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ASYMPTOTICS FOR THE MINIMIZATION OF A GINZBURG–LANDAU ENERGY IN n DIMENSIONS

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We prove that minimizers $u_\varepsilon \in W^{1,n}$ of the functional

$$E_\varepsilon(u) = \frac{1}{n}\int_\Omega |\nabla u|^n\,dx + \frac{1}{4\varepsilon^n}\int_\Omega (1-|u|^2)^2\,dx, \qquad \Omega \subset \mathbb{R}^n,\ n \ge 3,$$

which satisfy the Dirichlet boundary condition $u_\varepsilon = g$ on $\partial\Omega$ for $g : \partial\Omega \to S^{n-1}$ with zero topological degree, converge in $W^{1,n}$ and C^α_{loc} for any $\alpha < 1$ —upon passing to a subsequence $\varepsilon_k \to 0$—to some minimizing n-harmonic map. This is a generalization of an earlier result obtained for $n = 2$ by Bethuel, Brezis, and Hélein.

An example of nonunique asymptotic behaviour (which cannot occur in two dimensions if $\deg g = 0$) is presented.

1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, simply connected domain with smooth boundary. We consider the following *energy* of Ginzburg–Landau type:

$$E_\varepsilon(u) := \frac{1}{n}\int_\Omega |\nabla u|^n\,dx + \frac{1}{4\varepsilon^n}\int_\Omega (1-|u|^2)^2\,dx, \tag{1}$$

defined for maps $u \in W^{1,n}(\Omega, \mathbb{R}^n)$. For a fixed smooth boundary condition $g : \partial\Omega \to S^{n-1}$, write

$$W^{1,n}_g := \{u \in W^{1,n}(\Omega, \mathbb{R}^n) \mid u = g \text{ on } \partial\Omega\}.$$

It is easily seen that the minimum

$$\min_{u \in W^{1,n}_g} E_\varepsilon(u) \tag{2}$$

is achieved by some map u_ε which solves the Euler–Lagrange system

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$$-\mathrm{div}(|\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon^i) = \frac{1}{\varepsilon^n} u_\varepsilon^i (1 - |u_\varepsilon|^2), \quad i = 1, \ldots, n. \tag{3}$$

In general, u_ε does not have to be unique (see examples in Section 2).

In this paper, we study the asymptotic behaviour of u_ε as $\varepsilon \to 0$, for $n \geq 3$, and under the assumption that the boundary condition $g : \partial\Omega \to S^{n-1}$ is topologically trivial, i.e. $\deg g = 0$. Our results are formulated in Section 1.2 below.

1.1. *The two-dimensional case.* For $n = 2$, the functional (1) and its various analogues are often considered in the theory of phase fluctuations and vortex models of phase transitions.

In diverse applications, the complex-valued function u_ε is related to

- the density of superconducting electrons in type II superconductors; $|u_\varepsilon| \simeq 1$ corresponds to the superconducting state and $|u_\varepsilon| \simeq 0$ corresponds to the normal state;
- the two-dimensional magnetization vector in magnets;
- the condensate wavefunction in superfluids.

The parameter ε has the dimension of length and is usually *very* small in relevant physical applications. Therefore, it is of great mathematical interest to analyze the asymptotic behaviour of u_ε as $\varepsilon \to 0$ (even if the limiting problem has no physical meaning).

In a series of recent works [1–4] Bethuel, Brezis and Hélein have proved, among other things, the following results concerning the asymptotic behaviour of u_ε.

"*Easy*" *case*. Assume that $g : \partial\Omega \to S^1$ is smooth, and $\deg g = 0$. Then the asymptotic behaviour of the whole family $\{u_\varepsilon \mid \varepsilon > 0\}$ is fully described by the following

THEOREM 1 (Bethuel, Brezis, Hélein [1, 2]). *Let $n = 2$ and let u_ε be a minimizer of E_ε in $W^{1,2}_g(\Omega, \mathbb{R}^2)$. Denote by u_0 the unique S^1-valued harmonic mapping with $u_0|_{\partial\Omega} = g$. As $\varepsilon \to 0$, u_ε tends to u_0 in $C^{1,\alpha}(\overline{\Omega})$ for any $\alpha < 1$, and in $C^k(K)$ for any natural k and any compact $K \subset \Omega$.*

Tough case. If the boundary condition $g : \partial\Omega \to S^1$ has nonzero degree, $\deg g = d > 0$, then the situation is more complicated. The main difficulty stems from the fact that $E_\varepsilon(u_\varepsilon)$ blows up like $|\log\varepsilon|$ for $\varepsilon \to 0$. Nevertheless, one can prove the following.

THEOREM 2 (Bethuel, Brezis, Hélein [3, 4]). *Assume that $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$ is starshaped. Then one can find a sequence $\varepsilon_k \to 0$, precisely d points $a_1, \ldots, a_d$ in Ω, and a smooth harmonic map $u_* : \Omega \setminus \{a_1, \ldots, a_d\} \to S^1$*

satisfying $u_* = g$ *on* $\partial\Omega$ *such that*

$$u_{\varepsilon_k} \to u_* \qquad \text{in } C^{1,\alpha}(\overline{\Omega} \setminus \{a_1, \ldots, a_d\}) \text{ for any } \alpha \in (0,1),$$
$$\text{and in } C^k_{\mathrm{loc}}(\Omega \setminus \{a_1, \ldots, a_d\}) \text{ for any } k.$$

Moreover, u_* *coincides with the so-called canonical harmonic map*, *i.e.*

$$u_*(z) = \prod_{j=1}^{d} \left(\frac{z - a_j}{|z - a_j|} \right) \exp(i\varphi),$$

where $\varphi : \Omega \to \mathbb{R}$ *is a harmonic function with* $\exp(i\varphi) = \prod_{j=1}^{d} \big(\frac{|z-a_j|}{z-a_j} \big) g$ *on* $\partial\Omega$.

Another theorem from [3, 4] states that the configuration $\{a_1, \ldots, a_d\}$ of singularities of u_* minimizes a function of d complex variables,

$$W : \underbrace{\Omega \times \ldots \times \Omega}_{d\,\text{times}} \to \mathbb{R},$$

the so-called *renormalized energy*. To give an explicit formula for W, one has first to solve a linear Neumann problem (see [4, Section I.4] for more details).

The amazingly beautiful analysis of [4] (which should be viewed from an applied point of view as a first rigorous approach to a *model problem* in two spatial dimensions) heavily depends on the linearity of the Laplace operator and on the nice structure of the multiplicative group of complex numbers with modulus 1 that we have on S^1. In some sense, it also uses the fact that all regular maps on the boundary of a two-dimensional domain are quasiconformal. To deal with more realistic problems arising in the modelling of phase transitions in magnetic and superconducting media, it seems necessary to gain first some insight into mathematical phenomena occurring for more general variational problems, with more complicated nonlinearities in their Euler–Lagrange equations. Some of relevant open problems are listed in the closing chapter of [4]. One of them, Problem 17, is to describe the asymptotic behaviour of u_ε in the general case, for arbitrary n.

1.2. *The results.* Our result, concerning $n \geq 3$ and the easier case $\deg g = 0$, is stated below (see Theorem 5). Roughly speaking, we prove that some sequence u_{ε_k} converges to a minimizing n-harmonic map in the topologies of $W^{1,n}$ (this is pretty simple) and C^α_{loc}, for any $\alpha < 1$ (convergence in Hölder norms presents a more delicate problem).

