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NULL-FAMILIES OF SUBSETS OF MONOTONICALLY NORMAL COMPACTA

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The paper deals with compacta satisfying high separation axioms: perfect normality and monotone normality. By a result of A. J. Ostaszewski, [4, Theorem 1], each separable monotonically normal compactum is perfectly normal.

With additional set-theoretic assumptions, like most often the Continuum Hypothesis, one is able to construct a wide variety of perfectly normal compacta. Yet another space in that class is obtained in Example 1. The separable, perfectly normal, zero-dimensional and compact space X constructed there admits a (continuous) fully closed mapping f onto the Cantor set C such that $f^{-1}(t)$ consists of exactly three points for all but countably many points $t \in C$. The reader may find more information and problems concerning perfectly normal compacta and constructions of spaces in survey papers [3] and [6].

In contrast, no set-theoretic conditions are known (so far?) under which there would exist a separable monotonically normal compactum not being the continuous image of the double arrow space. Our main result implies that the space X of Example 1 is not monotonically normal. More generally, no separable space obtained by "resolving" uncountably many points of a compact space into at least three-point spaces can be monotonically normal.

Let $\mathbf{A}$ be a collection of subsets of a compact space X. We shall say that $\mathbf{A}$ is a *null-family* in X if, for each open covering $\mathbf{U}$ of X, the subcollection of all $F \in \mathbf{A}$ which are contained in no $V \in \mathbf{U}$ is finite. By the compactness of X, it is possible to show that $\mathbf{A}$ is a null-family in X if and only if for every two disjoint closed subsets G and H of X the pair of inequalities $F \cap G \neq \emptyset \neq F \cap H$ is valid for finitely many $F \in \mathbf{A}$ only.

An easy proof of the following lemma is left to the reader.

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LEMMA. *If* $\mathbf{A}$ *is a null-family of finite subsets of a compact space* X, F *is an open subset of* X *and* $x \in F$, *then the set*

$$G = \{x\} \cup \Big(F - \bigcup\{B \in \mathbf{A} : B \not\subset F\}\Big)$$

is an open subset of X.

We shall say that a continuous mapping $f : X \to Y$ of a compactum X onto a Hausdorff space Y is *fully closed* if the collection $\{f^{-1}(y) : y \in Y\}$ is a null-family in X.

Since the terminology concerning monotone normality is not fixed, we need to introduce the following definition: Let X be a T_1-space and M be an operator which assigns an open subset $M(x, U)$ of X to each ordered pair (x, U) consisting of a point $x \in X$ and its open neighbourhood U in X. We shall say that M is a *monotone normality operator* on X if

(1) $x \in M(x, U) \subset U$,
(2) if $x \in U \subset U'$ then $M(x, U) \subset M(x, U')$, and
(3) if $x \neq x'$ then $M(x, X - \{x'\}) \cap M(x', X - \{x\}) = \emptyset$.

The following theorem solves a problem of the first-named author (see [5, Problem 212]).

THEOREM 1. *Let* X *be a compact*, *separable and monotonically normal space. Suppose that* $\mathbf{A}$ *is a null-family of pairwise disjoint subsets of* X *such that* $|A| \geq 3$ *for each* $A \in \mathbf{A}$. *Then* $\mathbf{A}$ *is at most countable.*

Proof. Let M denote a monotone normality operator on X, and let S be a countable dense subset of X.

Let $A \in \mathbf{A}$. Let x_A^1, x_A^2 and x_A^3 be distinct points of A and $L_A = \{x_A^1, x_A^2, x_A^3\}$. Let $E_A^i = M(x_A^i, \{x_A^i\} \cup (X - L_A))$ for each $i \in \{1, 2, 3\}$. Then E_A^i, $i = 1, 2, 3$, are open pairwise disjoint subsets of X.

