain
\[
\gamma(\nabla^N_X \tilde{B}, \tilde{Y}) \circ \pi = (\tilde{X} \gamma(\tilde{B}, \tilde{Y}) - \gamma(\tilde{B}, \nabla^N_X \tilde{Y})) \circ \pi
= X g(B, Y) - g(B, \nabla^M_X Y) = g(\nabla^M_X B, Y) = 0,
\]
where $\tilde{X}, \tilde{Y}$ are any vector fields tangent to $N$, and $X, Y$ are their horizontal lifts. Hence we obtain $\nabla^N_X \tilde{B} = 0$, that is, $N$ is a generalized Hopf manifold.

Remark 2. In this theorem, let $L$ be a locally conformal Kaehler manifold and $M$ a totally umbilical CR-submanifold of $L$ and the Lee vector field $B \in T^h M$. It is known that if $B$ is tangent to $M$, then a totally umbilical proper CR-submanifold $M$ of $L$ is totally geodesic [6]. For $X, Y \in T^h M$, we have $A_X Y = \frac{1}{2} \mathcal{V}[X, Y]$ (see [11]). Therefore, using (19), we see that the horizontal distribution $T^h M$ is integrable and the integral submanifolds are totally geodesic.

4. Proof of Theorem 3. Since $B \in T^M^\perp$, for any vector field $X$ tangent to $M$, we have $\omega(X) = 0$. Since $M$ is a CR-submanifold of $L$, (5) implies
\[
\nabla_X JY + \frac{1}{2} g(X, JY) B = J \nabla_X Y + \frac{1}{2} g(X, Y) JB,
\]
where $X$ and $Y$ are horizontal vector fields. Using (17) and (28), we obtain
\[
\mathcal{H} \nabla^M_X JY = J \mathcal{H} \nabla^M_X Y \in T^h M,
\]
\[
A_X JY = J \sigma(X, Y) + \frac{1}{2} g(X, Y) JB \in T^v M,
\]
\[
\sigma(X, JY) + \frac{1}{2} g(X, JY) B = JA_X Y \in T^M^\perp,
\]
where $X$ and $Y$ are any horizontal vector fields on $M$.

Since $\pi_*$ is a complex isometry, we have $\pi_* \circ J = J' \circ \pi_*$. Therefore, if $X$ is a basic vector field, $JX$ is also a basic vector field. Using Lemma 1, (15) and (29), we have
\[
\nabla^N_X J' \tilde{Y} = J' \nabla^N_X \tilde{Y}.
\]
Hence $N$ is a Kaehler manifold.

5. Proof of Theorem 4. Since $L$ is a $P_0 K$-manifold, we have $\mathcal{W} R = 0$ and $\nabla \omega = 0$. We set $c := |\omega|/2$. Since $\nabla \omega = 0$, we have $\nabla B = 0$ and $c = \text{constant}$ (see [17]). From (8), we have
\[
R^L(X, Y) Z = \frac{1}{4} \left[ [\omega(X) Y - \omega(Y) X] \omega(Z) \\
+ [g(X, Z) \omega(Y) - g(Y, Z) \omega(X)] B \\
+ \frac{1}{2} g(Y, Z) X - g(X, Z) Y \right].
\]
Using $\nabla \omega = 0$ and $\nabla B = 0$, we obtain $\nabla R^L = 0$ (see [6]). Since $B \in T^M^\perp$, using (10) and (32), for any vector fields $X, Y, Z$ and $W$ tangent to $M$, we
Hence the reflections \( \varphi \) have curvature tangent vector. \( \tilde{M} \) vector is parallel, the second fundamental form is parallel. Thus \( M \) is a totally geodesic submanifold of \( L \) and the mean curvature vector is parallel, the second fundamental form is parallel. Thus \( M \) is a locally symmetric space. Using (33) and \( M \), we get (36)

\[
\frac{1}{2} - 1 = \frac{1}{2}
\]

From (30) and (31), we get

\[
\frac{1}{2} \cdot 2 = 1
\]

6. Proof of Theorem 5. Since \( L \) is a \( P_0K \)-manifold, the maximal integral submanifold \( M \) of \( L \) is a totally geodesic submanifold of \( L \) (see [17]). From (33), we have

\[
R^M(W, Z, X, Y) = c^2(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) + g(\sigma(Y, Z), \sigma(X, W)) - g(\sigma(X, Z), \sigma(Y, W)).
\]