Remark. For the tougher case $\deg g = d \neq 0$, Min Chun Hong [12] has proved that a subsequence of u_ε converges weakly in $W^{1,n}(G, \mathbb{R}^n)$, where $G = (\Omega$ except $|d|$ distinct points), to some map u_*.

After having completed this work, the author obtained a very interesting paper [9] of Z. Han and Y. Li, which contains a result more general than that of Hong. Han and Li prove that u_{ε_k} converges to u_* strongly in $C^0_{\mathrm{loc}}(\overline{\Omega}\setminus\{a_1,\ldots,a_{|d|}\})$. Their method of proof is different from ours; in particular, differentiation of systems (which we use to obtain gradient bounds in Section 3) is replaced by a new regularity theorem, obtained via perturbation arguments, for p-harmonic systems with Hölder continuous coefficients.

Whether, in the nonzero degree case for $n \ge 3$, u_{ε_k} converges to u_* in better norms, say at least C^α, seems to be still an open question.

Let us first formulate some definitions. Consider the n-Dirichlet integral

$$\mathbf{I}_n(u) = \int_\Omega |\nabla u|^n\,dx.$$

We say that a map $u \in W^{1,n}_g(\Omega, S^{n-1})$ is (*weakly*) *n-harmonic* iff it is a critical point of $\mathbf{I}_n$ in the class $W^{1,n}_g(\Omega, S^{n-1})$ with respect to variations in the range. Computing the derivative

$$\frac{d}{dt}\bigg|_{t=0} \mathbf{I}_n\bigg(\frac{u+t\varphi}{|u+t\varphi|}\bigg)$$

for an arbitrary $\varphi \in C^\infty_0(\Omega,\mathbb{R}^n)$, one can easily check that n-harmonic maps are precisely the weak solutions of the nonlinear degenerate elliptic system

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u) = u|\nabla u|^n. \tag{4}$$

By a *minimizing n-harmonic map* we mean here any minimizer of $\mathbf{I}_n$ in the class $W^{1,n}_g(\Omega, S^{n-1})$. For a fixed $g : \partial\Omega \to S^{n-1}$ with $\deg g = 0$, we denote by $\mathcal{M}_g$ the set of all minimizing n-harmonic maps u with $u = g$ on $\partial\Omega$.

PROPOSITION 3. *If $\{u_\varepsilon \mid \varepsilon > 0\}$ is a family of minimizers for the problem* (2), *then one can choose a sequence $\varepsilon_k \to 0$, and a minimizing n-harmonic map $u_* \in \mathcal{M}_g$ such that $u_{\varepsilon_k} \to u_*$ strongly in $W^{1,n}(\Omega,\mathbb{R}^n)$ and a.e. as $k \to \infty$.*

Proof. Pick a minimizing n-harmonic map $w \in \mathcal{M}_g$. (This is the only place in the proof where the assumption $\deg g = 0$ is used: it implies that $W^{1,n}_g(\Omega, S^{n-1})$ is nonempty!)

Since $|w| = 1$ a.e., we have

$$\frac{1}{n}\int_\Omega |\nabla u_\varepsilon|^n\,dx + \frac{1}{4\varepsilon^n}\int_\Omega (1-|u_\varepsilon|^2)^2\,dx \le E_\varepsilon(w) = \frac{1}{n}\int_\Omega |\nabla w|^n\,dx. \tag{5}$$

Hence, the family $\{u_\varepsilon \mid \varepsilon > 0\}$ is bounded in $W^{1,n}(\Omega,\mathbb{R}^n)$. Thus, one can find a sequence $\varepsilon_k \to 0$ and a map $u_* \in W^{1,n}(\Omega,\mathbb{R}^n)$ such that

$$\begin{aligned} &\nabla u_{\varepsilon_k} \rightharpoonup \nabla u_* \quad &&\text{weakly in } L^n(\Omega,\mathbb{R}^{n^2}),\\ &u_{\varepsilon_k} \to u_* &&\text{strongly in } L^n(\Omega,\mathbb{R}^n) \text{ and a.e.} \end{aligned} \tag{6}$$

By (5) and (6),

$$\int_\Omega |\nabla u_*|^n\,dx \le \liminf_{k\to\infty} \int_\Omega |\nabla u_{\varepsilon_k}|^n\,dx \le \limsup_{k\to\infty} \int_\Omega |\nabla u_{\varepsilon_k}|^n\,dx \le \int_\Omega |\nabla w|^n\,dx. \tag{7}$$

Moreover,

$$\int_\Omega (1-|u_{\varepsilon_k}|^2)^2\,dx \le C\varepsilon_k^n \overset{k\to\infty}{\longrightarrow} 0. \tag{8}$$

Thus, $|u_*| = 1$ a.e., and $u_* \in W^{1,n}_g(\Omega, S^{n-1})$. By (7), we have $u_* \in \mathcal{M}_g$, $\int_\Omega |\nabla u_*|^n\,dx = \int_\Omega |\nabla w|^n\,dx$, and

$$\begin{aligned}\lim_{k\to\infty} \int_\Omega |\nabla u_{\varepsilon_k}|^n\,dx &= \int_\Omega |\nabla u_*|^n\,dx,\\ \lim_{k\to\infty} \frac{1}{4\varepsilon_k^n} \int_\Omega (1-|u_{\varepsilon_k}|^2)^2\,dx &= 0.\end{aligned} \tag{9}$$

Now, recall a classical theorem of functional analysis.

THEOREM 4. *If a sequence of functions $(f_k) \subset L^p$, $1 < p < \infty$, converges weakly to $f \in L^p$ and $\lim \|f_k\|_p = \|f\|_p$, then $\|f_k - f\|_p \to 0$ as $k \to \infty$.*

This result combined with (9) implies immediately that $u_{\varepsilon_k} \to u_*$ strongly in $W^{1,n}$. ■

Remark. In general, it may happen that, for a fixed boundary condition g, different sequences of minimizers u_ε converge to different limits. (See examples in Section 2.) Such nonunique asymptotic behaviour (in the "easy" case of a topologically trivial boundary condition) is excluded for $n = 2$.

The next theorem is the main result of this paper.

THEOREM 5. *Let u_{ε_k} be, as in Proposition 3, a sequence of minimizers of E_{ε_k} which converges to some minimizing n-harmonic map u_* in the $W^{1,n}$-norm. Then, for any $\alpha \in (0,1)$ and any compact subset K of Ω, u_{ε_k} tends to u_* in the space $C^\alpha(K)$.*

Let us give here a rough idea of the proof of Theorem 5 (the details are presented in Section 3). The whole proof is similar in spirit to [2], though some details are technically more complicated due to the degeneracy of the n-harmonic operator $L_n(u) := \operatorname{div}(|\nabla u|^{n-2}\nabla u)$.