Let $i \in \{1, 2, 3\}$. Let F_A^i be an open neighbourhood of x_A^i such that $\mathrm{cl}(F_A^i) \subset E_A^i$. Let $G_A^i = \{x_A^i\} \cup (F_A^i - \bigcup\{L_B : B \in \mathbf{A} \text{ and } L_B \not\subset F_A^i\})$. By Lemma, G_A^i is an open subset of X, because $\{L_B : B \in \mathbf{A}\}$ is a null-family of finite subsets of X. Finally, let $H_A^i = M(x_A^i, G_A^i)$. Thus, $x_A^i \in H_A^i \subset G_A^i \subset F_A^i \subset \mathrm{cl}(F_A^i) \subset E_A^i$.

Since S is countable and dense, there exist $s_1 \in S$ and an uncountable subcollection $\mathbf{B}$ of $\mathbf{A}$ such that $s_1 \in H_A^1$ for each $A \in \mathbf{B}$. Similarly, there exist $s_2 \in S$ and an uncountable subcollection $\mathbf{C}$ of $\mathbf{B}$ such that $s_2 \in H_A^2$ for each $A \in \mathbf{C}$, and there exist $s_3 \in S$ and an uncountable subcollection $\mathbf{D}$ of $\mathbf{C}$ such that $s_3 \in H_A^3$ for each $A \in \mathbf{D}$.

Thus, $\mathbf{D}$ is an uncountable subfamily of $\mathbf{A}$ and $s_1, s_2, s_3 \in S$ are points such that $s_i \in H_A^i$ for each $A \in \mathbf{D}$ and $i \in \{1, 2, 3\}$. Let $B \in \mathbf{D}$. Since $\mathbf{D}$ is an infinite null-family, and the sets $\mathrm{cl}(F_B^i)$, $i = 1, 2, 3$, are pairwise disjoint, there exists $C \in \mathbf{D}$ such that C meets at most one of the sets $\mathrm{cl}(F_B^i)$. Say, $C \cap \mathrm{cl}(F_B^2) = \emptyset = C \cap \mathrm{cl}(F_B^3)$.

By the definition of the sets G_C^i, $i = 1, 2, 3$, L_B meets at most one of them. We assume that $L_B \cap G_C^3 = \emptyset$ (if $L_B \cap G_C^2 = \emptyset$, the argument is analogous with 3 replaced by 2 everywhere below). Then $x_B^3 \notin G_C^3$, and so $H_C^3 = M(x_C^3, G_C^3) \subset M(x_C^3, X - \{x_B^3\})$.

Since $C \cap G_B^3 \subset C \cap \mathrm{cl}(F_B^3) = \emptyset$, it follows that $x_C^3 \notin G_B^3$. Therefore, $H_B^3 = M(x_B^3, G_B^3) \subset M\big(x_B^3, X - \{x_C^3\}\big)$. Since M is a monotone normality operator, $M(x_B^3, X - \{x_C^3\}) \cap M(x_C^3, X - \{x_B^3\}) = \emptyset$, which implies that $H_B^3 \cap H_C^3 = \emptyset$. But $B, C \in \mathbf{D}$, and so $s_3 \in H_B^3 \cap H_C^3$, a contradiction which concludes the proof.

A fairly general method of constructing perfectly normal compacta is due to Filippov, [2]. A similar and more general method of constructing compact spaces was introduced by Fedorchuk, [1]. A nice presentation of the method can be found in [6] (see the subsections 3.1.32–3.1.37 and 3.4.1–3.4.10). The construction of Example 1, below, is using Fedorchuk's method.

Roughly speaking, in Fedorchuk's method, one starts with a compact space Z and an appropriate collection $\{Y_z : z \in Z\}$ of compact spaces. Then each point $z \in Z$ is "resolved" into a copy of Y_z. The resulting space X is compact and the natural projection $\pi : X \to Z$ is a fully closed mapping.

Recall that a subset L of a compact metric space Z is said to be a *Lusin set* in Z if L is uncountable and the intersection $L \cap A$ is a countable set, for each nowhere dense subset A of Z. It is well known that the Continuum Hypothesis implies the existence of Lusin sets.

