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ON CONTINUOUS ACTIONS COMMUTING WITH ACTIONS OF POSITIVE ENTROPY

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Let F and G be finitely generated groups of polynomial growth with the degrees of polynomial growth $d(F)$ and $d(G)$ respectively. Let $S = \{S^f\}_{f\in F}$ be a continuous action of F on a compact metric space X with a positive topological entropy $h(S)$. Then (i) there are no expansive continuous actions of G on X commuting with S if $d(G) < d(F)$; (ii) every expansive continuous action of G on X commuting with S has positive topological entropy if $d(G) = d(F)$.

1. Introduction and results. In this note we will be concerned with actions of discrete groups by homeomorphisms of a compact metric space (X, ϱ). Such an action $T = \{T^g\}_{g\in G}$, where $g \to T^g$ is a homomorphism of a discrete group G into the group $\mathrm{Homeo}(X)$ of homeomorphisms of X, is said to be *expansive* if there exists a constant $c > 0$ (called an *expansive constant*) such that for every pair of distinct points $x, y \in X$ there exists $g \in G$ such that $\varrho(T^g x, T^g y) > c$.

The dynamics of expansive $\mathbb{Z}$ actions (= expansive homeomorphisms) has been studied in numerous papers and seems fairly well understood by now. In particular, there have been discovered certain topological obstructions to the expansiveness. For instance, R. Mañé [M] proved that expansive homeomorphisms exist only on spaces with finite topological dimension. A. Fathi [F] showed that expansive homeomorphisms with zero topological entropy exist only on zero-dimensional spaces. It is also known that there are no expansive homeomorphisms on certain manifolds: the circle S^1 (see [W]), the sphere S^2 etc. [L]. Much less is known about expansive actions of larger groups, in particular $\mathbb{Z}^n$ with $n > 1$. The above mentioned obstructions obviously fail to extend to continuous actions of $\mathbb{Z}^n$, $n > 1$. Indeed, for example the $\mathbb{Z}^2$ action generated by two commuting hyperbolic toral automorphisms is expansive, has zero entropy, but lives on a space of positive topological

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dimension. Also, consider the following $\mathbb{Z}^2$-subshift of finite type with the circle $S = \mathbb{R}/\mathbb{Z}$ as an alphabet: $X = \{\underline{x} \in S^{\mathbb{Z}^2} : 3x_{ij} + x_{i+1,j} + x_{i,j+1} = 0$ for all $(i,j) \in \mathbb{Z}^2\}$. It can be shown [S] that the natural action of $\mathbb{Z}^2$ on X by shifts is expansive, but X clearly has infinite topological dimension. It is apparently an unexplored question whether there exist expansive $\mathbb{Z}^n$ actions with $n > 1$ on S^1, S^2 and other manifolds.

Here we give some obstructions to the expansiveness of an action of a finitely generated group with an additional condition that the action commutes with another action which has positive entropy. An important role in our results is played by the rate of polynomial growth of a finitely generated group. Let us recall the definition (see e.g. [B], [G]). Let G be a discrete group generated by a finite set $\Gamma = \{\gamma_1, \ldots, \gamma_s\} \subset G$. For every $g \in G$ define $\|g\|_\Gamma = \min\{l : g = \gamma_{j_1}^{\sigma_1}\gamma_{j_2}^{\sigma_2}\ldots\gamma_{j_l}^{\sigma_l}\}$, where $\sigma_k \in \{-1,1\}$, $1 \le k \le l$, i.e. $\|g\|_\Gamma$ is the minimal length of a word expressing g as a product of the generators and their inverses. Then we define the (finite) ball of radius $m \ge 0$ to be $B_{G,\Gamma}(m) = \{g \in G : \|g\|_\Gamma \le m\}$. The group G is said to have *polynomial growth* ([B], [G]) if the value

$$d(G) = \limsup_{m\to\infty} \frac{\log|B_{G,\Gamma}(m)|}{\log m}$$

is finite. The number $d(G)$ is called the *degree of polynomial growth* of the group G and does not depend on the choice of generators. For example, it is easy to see that $d(\mathbb{Z}^s) = s$. It can be proved (see [B], [G]) that $d(G)$ is always an integer and that there exist constants $C_1, C_2 > 0$ such that