Since \( M \) is a totally umbilical submanifold of \( L \) and the mean curvature vector is parallel, the second fundamental form is parallel. Thus \( M \) is a locally symmetric space. Using (33) and \( M \), we obtain

\[
R^M(X, Y, Z, V) = 0.
\]

Moreover, since \( \sigma(X, Y) = g(X, Y)\varphi \) and \( B \in TM^\perp \), the fibers of \( \pi \) are totally geodesic [6]. Hence the reflections \( \varphi_{x-1} \) with respect to the fibers are isometries [4]. Therefore \( N \) is a locally symmetric space [4], [9]. From Theorem 3, \( N \) is a Kaehler manifold. Using (32), for any horizontal unit vector \( X \), we get

\[
H^N(X) = c^2. \text{ Thus, from (24), we have } H^N(\tilde{X}) > 0, \text{ where } \tilde{X} \text{ is any unit tangent vector.}
\]
Since $L$ is complete and $M$ is a totally geodesic submanifold of $L$, $M$ is complete. Since $M$ is complete and $\pi : M \to N$ is a Riemannian submersion, $N$ is complete [11]. From Theorem 3, $N$ is a Kaehler manifold.

It is known that a complete Kaehler manifold with constant scalar curvature and with positive sectional curvature is isometric to the complex projective space $P_m(\mathbb{C})$ (see [1]). Therefore $N$ is isometric to $P_m(\mathbb{C})$.

7. Proof of Theorem 6. For the Lee vector field $B$, we set

$$C := B/\sqrt{g(B, B)}.$$  

We define a vector field $V$, a 1-form $\eta$ and a tensor field $\phi$ of type $(1, 1)$ on $M$ by

$$V = J\alpha, \quad \eta(X) = g(X, V), \quad JX = \phi X - \eta(X)C.$$  

Since $L$ is a Hermitian manifold, $(M, \phi, V, \eta, g)$ admits an almost contact metric structure [2], [17].

Let $H X$ and $V X$ be the $T^h M$ part and $T^v M$ part of $X \in TM$ respectively. We set $\sigma(X, Y) = -k(X, Y)C$. From (5), for any vector field $X$ in $T^h M$, we obtain

$$\nabla V JX = J \nabla V X.$$  

Using $\nabla V X = \nabla^M V X - k(V, X)C$, by (39), we have the following equations:

$$\mathcal{H} \nabla^M V X = J \mathcal{H} \nabla^M V X \in T^h M,$$

$$\nabla^M V X = -k(V, X) \in T^v M,$$

$$-k(V, JX)C = J \nabla^M V X \in TM^\perp,$$

where $X$ is any vector field in $T^h M$. From (38) and (40), for any vector fields $X$ and $Y$ in $T^h M$, we obtain

$$g((\nabla^M V \phi)X, Y) = g(\nabla^M V \phi X - \phi \nabla^M V X, Y)$$

$$= g(\mathcal{H} \nabla^M V X - J \mathcal{H} \nabla^M V X, Y) = 0.$$  

From the $T^h M$ part of (5) and (38), for any vector fields $X$ and $Y$ in $T^h M$, we obtain

$$\mathcal{H} \nabla^M X \phi Y = \phi \mathcal{H} \nabla^M X Y.$$  

Now, for any vector fields $X$ and $Y$ tangent to $M$, we assume $k(X, Y) = \left(\frac{1}{2} \sqrt{\omega(B)} - 1\right)g(X, Y) + \alpha \eta(X)\eta(Y)$. Let $V$ and $W$ be any vector fields in $T^v M$ and $X$ be any vector field in $T^h M$. From (42), we obtain $\nabla^M V X = 0$, because $k(X, V) = 0$. Using (5), we obtain $g(\mathcal{H} \nabla^M V X, X) = g(\mathcal{H} \nabla^M V JX, X) = -g(\sigma(X, X), JW)$. Hence, we get $\mathcal{H} \nabla^M V W = 0$.