First, we prove a maximum principle for weak solutions of (3). Then we obtain a Caccioppoli estimate which, after a standard reasoning, yields the local boundedness of $|\nabla u_\varepsilon|$ by a constant $C(\varepsilon)$. We also check that $C(\varepsilon)$ behaves—for small ε—like ε^{-1}. Next, we exploit the strong covergence of

∇u_ε in L^n to obtain a Caccioppoli estimate with constants independent of ε (this is the crucial point of the proof; we modify here a trick which for $n = 2$ is due to Bethuel, Brezis, and Hélein [2]). Then, after a Moser iteration, we prove that, for $1 < q < \infty$, the estimate $\|\nabla u_\varepsilon\|_{q,K} \le C(n, q, K)$ holds on all compact subsets K of Ω. The final argument is provided by the Rellich–Kondrashov compactness theorem and the Sobolev imbedding theorem.

Theorem 5 is of course not a final result. We expect that it is possible to obtain at least local $C^{1,\alpha}$ convergence, and C^α convergence of u_{ε_k} on $\overline{\Omega}$, also in the case of variable boundary data g_ε sufficiently close to a fixed map $g_0 : \partial\Omega \to S^{n-1}$. This would allow improving the results obtained by Han–Li [9] and Hong [12] for the nonzero degree case.

The notation throughout the paper is either standard or selfexplanatory. The barred integral $⨍_A f\,dx$ denotes the average value f_A of a function f over a measurable set A, $f_A := |A|^{-1} \int_A f\,dx$. By B_r or $B(a, r)$ we denote the Euclidean ball in $\mathbb{R}^n$ of radius r, centered at a. The length of the gradient is defined by the formula

$$|\nabla u|^2 := \sum_{1\le i,j\le n} \left(\frac{\partial u^i}{\partial x_j}\right)^2.$$

Finally, C denotes a general constant which may change from one line to another; $C(n)$ denotes a constant depending *only* on n.

2. Planar n-harmonic maps. In this section, we show an example of nonuniqueness of minimizing n-harmonic maps from B^n into S^{n-1}. The general idea follows the papers of Hardt and Kinderlehrer [10], and Hardt, Kinderlehrer, and Lin [11]. Some technicalities are simplified here.

For simplicity, let $n = 3$ and write, for some $q > 0$ to be specified later,

$$g(x) = (\cos qx_3, \sin qx_3, 0) \quad \text{for } (x_1, x_2, x_3) \in \partial B^3 \equiv S^2. \tag{10}$$

Then define $v : B^3 \to S^2$ by the same formula. By a straightforward calculation one verifies that, for every positive q, v is a smooth S^2-valued 3-harmonic map with $|\nabla v| \equiv q$ and $\mathbf{I}_3(v) = 4\pi q^3/3$. We shall show that v is *not* minimizing in the class $W^{1,3}_g(B^3, S^2)$, and that a minimizing $u = (u^1, u^2, u^3) \in W^{1,3}_g(B^3, S^2)$ cannot be planar, i.e. its coordinate u^3 must be nonzero on a set of nonzero Lebesgue measure.

To this end, pick a cutoff function $\eta \in C_0^\infty(B^3)$ with $\eta \equiv \pi/2$ on a smaller ball $B(0, 1 - \mu)$ and $|\nabla\eta| \le 2/\mu$. Put

$$w(x) = (\cos\eta(x)\,\cos qx_3, \cos\eta(x)\,\sin qx_3, \sin\eta(x)).$$

Then, for μ small and q large, $\mathbf{I}_3(w) < \mathbf{I}_3(v)$. Indeed, $|\nabla w|^2 = \cos^2(\eta)|\nabla v|^2 + |\nabla\eta|^2$. Therefore, by the elementary inequality $(a + b)^s \le 2^{s-1}(a^s + b^s)$, we obtain

$$\begin{aligned}\mathbf{I}_3(w) = \int\limits_{B^3} |\nabla w|^3\,dx &\le \sqrt{2}\int\limits_{B^3} (\cos^3(\eta)|\nabla v|^3 + |\nabla\eta|^3)\,dx \\ &\le \frac{4\pi\sqrt{2}}{3}(1-(1-\mu)^3)\left(q^3 + \frac{8}{\mu^3}\right) < 200(\mu q^3 + \mu^{-2}).\end{aligned}$$

It is clear that, for μ small and q large, the last expression does not exceed $\mathbf{I}_3(v) = 4\pi q^3/3$. In fact, it is enough to take e.g. $q = \mu^{-1} = 100$. Therefore v is not minimizing.

Suppose now that $w = (w^1, w^2, 0) \in W^{1,3}_g(B^3, S^2)$ were a minimizing 3-harmonic map. By a theorem of Bethuel and Zheng [5, Lemma 1] one could write $w^1 = \cos\theta$, $w^2 = \sin\theta$ for some real-valued function θ of class $W^{1,3}$. Since w satisfies (4) for $n = 3$, one can easily compute that θ is a solution of the unconstrained 3-harmonic equation,

$$\operatorname{div}(|\nabla\theta|\nabla\theta) = 0.$$

Moreover, the boundary condition implies

$$\theta(x_1, x_2, x_3) - qx_3 \in \{2k\pi \mid k \in \mathbb{Z}\} \qquad \text{for } (x_1, x_2, x_3) \in S^2.$$

But if the trace of a Sobolev function has all its values in a discrete subset of $\mathbb{R}$, then it is constant. Hence, without loss of generality we can assume that $\theta(x) = qx_3$ on S^2. By the monotonicity of the 3-harmonic operator we conclude that $\theta(x) = qx_3$ in B^3 (there can only be one 3-harmonic function with given boundary values, and all linear functions are 3-harmonic). Hence, $w \equiv v$, and since v is not a minimizer, this is a contradiction. Therefore, no minimizer can be planar in our case.

Take now a minimizer $\psi = (\psi^1, \psi^2, \psi^3)$. Since $|\nabla(|u|)| \le |\nabla u|$, the above reasoning implies that $W^{1,3}_g$ contains two *distinct minimizers* for $\mathbf{I}_3$, i.e. $(\psi^1, \psi^2, +|\psi^3|)$ and $(\psi^1, \psi^2, -|\psi^3|)$, and one nonminimizing planar 3-harmonic map. Moreover, if one minimizes E_ε in the class $W^{1,3}_g$, there are, for small values of ε, at least two distinct minimizers $(u^1_\varepsilon, u^2_\varepsilon, \pm|u^3_\varepsilon|)$. Since minimizers of $\mathbf{I}_3$ are not planar, subsequences of $(u^1_\varepsilon, u^2_\varepsilon, +|u^3_\varepsilon|)$ and $(u^1_\varepsilon, u^2_\varepsilon, -|u^3_\varepsilon|)$ cannot converge to the same limit.

It is clear that one can use the same reasoning to construct examples of nonuniqueness of minimizers for $\mathbf{I}_n$ in the class $W^{1,n}(B^n, S^{n-1})$.

The example presented here raises obvious questions. Can the cardinality of $\mathcal{M}_g$ be an arbitrary natural number? Can $\mathcal{M}_g$ be an infinite set? (For the case of harmonic maps, an example of a boundary condition $g : S^2 \to S^2$ for which there exist uncountably many minimizing harmonic mappings $u : B^3 \to S^2$ with $u = g$ on S^2 has been given in [11].) Can one characterize those $u \in \mathcal{M}_g$ which can be limits of subsequences of u_ε? These problems shall be object of our further study.