$$C_1 m^d \le |B_{G,\Gamma}(m)| \le C_2 m^d, \qquad m \ge 1. \tag{1}$$

The set of generators being fixed, we will drop the subscript Γ and write simply $B_G(m)$ for the ball in G. It is easy to see that a group G of polynomial growth is amenable, since the sequence of balls $\{B_G(m)\}_{m\ge1}$ is a Følner sequence (see e.g. [MO]). So, if $T = \{T^g\}_{g\in G}$ is a continuous action of G on a metric compactum (X, ϱ), then the topological entropy of T can be computed as follows [MO]. Given a finite set $E \subset G$ and $\varepsilon > 0$ we say that a set $A \subset X$ is (T, E, ε)-*separated* if for any $x, y \in A$ with $x \ne y$ there exists $g \in E$ such that $\varrho(T^g x, T^g y) > \varepsilon$. We denote by $Z(T, E, \varepsilon)$ the maximum cardinality of a (T, E, ε)-separated set and define

$$H(T,\varepsilon) = \limsup_{m\to\infty} |B_G(m)|^{-1} \log Z(T, B_G(m), \varepsilon).$$

The topological entropy is then defined by

$$h(T) = \lim_{\varepsilon\to 0} H(T,\varepsilon).$$

The following theorem is a strengthening of the results of [Sh].

THEOREM 1.1. *Let $S = \{S^f\}_{f\in F}$ and $T = \{T^g\}_{g\in G}$ be commuting* (*i.e.*

$S^f \circ T^g = T^g \circ S^f$ for all $f \in F$ and $g \in G$) continuous actions of finitely generated groups F and G with polynomial growth on a compact metric space (X, ϱ). Suppose, in addition, that $h(S) > 0$ and let $d(F)$, $d(G)$ stand for the degrees of polynomial growth of F and G. Then

(i) *if $d(G) < d(F)$, then the action T cannot be expansive*;
(ii) *if $d(G) = d(F)$ and T is expansive, then $h(T) > 0$.*

Here are some immediate consequences of the theorem.

COROLLARY 1.2. *Let the action $S = \{S^f\}_{f\in F}$ be as above with $h(S) > 0$ and $d(F) > 1$. Then the centralizer $C(S) = \{T \in \mathrm{Homeo}(X) : T \circ S^f = S^f \circ T$ for all $f \in F\}$ contains no expansive homeomorphisms.*

Proof. Follows directly from Theorem 1.1(i). ■

COROLLARY 1.3. *Let $S = \{S^f\}_{f\in\mathbb{Z}^s}$ be a continuous action of $\mathbb{Z}^s$ on a metric compactum (X, ϱ) and let $h(S) > 0$. Then for any subgroup H of infinite index in $\mathbb{Z}^s$ the corresponding subaction $S_H = \{S^f\}_{f\in H}$ is non-expansive.*

Proof. Notice that since H has infinite index in $\mathbb{Z}^s$, we have $d(H) < d(\mathbb{Z}^s) = s$ (cf. [G]), and apply Theorem 1.1(i) to the pair of actions S and $T = S_H$. ■

The last result can be formulated in terms of expansive directions introduced by M. Boyle and D. Lind in [BL]. Let $S = \{S^f\}_{f\in\mathbb{Z}^s}$ be a continuous action of $\mathbb{Z}^s$ on a compact metric space (X, ϱ). A q-plane $P \subset \mathbb{R}^s$ is *expansive* (for the action S) if there exist constants $C > 0$ and $\lambda > 0$ such that for any $x, y \in X$ with $x \neq y$ there exists $v \in \mathbb{Z}^s$ with $\mathrm{dist}(v, P) \le \lambda$ such that $\varrho(S^v x, S^v y) > C$ (here the group $\mathbb{Z}^s$ is meant to be imbedded in $\mathbb{R}^s$ as its integer lattice and dist stands for the Euclidean distance in $\mathbb{R}^s$). It is easy to see that if $P \cap \mathbb{Z}^s = H$, where H is a subgroup of rank $q < s$ in $\mathbb{Z}^s$, then P is expansive if and only if the subaction $S_H = \{S^f\}_{f\in H}$ is expansive in the usual sense. In [BL] the authors introduce the set of expansive q-planes $E_S(q, s) = \{P \in G(q, s) : P$ is expansive for the action $S\}$, where $G(q, s)$ is the Grassmannian manifold of q-planes in $\mathbb{R}^s$, and ask what are possible sets $E_S(q, s)$ for continuous $\mathbb{Z}^s$ actions. They show, in particular, that $E_S(q, s)$ is always open in $G(q, s)$.