We shall prove that $(M, \phi, V, \eta, g)$ admits a Sasakian structure. Let $X$, $Y$ and $Z$ be any vector fields tangent to $M$. Using (44) and the above result,
Thus, by (43) and (45), we have

\[ g(\nabla^M_X \phi Y, Z) = g(\nabla^M_X \phi Y, \mathcal{H} Z) + g(\nabla^M_X \phi Y, V Z) \]

\[ = g(\nabla^M_X \phi Y, \mathcal{H} Z) - g(\phi \nabla^M_X Y, \mathcal{H} Z) + g(\nabla^M_X \phi Y, V Z) - g(\phi \nabla^M_X Y, V Z) \]

\[ = g(\nabla^M_{\mathcal{H}X} \phi \mathcal{H} Y, \mathcal{H} Z) + g(\nabla^M_{\mathcal{H}X} \phi \mathcal{V} Y, \mathcal{H} Z) + g(\nabla^M_{\mathcal{H}X} \phi \mathcal{H} Y, \mathcal{V} Z) + g(\nabla^M_{\mathcal{H}X} \phi \mathcal{V} Y, \mathcal{V} Z) - g(\phi \nabla^M_{\mathcal{H}X} \mathcal{H} Y, \mathcal{V} Z) \]

\[ - g(\phi \nabla^M_{\mathcal{H}X} \mathcal{V} Y, \mathcal{V} Z) \]

\[ = g(\nabla^M_{\mathcal{V}X} \phi \mathcal{H} Y, \mathcal{H} Z) + g(\mathcal{V}, \mathcal{V} Z)g(\nabla^M_{\mathcal{H}X} \phi \mathcal{H} Y, \mathcal{V}) - g(\mathcal{V}, \mathcal{V} Y)g(\nabla^M_{\mathcal{H}X} \phi \mathcal{H} Z, \mathcal{V} \mathcal{Z}) \]

Using the $T^*M$ part of (5) and the assumption, we obtain

(45) \[ g(\nabla^M_{\mathcal{H}X} \phi \mathcal{H} Y, \mathcal{V}) = g(\mathcal{V} \nabla^M_{\mathcal{H}X} \mathcal{J} \mathcal{H} Y, \mathcal{V}) \]

\[ = -k(\mathcal{H} X, \mathcal{H} Y) + \frac{1}{2}g(\mathcal{H} X, \mathcal{H} Y)\sqrt{\omega(B)} \]

\[ = g(\mathcal{H} X, \mathcal{H} Y). \]

Thus, by (43) and (45),

(46) \[ g(\nabla^M_X \phi Y, Z) = g(\mathcal{V}, \mathcal{V} Z)g(\mathcal{H} X, \mathcal{H} Y) - g(\mathcal{V}, \mathcal{V} Y)g(\mathcal{H} X, \mathcal{H} Z). \]

On the other hand,

\[ g(g(X, Y)V - \eta(Y) X, Z) = g(\mathcal{H} X, \mathcal{H} Y)g(\mathcal{V}, \mathcal{V} Z) + g(\mathcal{V} X, \mathcal{V} Y)g(\mathcal{V}, \mathcal{V} Z) - g(\mathcal{H} X, \mathcal{H} Z)g(\mathcal{V}, \mathcal{V} Y) - g(\mathcal{V} X, \mathcal{V} Z)g(\mathcal{V}, \mathcal{V} Y) \]

\[ = g(\mathcal{V}, \mathcal{V} Z)g(\mathcal{H} X, \mathcal{H} Y) - g(\mathcal{V}, \mathcal{V} Y)g(\mathcal{H} X, \mathcal{H} Z). \]

Therefore

(47) \[ \nabla^M_X \phi Y = g(X, Y)V - \eta(Y) X. \]