3. Gradient bounds and Hölder estimates for u_ε. This section contains the proof of Theorem 5.

First, we prove a maximum principle for local weak solutions of (3). Then we differentiate the system (3) and derive a Caccioppoli inequality (Lemma 8) for $w_\varepsilon = |\nabla u_\varepsilon|^2$. Finally, we show how to obtain Caccioppoli type estimates with constants independent of ε. This allows carrying out a Moser iteration procedure and applying the Rellich–Kondrashov and Sobolev theorems to conclude the proof of Theorem 5.

LEMMA 6. *If* $u \in W^{1,n}_g(\Omega, \mathbb{R}^n)$ *solves the system*

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u^i) = \frac{1}{\varepsilon^n}u^i(1-|u|^2), \qquad i=1,\ldots,n,$$

and $g : \partial\Omega \to S^{n-1}$, *then* $|u| \le 1$ *a.e.*

Proof. For any nonnegative testing function φ which vanishes on $\partial\Omega$, we have

$$\int_\Omega |\nabla u|^{n-2}\nabla u^i \cdot \nabla(\varphi u^i)\,dx = \frac{1}{\varepsilon^n}\int_\Omega \varphi(u^i)^2(1-|u|^2)\,dx.$$

Summing this with respect to $i=1,\ldots,n$ and dropping the nonnegative term $\int_\Omega \varphi|\nabla u|^n$ leads to the inequality

$$\frac{1}{2}\int_\Omega |\nabla u|^{n-2}\nabla(|u|^2-1)\cdot\nabla\varphi\,dx \le \frac{1}{\varepsilon^n}\int_\Omega |u|^2(1-|u|^2)\varphi\,dx.$$

Set $a(x) = \frac{1}{2}|\nabla u|^{n-2}$, $b(x) = |u|^2/\varepsilon^n$, and $v = |u|^2-1$. Rewriting the last inequality, we obtain

$$\int_\Omega a(x)\nabla v\cdot\nabla\varphi\,dx + \int_\Omega b(x)v\varphi\,dx \le 0.$$

Putting now $\varphi = \min(k, v_+)$ for a fixed $k \in \mathbb{R}_+$, and defining $A_k := \{x \in \Omega \mid v_+(x) < k\}$, we obtain

$$\int_{A_k} a(x)|\nabla v_+|^2\,dx + \int_{A_k} b(x)(v_+)^2\,dx \le 0.$$

Since a and b are nonnegative, this clearly implies $v_+ = 0$ a.e. Hence, $|u|^2 \le 1$ a.e. ∎

The next lemma justifies the differentiation (in the weak sense) of both sides of the system (3) with respect to x.

LEMMA 7. *For any* $\varepsilon > 0$, *the mapping* $G_\varepsilon := |\nabla u_\varepsilon|^{(n-2)/2}\nabla u_\varepsilon$ *is of class* $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n^2})$.

The proof of this result is quite standard (one has to consider the difference quotients of G_ε, see e.g. [13] or [6]) and we omit it here.

LEMMA 8. *If* u_ε *is a weak solution to* (3), $\beta \ge 0$, $n \ge 3$, *and* $w_\varepsilon := |\nabla u_\varepsilon|^2$, *then, for any* $\zeta \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla(w_\varepsilon^{(n+\beta)/4})|^2 \zeta^2\,dx \le \frac{2(n+\beta)}{\varepsilon^n} \int_\Omega (1-|u_\varepsilon|^2) w_\varepsilon^{(\beta+2)/2}\zeta^2\,dx + 9n^2 \int_\Omega w_\varepsilon^{(n+\beta)/2} |\nabla\zeta|^2\,dx.$$

Proof. By Lemma 7, we may differentiate both sides of (3) with respect to x_j to obtain

$$-\operatorname{div}\left[|\nabla u_\varepsilon|^{n-2}\nabla\left(\frac{\partial u_\varepsilon^i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}(|\nabla u_\varepsilon|^{n-2})\nabla u_\varepsilon^i\right] = \frac{\partial f_{\varepsilon,i}}{\partial x_j}, \tag{11}$$

where $i, j = 1, \ldots, n$ and $f_{\varepsilon,i} := \varepsilon^{-n} u_\varepsilon^i (1-|u_\varepsilon|^2)$. Equation (11) means that for any testing function $\phi_{ij} \in W^{1,n}$ with compact support we have

$$\int_\Omega |\nabla u_\varepsilon|^{n-2}\nabla\left(\frac{\partial u_\varepsilon^i}{\partial x_j}\right)\cdot\nabla\phi_{ij}\,dx + \int_\Omega \frac{\partial}{\partial x_j}\left(|\nabla u_\varepsilon|^{n-2}\right)\nabla u_\varepsilon^i\cdot\nabla\phi_{ij}\,dx = \int_\Omega \frac{\partial f_{\varepsilon,i}}{\partial x_j}\phi_{ij}\,dx. \tag{12}$$

Now, pick a smooth nonnegative cutoff function $\zeta \in C_0^\infty(\Omega)$, and take $\phi_{ij} = \zeta^2 |\nabla u_\varepsilon|^\beta \partial u_\varepsilon^i/\partial x_j$, where $\beta \ge 0$, and insert this into (12). Since all the integrations on the left hand side are performed only on the set $\{|\nabla u_\varepsilon| > 0\}$, we are allowed to assume that Lemma 7 implies the existence of $D^2 u_\varepsilon \in L^n_{\mathrm{loc}}$. A simple but tedious computation leads to the following two equalities:

$$\sum_{1\le i,j\le n} \int_\Omega |\nabla u_\varepsilon|^{n-2}\nabla\left(\frac{\partial u_\varepsilon^i}{\partial x_j}\right)\cdot\nabla\phi_{ij}\,dx = \sum_{1\le i,j\le n}\int_\Omega w_\varepsilon^{(n+\beta-2)/2}\zeta^2\left|\nabla\left(\frac{\partial u_\varepsilon^i}{\partial x_j}\right)\right|^2 dx + \frac{\beta}{4}\int_\Omega w_\varepsilon^{(n+\beta-4)/2}\,|\nabla w_\varepsilon|^2\,\zeta^2\,dx + \int_\Omega w_\varepsilon^{(n+\beta-2)/2}\nabla w_\varepsilon\cdot\zeta\nabla\zeta\,dx; \tag{13}$$

$$\begin{aligned}(14)\qquad &\sum_{1\le i,j\le n}\int_\Omega \frac{\partial}{\partial x_j}\Big(|\nabla u_\varepsilon|^{n-2}\Big)\nabla u_\varepsilon^i\cdot\nabla\phi_{ij}\,dx\\ &= \frac{n-2}{4}\int_\Omega w_\varepsilon^{(n+\beta-4)/2}|\nabla w_\varepsilon|^2\zeta^2\,dx\\ &\quad+\frac{(n-2)\beta}{4}\int_\Omega w_\varepsilon^{(n+\beta-6)/2}\zeta^2\sum_{1\le i\le n}(\nabla u_\varepsilon^i\cdot\nabla w_\varepsilon)^2\,dx\\ &\quad+(n-2)\int_\Omega w_\varepsilon^{(n+\beta-4)/2}\zeta\sum_{1\le i\le n}(\nabla u_\varepsilon^i\cdot\nabla w_\varepsilon)(\nabla u_\varepsilon^i\cdot\nabla\zeta)\,dx.\end{aligned}$$