COROLLARY 1.4. *If $h(S) > 0$, then $E_S(q, s) = \emptyset$ for all $q < s$.*

Proof. Let $G_{\mathrm{rat}}(q, s) \subset G(q, s)$ $(q < s)$ be the subset of "rational" q-planes $P \subset \mathbb{R}^s$, i.e. ones spanned by some integer vectors $h_1, \ldots, h_q \in \mathbb{Z}^s \subset \mathbb{R}^s$. From Corollary 1.3 it follows that $G_{\mathrm{rat}}(q, s) \cap E_S(q, s) = \emptyset$ $(q < s)$ for every continuous $\mathbb{Z}^s$ action S with $h(S) > 0$. But since $G_{\mathrm{rat}}(q, s)$ is dense in $G(q, s)$ and $E_S(q, s)$ is open [BL], it follows that $E_S(q, s) = \emptyset$. ■

2. Proof of Theorem 1.1. Let the assumption of the theorem hold and suppose, in addition, that the action T is expansive with an expansive constant $c > 0$. By a straightforward compactness argument one shows (cf. Lemma 1 of [Sh]) that for any $\delta > 0$ there exists $N \geq 1$ such that $\varrho(x,y) \geq \delta$ implies $\varrho^{(T)}_{B_G(N)}(x,y) > c$, where $\varrho^{(T)}_E(x,y)$ stands for the *Bowen metric* $\varrho^{(T)}_E(x,y) = \max\{\varrho(T^g x, T^g y) : g \in E\}$ corresponding to a finite set $E \subset G$. This also means that there exists $M \geq 1$ such that $\varrho^{(S)}_{B_F(1)}(x,y) \geq c$ implies $\varrho^{(T)}_{B_G(M)}(x,y) > c$. Denote the open Bowen ball by $O^{(T)}_E(x,\varepsilon) = \{y \in X : \varrho^{(T)}_E(x,y) < \varepsilon\}$ for a finite $E \subset G$ and $\varepsilon > 0$.

LEMMA 2.1. *There exists an integer $M \geq 1$ such that the inclusion $O^{(T)}_{B_G(M)}(x,c) \subset O^{(S)}_{B_F(1)}(x,c)$ holds for all $x \in X$.*

Using the fact that S and T commute we can "stretch" the previous result to get the following.

LEMMA 2.2. *The inclusion $O^{(T)}_{B_G(Mm)}(x,c) \subset O^{(S)}_{B_F(m)}(x,c)$ holds for all $x \in X$, $m \geq 1$.*

Proof. Let us prove the lemma by induction on m. The case $m = 1$ is established by Lemma 2.1. Suppose the inclusion holds for $m = k$, i.e. for all $x \in X$ we have

$$O^{(T)}_{B_G(Mk)}(x,c) \subset O^{(S)}_{B_F(k)}(x,c). \tag{2}$$

Let $y \in O^{(T)}_{B_G(M(k+1))}(x,c)$. Since for any positive integers r_1, r_2,

$$B_G(r_1)B_G(r_2) = B_G(r_1 + r_2), \tag{3}$$

for every $g \in B_G(Mk)$ we have $T^g y \in O^{(T)}_{B_G(M)}(T^g x, c)$ and, by Lemma 2.1, $T^g y \in O^{(S)}_{B_F(1)}(T^g x, c)$. This means $\varrho(S^f T^g x, S^f T^g y) < c$ for all $f \in B_F(1)$, $g \in B_G(Mk)$ and, since S and T commute, implies $S^f y \in O^{(T)}_{B_G(Mk)}(S^f x, c)$ for all $f \in B_F(1)$. In view of (2) we now have $S^f y \in O^{(S)}_{B_F(k)}(S^f x, c)$, $f \in B_F(1)$. This, because of (3), means $y \in O^{(S)}_{B_F(k+1)}(x,c)$. Thus, $O^{(T)}_{B_G(M(k+1))}(x,c) \subset O^{(S)}_{B_F(k+1)}(x,c)$, which completes the proof. ∎

This lemma clearly implies that $Z(T, B_G(Mm), c) \geq Z(S, B_F(m), c)$, $m \geq 1$ (for the notations see Section 1). Now we have

$$H(T,c) \geq H(S,c) \liminf_{m\to\infty} \frac{|B_F(m)|}{|B_G(Mm)|}\,. \tag{4}$$

Since $h(S) > 0$, we can assume the expansive constant $c > 0$ to be so small that $H(S,c) > 0$. In view of (1) we can see that

$$\liminf_{m\to\infty} \frac{|B_F(m)|}{|B_G(Mm)|} = \infty \quad \text{if } d(G) < d(F),$$

and (4) gives $H(T,c) = \infty$, which is well known to be impossible (see e.g. [W]). This contradiction proves part (i) of the theorem.

If $d(G) = d(F)$, then $\liminf_{m\to\infty} |B_F(m)|/|B_G(Mm)|$ equals a positive finite constant K and we conclude from (4) that $h(S) > 0$ implies $h(T) > 0$, proving (ii).

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