Hence $(M, \phi, V, \eta, g)$ is a Sasakian manifold.
Conversely, assume that \((M, \phi, V, \eta, g)\) is a Sasakian manifold. Let \(X\) and \(Y\) be any vector fields tangent to \(M\). From (9) and (38), we obtain
\[
\nabla_X JY - J\nabla_X Y = \nabla_X (\phi Y - \eta(Y)C) - J(\nabla^M_X Y + \sigma(X, Y))
\]
\[
= \nabla_X \phi Y - \nabla_X (\eta(Y)C) - \phi \nabla^M_X Y + \eta(\nabla^M_X Y)C - J\sigma(X, Y)
\]
\[
= (\nabla^M_X \phi) Y - k(X, \phi Y) C - X\eta(Y) C
\]
\[
- \eta(Y) \nabla_X C + \eta(\nabla^M_X Y) C + k(X, Y) V.
\]
On the other hand, by (5),
\[
\nabla_X JY - J\nabla_X Y = \frac{1}{2} \omega(JY) X - \frac{1}{2} g(X, JY) B + \frac{1}{2} g(X, Y) JB
\]
\[
= - \frac{1}{2} \sqrt{\omega(B)} \eta(Y) X - \frac{1}{2} g(X, \phi Y) B + \frac{1}{2} \sqrt{\omega(B)} g(X, Y) V.
\]
From these equations and (47), we have
\[
g(X, Y) V - \eta(Y) X - k(X, \phi Y) C - X\eta(Y) C
\]
\[
- \eta(Y) \nabla_X C + \eta(\nabla^M_X Y) C + k(X, Y) V
\]
\[
= - \frac{1}{2} \sqrt{\omega(B)} \eta(Y) X - \frac{1}{2} g(X, \phi Y) B + \frac{1}{2} \sqrt{\omega(B)} g(X, Y) V.
\]
The \(V\) component of this equation is
\[
g(X, Y) - \eta(Y) \eta(X) - \eta(Y) g(\nabla_X C, \eta) + k(X, Y)
\]
\[
= - \frac{1}{2} \sqrt{\omega(B)} \eta(Y) \eta(X) + \frac{1}{2} \sqrt{\omega(B)} g(X, Y).
\]
Thus
\[
k(X, Y) = \left( \frac{1}{2} \sqrt{\omega(B)} - 1 \right) g(X, Y)
\]
\[
- \left( \frac{1}{2} \sqrt{\omega(B)} - 1 \right) \eta(X) \eta(Y) + \eta(Y) g(\nabla_X C, V).
\]
Since \(k(X, Y)\) is symmetric, we have \(\eta(Y) g(\nabla_X C, V) = \eta(X) g(\nabla_Y C, V)\). This equation shows that \(g(\nabla_X C, V) = \beta \eta(X)\), where \(\beta\) is a function. We set \(\alpha = - \frac{1}{2} \sqrt{\omega(B)} + 1 + \beta\); then we have
\[
k(X, Y) = \left( \frac{1}{2} \sqrt{\omega(B)} - 1 \right) g(X, Y) + \alpha \eta(X) \eta(Y).
\]

8. Examples. (I) Let \((M, \phi, V, \eta, g)\) be a Sasakian manifold and \(S^1\) the circle with length element \(\omega = dt\). Then \(S^1 \times M\) is a generalized Hopf manifold with metric \(\omega^2 + g\) and Lee form \(\omega\) (see [17]).

Let \(\mathbb{C}^{n+m}\) be the complex vector space of all \((n+m)\)-tuples of complex numbers \(z = (z_1, \ldots, z_{n+m})\) and \(a_{kj}\) be positive integers and \(\alpha_{kj}\) be real numbers, \(k = 1, \ldots, m, j = 1, \ldots, n + m\). Let
\[
f_k(z_1, \ldots, z_{n+m}) = \sum_{j=1}^{n+m} \alpha_{kj} z_j^{a_{kj}}, \quad k = 1, \ldots, m,
\]
be a collection of complex polynomials. Let \(F = \bigcap_{k=1}^{m} f_k^{-1}(0)\). Let \(d_k = \text{LCM}(a_{k1}, a_{k2}, \ldots, a_{kn+m})\), \(q_{kj} = d_k / a_{kj}\). Suppose that
(i) $F$ is a complete intersection of the $f_k^{-1}(0)$.
(ii) $F$ has an isolated singularity at the origin.
(iii) $q_k$ is independent of $k$ (let $q_j = q_k$).

Let $B^{2n-1} = F \cap S^{2(n+m) - 1} \subset \mathbb{C}^{n+m}$. Then $B^{2n-1}$ is called a generalized Brieskorn manifold [12]. It is a $(2n-1)$-dimensional submanifold in $S^{2(n+m) - 1}$. Let $(S^{2(n+m) - 1}, \phi, V, \eta, \tilde{g})$ be the unit sphere with the standard Sasakian structure and imbedded in $\mathbb{C}^{n+m}$. Denoting by $x_1, y_1, \ldots, x_n, y_n, x_{n+1}, \ldots, x_{n+m}$ the real coordinates of $\mathbb{C}^{n+m}$ such that $z_j = x_j + \sqrt{-1} y_j$ ($j = 1, \ldots, n + m$), we define a real vector field $\tilde{V}$ on $\mathbb{C}^{n+m}$ by

$$\tilde{V} = \sum_{j=1}^{n+m} A_j (x_j \partial/\partial y_j - y_j \partial/\partial x_j),$$

where $A_j = \gamma q_j$ for a positive constant $\gamma$ ($j = 1, \ldots, n + m$). We set