Dealing with the right hand side of (12), we obtain

$$\begin{aligned}(15)\qquad &\sum_{1\le i,j\le n}\int_\Omega \frac{\partial f_{\varepsilon,i}}{\partial x_j}\phi_{ij}\,dx\\ &= \frac{1}{\varepsilon^n}\int_\Omega (1-|u_\varepsilon|^2)w_\varepsilon^{(\beta+2)/2}\zeta^2\,dx\\ &\quad-\frac{2}{\varepsilon^n}\sum_{1\le i,j,k\le n}\int_\Omega w_\varepsilon^{\beta/2}\zeta^2u_\varepsilon^iu_\varepsilon^k\frac{\partial u_\varepsilon^i}{\partial x_j}\frac{\partial u_\varepsilon^k}{\partial x_j}\,dx\\ &\le \frac{1}{\varepsilon^n}\int_\Omega (1-|u_\varepsilon|^2)w_\varepsilon^{(\beta+2)/2}\zeta^2\,dx.\end{aligned}$$

Putting (13)–(15) together, we obtain an inequality of the form

$$I_1+I_2+I_3\le J_1+J_2+J_3, \tag{16}$$

where

$$\begin{aligned}I_1&=\frac{n+\beta-2}{4}\int_\Omega w_\varepsilon^{(n+\beta-4)/2}|\nabla w_\varepsilon|^2\zeta^2\,dx,\\ I_2&=\frac{(n-2)\beta}{4}\int_\Omega w_\varepsilon^{(n+\beta-6)/2}\zeta^2\sum_{1\le i\le n}(\nabla u_\varepsilon^i\cdot\nabla w_\varepsilon)^2\,dx,\\ I_3&=\sum_{1\le i,j\le n}\int_\Omega w_\varepsilon^{(n+\beta-2)/2}\zeta^2\left|\nabla\left(\frac{\partial u_\varepsilon^i}{\partial x_j}\right)\right|^2dx,\end{aligned}\tag{17}$$

and

$$\begin{aligned}J_1&=\frac{1}{\varepsilon^n}\int_\Omega (1-|u_\varepsilon|^2)w_\varepsilon^{(\beta+2)/2}\zeta^2\,dx,\\ J_2&=\int_\Omega w_\varepsilon^{(n+\beta-2)/2}|\nabla w_\varepsilon|\zeta|\nabla\zeta|\,dx,\\ J_3&=(n-2)\int_\Omega w_\varepsilon^{(n+\beta-4)/2}\zeta\sum_{1\le i\le n}|\nabla u_\varepsilon^i\cdot\nabla w_\varepsilon|\,|\nabla u_\varepsilon^i\cdot\nabla\zeta|\,dx.\end{aligned}\tag{18}$$

The Cauchy–Schwarz inequality readily implies that $J_3 \le (n-2)J_2$. Hence, dropping the nonnegative terms I_2 and I_3 on the left hand side of (16), we obtain

$$I_1 \le J_1 + (n-1)J_2. \tag{19}$$

Employing the inequality $ab \le \delta a^2 + \frac{1}{4\delta}b^2$, valid for positive a, b, and δ, we check that

$$(n-1)J_2 \le \frac{1}{2}I_1 + \frac{2(n-1)^2}{n+\beta-2}\int_\Omega w_\varepsilon^{(n+\beta)/2}|\nabla\zeta|^2\,dx.$$

This estimate, combined with (19), leads to

$$\int_\Omega |\nabla(w_\varepsilon^{(n+\beta)/4})|^2\zeta^2\,dx \le \frac{2(n+\beta)}{\varepsilon^n}\int_\Omega (1-|u_\varepsilon|^2)w_\varepsilon^{(\beta+2)/2}\zeta^2\,dx + 9n^2\int_\Omega w_\varepsilon^{(n+\beta)/2}|\nabla\zeta|^2\,dx. \tag{20}$$

The proof of Lemma 8 is now complete. ∎

Applying the Moser and De Giorgi–Stampacchia iteration techniques in a way mimicking the arguments of DiBenedetto and Friedman [7] one can obtain from (20) the following standard result.

COROLLARY 9. *The function* $|\nabla u_\varepsilon|$ *is of class* $L^q_{\mathrm{loc}}(\Omega)$ *for any* $q \in (1,\infty]$. *Moreover*, *if* $K \subset \Omega$ *is compact*, *then* $\|\nabla u_\varepsilon\|_q \le C$ *for some constant* C *depending on* n, q, ε, $\mathrm{dist}(K,\partial\Omega)$, *and the boundary data* g. ∎

In fact, a careful verification shows that for u_ε with $\int_\Omega |\nabla u_\varepsilon|^n\,dx \le C$ we have $|\nabla u_\varepsilon| \le C/\varepsilon$ on any compact $K \subset \Omega$, with $C = C(n,K)$ (the interested reader may consult the proof of Lemma 12 in the Appendix). As a consequence of this estimate, we obtain the following.

LEMMA 10. *As* ε_k *tends to zero*, $|u_{\varepsilon_k}| \to 1$ *uniformly on compact subsets of* Ω.

Proof. Suppose that the lemma is false. Then, for some $\eta > 0$ and some compact $K \subset \Omega$, we can find a sequence of points $(x_j)_{j=1,2,\ldots} \subset K$ and $\varepsilon_j \to 0$ such that

$$|u_{\varepsilon_j}(x_j)| \le 1-\eta.$$

Set $2\delta = \mathrm{dist}(K,\partial\Omega)$. Since $|\nabla u_{\varepsilon_j}(x)| \le C/\varepsilon_j$ for all $x \in F$, where

$$F := \{x \in \Omega \mid \mathrm{dist}(x,\partial\Omega) \ge \delta\} \supset K,$$

we have

$$|u_{\varepsilon_j}(x)| \le 1 - \eta/2 \tag{21}$$

for small ε_j and for all $x \in B_j \equiv B(x_j, \eta\varepsilon_j/(2C))$. Thus, for large j,

$$\frac{1}{\varepsilon_j^n} \int_\Omega (1 - |u_{\varepsilon_j}|^2)^2\,dx \ge \frac{1}{\varepsilon_j^n} \int_{B_j} (1 - |u_{\varepsilon_j}|)^2\,dx \ge \omega(n)(\eta/2)^{n+2}C^{-n} \not\to 0$$

as $j \to 0$. This is a contradiction to (9). ∎

3.1. *Getting rid of* ε. Our main concern is now to show that $|\nabla u_\varepsilon|$ is locally bounded in any space L^q, $1 < q < \infty$, by a constant which *does not* depend on ε. The reasoning presented below is a version of a trick which, in the case $n = 2$, is due to Bethuel, Brezis and Hélein [2, steps A3 and A4 of the proof].