In Filippov's method, the base space Z is an uncountable metric compactum, and the set of resolved points $L = \{z \in Z : Y_z \text{ is non-degenerate}\}$ is a Lusin set in Z, while each fiber Y_z is a metric compactum and the projection $\pi : X \to Z$ is an irreducible mapping. The obtained space X is separable because Z is separable, and non-metrizable because L is uncountable. Perfect normality of X follows from the fact that L is a Lusin set in Z (see [2, Example II] or [6, 3.3.6]). Indeed, if F is a closed subset of X, then F differs from $\pi^{-1}(\pi(F))$ on countably many fibers Y_z only, where z belongs to the nowhere dense subset $\mathrm{bd}(\pi(F))$ of Z.

Example 1. Let C denote the usual Cantor set, $C \subset [0, 1]$, $0, 1 \in C$. Let A denote the set of all points of C which are left-isolated or right-isolated in C.

If $2^{\aleph_0} = \aleph_1$, then there exists a perfectly normal, separable and zero-dimensional compactum X which admits a fully closed map f onto C such that $|f^{-1}(t)| = 3$ for each $t \in C - A$ and $|f^{-1}(t)| = 1$ for each $t \in A$.

Since $\{f^{-1}(t) : t \in C - A\}$ is a null-family of pairwise disjoint subsets of X, Theorem 1 implies that X is not monotonically normal.

We remark that this example is related to a problem of S. Watson, [6, 3.4.10].

Let $\{C_\alpha : \alpha < \omega_1\}$ be an enumeration of all closed subsets of C which have no isolated points, with $C_0 = C$, and let $\{z_\alpha : \alpha < \omega_1\}$ be an enumeration of all points of $C - A$. For each $\alpha < \omega_1$, let $F^1_\alpha = C \cap [0, z_\alpha)$ and let A_α denote the set of all points of C_α which are left-isolated or right-isolated in C_α.

Let $\alpha < \omega_1$. Let $(D_n)_{n=1}^\infty$ be a sequence of sets such that

(a) each D_n coincides with C_β for some $\beta \le \alpha$ such that $z_\alpha \in C_\beta - A_\beta$,
(b) if $\beta \le \alpha$ and $z_\alpha \in C_\beta - A_\beta$, then the set $\{n : D_n = C_\beta\}$ is infinite.

Now, it is easy to construct by induction points $s_1, s_2, \ldots, t_1, t_2, \ldots \in (z_\alpha, 1] - C$ such that $s_{n+1} < t_n < s_n$, $s_n - z_\alpha < 1/n$ and $(s_{n+1}, t_n) \cap D_n \ne \emptyset \ne (t_n, s_n) \cap D_n$ for $n = 1, 2, \ldots$ Let $F^2_\alpha = C \cap \bigcup_{n=1}^\infty (s_{n+1}, t_n)$ and $F^3_\alpha = C \cap \bigcup_{n=1}^\infty (t_n, s_n)$. It follows that

(i) $F^2_\alpha \cup F^3_\alpha = C \cap (z_\alpha, 1]$,
(ii) $\mathrm{cl}(F^2_\alpha) \cap \mathrm{cl}(F^3_\alpha) = \{z_\alpha\}$, and
(iii) if $\varepsilon > 0$, $\beta \le \alpha$ and $z_\alpha \in C_\beta - A_\beta$, then $(z_\alpha, z_\alpha + \varepsilon] \cap F^i_\alpha \cap C_\beta \ne \emptyset$ for $i = 2, 3$.

Let x^i_α be a collection of new points, where $i = 1, 2, 3$ and $\alpha < \omega_1$. Let $X = A \cup \{x^i_\alpha : i = 1, 2, 3,\ \alpha < \omega_1\}$. Define $f : X \to C$ by the rules $f(x^i_\alpha) = z_\alpha$ and $f(x) = x$ if $x \in A$. Topologize X by taking all the sets $f^{-1}(F^i_\alpha) \cup \{x^i_\alpha\}$ and all the sets $f^{-1}(U)$, where U is an open subset of C, to be a subbasis of open sets in X. By (i) and (ii), it follows that X is compact (and Hausdorff) (see [6, 3.1.33]), separable (see [6, 3.1.37]) and zero-dimensional, and f is continuous and irreducible (see [6, 3.1.35]), and fully closed.