$$\mu = \tilde{V} - V, \quad \tilde{\eta} = (1 + \eta(\mu))^{-1} \eta, \quad \tilde{\phi}(X) = \phi(X - \tilde{\eta}(X)\tilde{V}),$$

$$\tilde{g}(X, Y) = (1 + \eta(\mu))^{-1} g(X - \tilde{\eta}(X)\tilde{V}, Y - \tilde{\eta}(Y)\tilde{V}) + \tilde{\eta}(X)\tilde{\eta}(Y),$$

where $X$ and $Y$ are vector fields on $S^{2(n+m) - 1}$. Then, by the theorem of Takahashi [13], $(S^{2(n+m) - 1}, \tilde{\phi}, \tilde{V}, \tilde{\eta}, \tilde{g})$ is also a Sasakian manifold. Let $\iota : B^{2n-1} \to S^{2(n+m) - 1}$ be the inclusion mapping. We define four tensor fields $(\tilde{\phi}, \tilde{V}, \tilde{\eta}, \tilde{g})$ on $B^{2n-1}$ by

$$\tilde{\phi} = \tilde{\phi}_{|B^{2n-1}}, \quad \tilde{V} = \tilde{V}_{|B^{2n-1}}, \quad \tilde{\eta} = \iota^* \tilde{\eta}, \quad \tilde{g} = \iota^* \tilde{g}.$$

Using calculations similar to those of [13], we can prove that every generalized Brieskorn manifold $(B^{2n-1}, \tilde{\phi}, \tilde{V}, \tilde{\eta}, \tilde{g})$ admits many Sasakian structures. Therefore, $S^1 \times B^{2n-1}$ is a generalized Hopf submanifold of the generalized Hopf manifold $S^1 \times S^{2(n+m) - 1}$.

(II) Let $E^{2n-1}(-3)$ be the Sasakian space form with constant $\phi$-sectional curvature $-3$ with standard Sasakian structure in a Euclidean space. Let $S^1(r_i)$ be a circle of radius $r_i$, $i = 1, \ldots, p$. A pythagorean product $E^{2(n-p) - 1}(-3) \times S^1(r_1) \times \ldots \times S^1(r_p)$ is a pseudo-umbilical generic submanifold of $E^{2n-1}(-3)$ ($p \geq 2$) (see [20]). Let $S^1$ be the circle with length element $\omega$. Then $\omega$ is the Lee form of the generalized Hopf manifold $S^1 \times E^{2n-1}(-3)$. Hence $S^1 \times E^{2(n-p) - 1}(-3) \times S^1(r_1) \times \ldots \times S^1(r_p)$ is a CR-submanifold of $S^1 \times E^{2n-1}(-3)$ satisfying the conditions (a)–(c) and $S^1 \times E^{2(n-p) - 1}(-3)$ is tangent to the Lee vector field of $S^1 \times E^{2n-1}(-3)$. The projection

$$\pi : S^1 \times E^{2(n-p) - 1}(-3) \times S^1(r_1) \times \ldots \times S^1(r_p) \to S^1 \times E^{2(n-p) - 1}(-3)$$

is a Riemannian submersion satisfying (d). $S^1 \times E^{2(n-p) - 1}(-3)$ is also a generalized Hopf manifold.

(III) The Hopf manifold $H^n_{\phi}$ is isometric to $S^1(1/\pi) \times S^{2n-1}$ (see [17]). $S^{2n-1}$ is a real hypersurface of $H^n_{\phi}$ and the Lee vector field of $H^n_{\phi}$ is
normal to $S^{2n-1}$. $S^{2n-1}$ is a CR-submanifold of $H^n_{e2}$ satisfying the conditions (a)–(c). $\pi: S^{2n-1} \to P_{n-1}(\mathbb{C})$ is a Riemannian submersion satisfying (d). From O’Neill [11], for orthonormal horizontal vectors $X, Y$, $AXY = -g(X, JY)JC$, where $J$ is an almost complex structure on $H^n_{e2}$ and $C$ is the unit normal vector to $S^{2n-1}$. The holomorphic sectional curvature $H$ of $P_{n-1}(\mathbb{C})$ is $H(\tilde{X}) = 1 + 3|AXJX|^2 = 4$, where $\tilde{X}$ is any unit vector tangent to $P_{n-1}(\mathbb{C})$ and $X = h.l.(\tilde{X})$.

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