Pick a cutoff function $\zeta \in C_0^\infty(\Omega)$, with $0 \le \zeta \le 1$, $\zeta \equiv 1$ on $B(a, r)$, $\zeta \equiv 0$ outside $B(a, R)$, and $|\nabla\zeta| \le 2/(R-r)$. By Lemma 10, one can assume that

$$|u_\varepsilon(x)|^2 \ge 1/2 \quad \text{for all } x \in \operatorname{supp}\zeta \text{ and all } \varepsilon \text{ sufficiently small.} \tag{22}$$

Thus, the system (3) implies

$$\frac{1 - |u_\varepsilon|^2}{\varepsilon^n} = -\sum_{1\le i\le n} \operatorname{div}(|\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon^i)\frac{u_\varepsilon^i}{|u_\varepsilon|^2}.$$

From this equality, computing the divergence and integrating by parts the term with the Laplacian Δu_ε^k, we obtain

$$\begin{aligned}
&\int_\Omega \frac{1 - |u_\varepsilon|^2}{\varepsilon^n} w_\varepsilon^{(\beta+2)/2}\zeta^2\,dx \qquad (23)\\
&\quad = -\int_\Omega w_\varepsilon^{(n+\beta)/2}\zeta^2 \sum_{1\le i\le n} \frac{u_\varepsilon^i \Delta u_\varepsilon^i}{|u_\varepsilon|^2}\,dx\\
&\qquad - \int_\Omega \sum_{1\le i\le n} \nabla(w_\varepsilon^{(n-2)/2}) \cdot \nabla u_\varepsilon^i \frac{w_\varepsilon^{(\beta+2)/2}\zeta^2 u_\varepsilon^i}{|u_\varepsilon|^2}\,dx\\
&\quad \le \underbrace{\int_\Omega \sum_{1\le i\le n} \nabla\biggl(w_\varepsilon^{(n+\beta)/2}\frac{\zeta^2 u_\varepsilon^i}{|u_\varepsilon|^2}\biggr) \cdot \nabla u_\varepsilon^i\,dx}_{K_1}\\
&\qquad + (n-2)\int_\Omega w_\varepsilon^{(n+\beta-1)/2}\zeta^2|\nabla w_\varepsilon|\,dx
\end{aligned}$$

(the last inequality follows from (22)). We decompose the integral K_1 into four terms as follows.

$$K_{11} = \sum_{1\le i\le n} \int_\Omega w_\varepsilon^{(n+\beta)/2} \frac{\zeta^2}{|u_\varepsilon|^2} \nabla u_\varepsilon^i \cdot \nabla u_\varepsilon^i \, dx,$$

$$K_{12} = 2 \sum_{1\le i\le n} \int_\Omega w_\varepsilon^{(n+\beta)/2} \frac{\zeta u_\varepsilon^i}{|u_\varepsilon|^2} (\nabla \zeta \cdot \nabla u_\varepsilon^i) \, dx,$$

$$K_{13} = \frac{n+\beta}{2} \sum_{1\le i\le n} \int_\Omega w_\varepsilon^{(n+\beta-2)/2} \frac{\zeta^2 u_\varepsilon^i}{|u_\varepsilon|^2} (\nabla w_\varepsilon \cdot \nabla u_\varepsilon^i) \, dx,$$

$$K_{14} = -2 \sum_{1\le i,j\le n} \int_\Omega w_\varepsilon^{(n+\beta)/2} \frac{\zeta^2}{|u_\varepsilon|^4} (u_\varepsilon^i \nabla u_\varepsilon^i \cdot u_\varepsilon^j \nabla u_\varepsilon^j) \, dx.$$

Obviously, $K_{14} \le 0$. Moreover, by (22) and Lemma 6, we have

$$|K_{11}| \le 2 \int_\Omega w_\varepsilon^{(n+\beta+2)/2} \zeta^2 \, dx, \tag{24}$$

$$\begin{aligned} |K_{12}| &\le 4n \int_\Omega w_\varepsilon^{(n+\beta+1)/2} \zeta |\nabla \zeta| \, dx \\ &\le 2n \int_\Omega \left(w_\varepsilon^{(n+\beta+2)/2} \zeta^2 + w_\varepsilon^{(n+\beta)/2} |\nabla \zeta|^2 \right) dx, \end{aligned} \tag{25}$$

$$|K_{13}| \le n(n+\beta) \int_\Omega w_\varepsilon^{(n+\beta-1)/2} |\nabla w_\varepsilon| \zeta^2 \, dx. \tag{26}$$

Now, combine the inequalities (23)–(26) with Lemma 8 to obtain the estimate

$$\begin{aligned} \int_\Omega |\nabla (w_\varepsilon^{(n+\beta)/4})|^2 \zeta^2 \, dx \le C \Big[&\int_\Omega w_\varepsilon^{(n+\beta+2)/2} \zeta^2 \, dx \\ &+ \int_\Omega w_\varepsilon^{(n+\beta-1)/2} |\nabla w_\varepsilon| \zeta^2 \, dx \\ &+ \int_\Omega w_\varepsilon^{(n+\beta)/2} |\nabla \zeta|^2 \, dx \Big]. \end{aligned} \tag{27}$$

Next, we use a standard trick to get rid of the integral containing $|\nabla w_\varepsilon|$ on the right hand side. To this end, check that

$$\begin{aligned} C \int_\Omega w_\varepsilon^{(n+\beta-1)/2} |\nabla w_\varepsilon| \zeta^2 \, dx &= C \int_\Omega w_\varepsilon^{(n+\beta-4)/4} |\nabla w_\varepsilon| \zeta \cdot w_\varepsilon^{(n+\beta+2)/4} \zeta \, dx \\ &\le \frac{1}{2} \int_\Omega |\nabla (w_\varepsilon^{(n+\beta)/4})|^2 \zeta^2 \, dx \\ &\quad + \frac{8C^2}{(n+\beta)^2} \int_\Omega w_\varepsilon^{(n+\beta+2)/2} \zeta^2 \, dx. \end{aligned}$$

Hence, absorbing the term with $|\nabla(w_\varepsilon^{(n+\beta)/4})|^2$ in the left hand side of (27), we obtain

$$\int_\Omega |\nabla(w_\varepsilon^{(n+\beta)/4})|^2\zeta^2\,dx \le C(n,\beta)\Big(\int_\Omega w_\varepsilon^{(n+\beta+2)/2}\zeta^2\,dx + \int_\Omega w_\varepsilon^{(n+\beta)/2}|\nabla\zeta|^2\,dx\Big). \tag{28}$$

An examination shows that one can take here e.g. $C(n,\beta) = 2\cdot 10^3 n^2(n+\beta)^2$. To deal with the first integral on the right hand side, set $p = 2n/(n+2)$ so that the Sobolev conjugate exponent of p is $p^* = np/(n-p) = 2$. Put $\alpha = (n+\beta+2)/4$. By the Sobolev imbedding theorem, we have

$$\begin{aligned}\int_\Omega w_\varepsilon^{(n+\beta+2)/2}\zeta^2\,dx &= \int_{\mathbb{R}^n} (w_\varepsilon^\alpha\zeta)^2\,dx \le C\Big(\int_{\mathbb{R}^n} |\nabla(w_\varepsilon^\alpha\zeta)|^p\,dx\Big)^{2/p}\\ &\le C(n)\alpha^2\Big(\int_{\mathbb{R}^n} w_\varepsilon^{\alpha p-p}|\nabla w_\varepsilon|^p\zeta^p\,dx\Big)^{2/p}\\ &\quad + C(n)\Big(\int_{\mathbb{R}^n} w_\varepsilon^{\alpha p}\zeta^p|\nabla\zeta|^p\,dx\Big)^{2/p}.\end{aligned}$$