It remains to prove that X is perfectly normal. It is enough to show that each decreasing family $\{G_\alpha : \alpha < \omega_1\}$ of closed subsets of X is eventually constant. In fact, observe that X has $2^{\aleph_0}$ closed subsets and each closed subset of X is the intersection of all its closed-open neighbourhoods. Suppose that H is a closed set in X and let $\{H_\alpha : \alpha < \omega_1\}$ be the collection of all closed-open sets which contain H. Let $G_\alpha = \bigcap_{\beta \le \alpha} H_\beta$ for each α. Then $\{G_\alpha : \alpha < \omega_1\}$ is a decreasing collection of closed subsets of X and $H = \bigcap_{\beta < \omega_1} G_\beta$. If there exists α such that $G_\beta = G_\alpha$ when $\alpha \le \beta < \omega_1$, then $H = \bigcap_{\beta \le \alpha} G_\beta = \bigcap_{\beta \le \alpha} H_\beta$, and so H is a G_δ-set in X.

Suppose that G_α, $\alpha < \omega_1$, are closed subsets of X and $G_\beta \supset G_\alpha$ if $\beta \le \alpha$. Let $G = \bigcap_{\alpha < \omega_1} G_\alpha$. Clearly, $\{f(G_\alpha) : \alpha < \omega_1\}$ is a decreasing collection of closed subsets of C. Since C is compact and metric, there exists $\gamma_0 < \omega_1$ such that $f(G_\alpha) = f(G_{\gamma_0})$ for each $\alpha \geq \gamma_0$. Let $P = f(G_{\gamma_0})$. Then $f(G_\alpha) = P$ and $G_\alpha \subset f^{-1}(P)$ for each $\alpha \geq \gamma_0$. Also, $f(G) = P$. We are going to prove that the set $f^{-1}(P) - G$ is countable.

If P is countable then $f^{-1}(P)$ is also countable. Suppose that P is uncountable. Let Q denote the unique closed subset of P such that Q has no isolated points and $P - Q$ is countable. Then there is $\alpha_0 < \omega_1$ such that $Q = C_{\alpha_0}$. If $\alpha \geq \alpha_0$ and $z_\alpha \in C_{\alpha_0} - A_{\alpha_0}$, then the property (iii) of the sets F^i_α implies that $x^1_\alpha, x^2_\alpha, x^3_\alpha \in H$ for each closed subset H of X such that $Q \subset f(H)$. Therefore, $f^{-1}(P) - G$ is contained in the countable set $f^{-1}((P-Q) \cup \{z_\beta : \beta < \alpha_0\})$. Hence, there exists γ_1 such that $\gamma_0 \leq \gamma_1 < \omega_1$ and $G_\alpha = G_{\gamma_1}$ for each $\alpha \geq \gamma_1$. This concludes the proof of perfect normality of X.

The following remark gives some extra information about X: Let $\mathbf{B}$ denote the collection of all two-point sets each of which consists of the end-points of a component of $[0,1]-C$. Clearly, $\mathbf{B}$ is a null-family in C. However, the collection of two-point sets $\{f^{-1}(G) : G \in \mathbf{B}\}$ is not a null-family in X.

EXAMPLE 2. Let Y denote the disjoint union of two points, $[0,1]$ and the double arrow space. Then Y is a monotonically normal compactum which admits a mapping h onto $[0,1]$ such that $|h^{-1}(t)| = 3$ for each $t \in [0,1]$. Obviously, h is not a fully closed map. It is rather easy to modify the construction and get a zero-dimensional space Z which has all the properties of Y which are listed here.

PROBLEM 1. Suppose that X is a separable monotonically normal compactum which admits a fully closed map f into $[0,1]$ such that $|f^{-1}(t)| \leq 2$ for each $t \in [0,1]$. Does it follow that X is a continuous image of the double arrow space?

PROBLEM 2. Does each monotonically normal compactum admit a fully closed map into a metric space? What happens in the cases when the compactum is also separable? zero-dimensional? both?

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