In order to see here again the expressions which appear in (28), note that

$$\alpha p - p = \frac{n}{n+2}\cdot\frac{n+\beta-4}{2} + \frac{2}{n+2}\cdot\frac{n}{2},$$

and apply the Hölder inequality with exponents $(n+2)/n$ and $(n+2)/2$ to estimate the integrals on the right hand side. This gives

$$\begin{aligned}\int_\Omega w_\varepsilon^{(n+\beta+2)/2}\zeta^2\,dx &\le C(n)\Big(\int_\Omega |\nabla(w_\varepsilon^{(n+\beta)/4})|^2\zeta^2\,dx\Big)\cdot\Big(\int_{\{\zeta\ne0\}} w_\varepsilon^{n/2}\,dx\Big)^{2/n}\\ &\quad + C(n)\Big(\int_\Omega w_\varepsilon^{(n+\beta)/2}|\nabla\zeta|^2\,dx\Big)\cdot\Big(\int_{\{\zeta\ne0\}} w_\varepsilon^{n/2}\,dx\Big)^{2/n}.\end{aligned} \tag{29}$$

Set $s = (n+\beta)/2$ and define

$$\Phi(s,\varepsilon) := \int_\Omega |\nabla(w_\varepsilon^{s/2})|^2\zeta^2\,dx, \tag{30}$$

$$\Psi(s,\varepsilon) := \int_\Omega w_\varepsilon^s|\nabla\zeta|^2\,dx, \tag{31}$$

(32) $$I(\varepsilon) := \int\limits_{\{\zeta\neq 0\}} w_\varepsilon^{n/2}\,dx \equiv \int\limits_{B_R} |\nabla u_\varepsilon|^n\,dx,$$

Using this notation and combining (28) with (29), we see that

(33) $$\Phi(s,\varepsilon) \le C_0 s^2 \Phi(s,\varepsilon) I(\varepsilon)^{2/n} + C_0 s^2 (I(\varepsilon)^{2/n}+1)\Psi(s,\varepsilon).$$

Here, the constant C_0 depends only on n.

Next, we proceed to prove the following.

LEMMA 11. *For any compact set $K\subset\Omega$ and any $q\in(1,\infty)$ there exists a constant C, depending only on n, q, K, and the boundary data g, such that*

(34) $$\|w^q_{\varepsilon_k}\|_{W^{1,2}(K)} \le C \quad \textit{for } k=1,2,\ldots$$

Here, ε_k stands for the sequence selected in Proposition 3.

Proof. Fix $q\in[n/2,\infty)$. Since, by Proposition 3, the sequence ∇u_{ε_k} converges strongly in L^n, we may choose $\eta_0\equiv\eta_0(q)>0$ and $\gamma_0\equiv\gamma_0(q)>0$ such that

(35) $$I(\varepsilon_k) \equiv \int\limits_{B_R\cap\Omega} |\nabla u_{\varepsilon_k}|^n\,dx \le \min(1,(2C_0s^2)^{-n/2})$$

for all $R<\eta_0$, all $\varepsilon_k\in(0,\gamma_0)$, and all $s\in[n/2,2q)$. Assume with no loss of generality that $R=\operatorname{diam}(\operatorname{supp}\zeta)<\eta_0$. Then, by (33),

(36) $$\Phi(s,\varepsilon_k) \le C(n)s^2\Psi(s,\varepsilon_k) \quad \text{for } s\in[n/2,2q) \text{ and } \varepsilon_k\in(0,\gamma_0).$$

To start a Moser iteration, we combine (36) with the Sobolev inequality,

$$\Big(\int\limits_{B_r} |f|^{2^*}\,dx\Big)^{1/2^*} \le C(n)\Big(\int\limits_{B_r} |\nabla f|^2\,dx\Big)^{1/2} + \frac{C(n)}{r}\Big(\int\limits_{B_r}|f|^2\,dx\Big)^{1/2}.$$

Here, we set $f=w_{\varepsilon_k}^{s/2}$. Taking into account the properties of ζ, we obtain the so-called weak reverse Hölder inequality,

(37) $$\Big(\int\limits_{B_r} w_{\varepsilon_k}^{s\mu}\,dx\Big)^{1/\mu} \le C(n)s^2\Big(\frac{1}{(R-r)^2}+\frac{1}{r^2}\Big)\int\limits_{B_R} w^s_{\varepsilon_k}\,dx,$$

where $0<r<R<\eta_0$, $n/2\le s\le 2q$, and $\mu=2^*/2=n/(n-2)>1$.

Set

$$m = \Big[\frac{\log(4q)-\log n}{\log n - \log(n-2)}\Big].$$

Fix $\varrho<\eta_0/2$ and define, for $j=0,1,\ldots,m+1$,

$$\varrho_j = \varrho(1+2^{-j}), \qquad s_j = \mu^j n/2.$$

Note that $s_{m+1} \ge 2q$. Put

$$H_j := \Big(\fint_{B_{\varrho_j}} w_{\varepsilon_k}^{s_j}\, dx \Big)^{1/s_j}.$$

Applying (37) with $s = s_j$, $R = \varrho_j$, and $r = \varrho_{j+1}$, we easily obtain, for $j = 1, \dots, m$,

$$H_{j+1} \le C_1(n)^{1/s_j} \cdot s_j^{2/s_j} \cdot 4^{j/s_j} H_j \le C_2(n)^{j/\mu^j} H_j.$$

Here one can take, for instance, $C_2(n) = (C_1(n)\mu^2 n^2)^{2/n}$. By induction, since $\mu > 1$, the last inequality implies

$$H_{m+1} \le C(n) \cdot C_2(n)^{\Sigma_{i=1}^m i/\mu^i} H_0 \le C_3(n) \Big(\fint_{B_{2\varrho}} |\nabla u_{\varepsilon_k}|^n\, dx \Big)^{2/n}. \tag{38}$$

As $s_{m+1} \ge 2q$, we can combine (38) with the Hölder inequality to see that

$$\Big(\fint_{B(a,\varrho)} w_{\varepsilon_k}^{2q}\, dx \Big)^{1/(2q)} \le 2^n C_3(n) \Big(\fint_{B(a,2\varrho)} |\nabla u_{\varepsilon_k}|^n\, dx \Big)^{2/n}.$$

Hence, by a covering argument, for any compact $K \subset \Omega$ the integral $\int_K w_{\varepsilon_k}^{2q}\, dx$ is bounded by a constant which depends only on n, q, g, and K. By (30), (31), and (36), the integral $\int_K |\nabla(w_{\varepsilon_k}^q)|^2\, dx$ does not exceed a constant which may depend on n, q, g, and K, but *not* on ε_k. This completes the proof of (34).

We are now in a position to prove Theorem 5.

By the Rellich–Kondrashov theorem, $W^{1,2}$ is compactly imbedded in L^2. Therefore, we may use (34) and a standard diagonal procedure to select from $(\varepsilon_k)_{k=1,2,\dots}$ a subsequence, denoted also by ε_k, such that

$$\begin{aligned} &|\nabla u_{\varepsilon_k}|^{p/2} \to |\nabla u_*|^{p/2} && \text{strongly in } L^2(K) \text{ and a.e.,} \\ &\nabla u_{\varepsilon_k} \rightharpoonup \nabla u_* && \text{weakly in } L^p(K, \mathbb{R}^{n^2}), \end{aligned} \tag{39}$$

for any compact $K \subset \Omega$ and any p of the form $p = nk$, $k = 1, 2, \dots$

Applying Theorem 4, we see that (39) implies

$$u_{\varepsilon_k} \to u_* \quad \text{in } W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n), \text{ for } p = 2n, 3n, 4n, \dots$$

By the Sobolev imbedding theorem, we conclude that

$$\|u_{\varepsilon_k} - u_*\|_{C^\alpha_{\mathrm{loc}}} \xrightarrow{k\to\infty} 0$$

for $\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$

Summarizing, we have proved that any sequence $u_{\varepsilon_k} \to u_*$ in $W^{1,n}$ contains a subsequence converging to u_* locally in C^α (for any $\alpha < 1$). Therefore, the whole sequence u_{ε_k} constructed in Proposition 3 converges to u_* locally in C^α. The proof of Theorem 5 is now complete. ∎

Remark. The assumption that u_ε minimizes E_ε was used only to establish the convergence $u_{\varepsilon_k} \to u_*$ in $W^{1,n}$. Therefore, if we just know that

$$-\operatorname{div}(|\nabla u_{\varepsilon_k}|^{n-2}\nabla u_{\varepsilon_k}) = \frac{1}{\varepsilon_k^n} u_{\varepsilon_k}(1-|u_{\varepsilon_k}|^2)$$

and, for some reason, there exists a map $v \in W^{1,n}$ such that

$$u_{\varepsilon_k} \xrightarrow{W^{1,n}} v,$$

then $u_{\varepsilon_k} \to v$ locally in C^α for any $\alpha < 1$.

Remark. To obtain convergence of u_{ε_k} in $C^{1,\alpha}$ on compact subsets of Ω, one needs to know, among other things, that

$$\int_{B_R} \frac{1-|u_{\varepsilon_k}|^2}{\varepsilon_k^n}\, dx \le CR^{n-\delta} \tag{40}$$

for some constant C independent of ε_k and some $\delta \in [0,1)$. The trick used in [2, Lemma 2] does not seem to be useful here.

A *global* gradient bound, $|\nabla u_{\varepsilon_k}| \le C$ on $\overline{\Omega}$ (with C independent of ε), would imply (40) with $\delta = 0$. It would then be possible to prove that the family $\{\nabla u_{\varepsilon_k}\}$ is (locally) precompact in C^α for some $\alpha > 0$, using Morrey–Campanato estimates, a classical regularity result of Karen Uhlenbeck [13], and the Dirichlet growth theorem [8].

Appendix. For the convenience of the reader and to render our exposition self-contained, we prove here a (presumably well known) local boundedness result for $|\nabla u_\varepsilon|$.

LEMMA 12. *If* $u_\varepsilon \in W_g^{1,n}(\Omega,\mathbb{R}^n)$ *is a weak solution to* (3), *then for any compact* $K \subset \Omega$, *there exists a constant* $C(n,K)$ *such that*

$$|\nabla u_\varepsilon(x)|^n \le \frac{C(n,K)}{\varepsilon^n}\Big(1 + \int_\Omega |\nabla u_\varepsilon|^n\, dx\Big) \quad \textit{for a.e. } x \in K.$$

The idea of the proof is taken from Bethuel, Brezis, and Hélein [2, Lemma A.1]—scaling is the main tool. Linear elliptic estimates for the Laplace operator are replaced by Moser iteration.

Proof. Introduce an auxiliary function v defined by the formula $v(x) = u_\varepsilon(\varepsilon x)$ (so that $|\nabla v| = \varepsilon|\nabla u_\varepsilon|$) for all x in

$$G := \Omega_\varepsilon \equiv \{x \mid x = y/\varepsilon,\ y \in \Omega\}.$$

Check that

$$-\operatorname{div}(|\nabla v|^{n-2}\nabla v) = v(1-|v|^2) \quad \text{on } G.$$

Of course, $|v| \le 1$. Write $h = |\nabla v|^2$. Repeating the proof of Lemma 8, and setting $s = (n+\beta)/2$, $s \ge n/2$, leads to the inequality

$$\int_G |\nabla(h^{s/2})|^2 \zeta^2 \, dx \le Cs \int_G h^{s+1-n/2} \zeta^2 \, dx + C \int_G h^s (1 + |\nabla \zeta|^2) \, dx, \tag{41}$$

where $\zeta \in C_0^\infty(G)$ satisfies $0 \le \zeta \le 1$, $\zeta \equiv 1$ on some ball $B_r \subset G$, $\zeta \equiv 0$ on $G \setminus B_R$, $|\nabla \zeta| \le 2/(R-r)$. Set now $w = \max(1, h)$. Assume that $\frac{1}{2}R < r < R \le 1$. Since $w^s \ge w^{s+1-n/2} \ge h^{s+1-n/2}$ and $|\nabla(w^{s/2})| \le |\nabla(h^{s/2})|$, we have

$$\int_{B_r} |\nabla(w^{s/2})|^2 \, dx \le \frac{Cs}{(R-r)^2} \int_{B_R} w^s \, dx.$$

Moser iteration (as in Section 3.1) yields

$$\Big(\fint_{B_{\varrho_j}} w^{s_j} \, dx \Big)^{1/s_j} \le C(n) \Big(\fint_{B_\varrho} w^{n/2} \, dx \Big)^{2/n}, \tag{42}$$

where $s_j = (n/(n-2))^j \cdot n/2$, $\varrho_j = (\varrho/2)(1 + 2^{-j})$, and $j = 0, 1, 2, \ldots$ Upon letting $j \to \infty$, we conclude that

$$\operatorname*{ess\,max}_{x \in B_{\varrho/2}} w(x) \le C \Big(\fint_{B_\varrho} w^{n/2} \, dx \Big)^{2/n}. \tag{43}$$

Now, fix $K \subset \Omega$ and let $K_\varepsilon = \{x \mid x = y/\varepsilon,\ y \in \Omega\}$. With no loss of generality we can take ε small enough and assume that $\varrho = 1 < \operatorname{dist}(K_\varepsilon, \partial G) = \operatorname{dist}(K, \partial\Omega)/\varepsilon$. For simplicity assume also that all the balls are centered at 0. Scaling the variables back from $x \in G$ to $y = \varepsilon x \in \Omega$, we obtain

$$\operatorname*{ess\,max}_{y \in B(0,\varepsilon/2)} \varepsilon^2 |\nabla u_\varepsilon(y)|^2 \le C \Big(\fint_{B(0,\varepsilon)} (\max(1, \varepsilon^2 |\nabla u_\varepsilon|^2))^{n/2} \, dy \Big)^{2/n}.$$

Raising both sides to the power $n/2$, and applying the elementary inequality $\max(a,b)^s \le 2^s (a^s + b^s)$, we conclude that

$$\varepsilon^n |\nabla u_\varepsilon(y)|^n \le C \int_{B(0,\varepsilon)} \Big(\frac{1}{\varepsilon^n} + |\nabla u_\varepsilon|^n \Big) \, dx \le C \Big(1 + \int_\Omega |\nabla u_\varepsilon|^n \, dx \Big).$$

This completes the proof of Lemma 12.

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