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MENGER CURVES IN PEANO CONTINUA

BY

P. KRUPSKI (WROCŁAW) AND H. PATKOWSKA (WARSZAWA)

1. Preliminaries. By a continuum we mean a metric compact connected space. A curve is a one-dimensional continuum. We denote by M_1^3 the Menger universal curve. It is topologically characterized as a Peano curve with no local separating points and no nonempty open planar subsets [1, 2] (this and other facts about M_1^3 as well as related notions can be found in [8]).

A metric space (X,ϱ) has the *disjoint arcs property* (*DAP*) if any two paths in X can be approximated by disjoint paths, i.e., if for each $\varepsilon > 0$ and for any two continuous maps $f,g : I = [0,1] \to X$ there exist continuous maps $f',g' : I \to X$ such that $f'(I) \cap g'(I) = \emptyset$ and $\widetilde{\varrho}(f,f') < \varepsilon$, $\widetilde{\varrho}(g,g') < \varepsilon$, where $\widetilde{\varrho}$ denotes the sup-norm metric induced by ϱ.

There is another characterization of M_1^3 as a Peano curve with the DAP [3].

Let ∂A denote the set of end-points of an arc A. An arc A in a Peano continuum X is said to be *approximately non-locally-separating* if for any region (i.e., open connected set) $V \subset X$ such that $V \cap A = A \setminus \partial A$ there exists an arc B such that $V \cap B = B \setminus \partial B$, $\partial A = \partial B$ and $V \setminus B$ is connected. An arc with end-points a and b ordered from a to b will be denoted by ab.

Recall that if X is a Peano continuum with no local separating points and no nonempty open planar subsets, then each open nonempty subset of X contains a complete five-point graph [8, Corollary 3.9.2].

The hyperspace of all subcontinua of a continuum X with the Hausdorff metric is denoted by $C(X)$. We shall consider the subspace $\mathbf{M} \subset C(X)$ consisting of all topological copies of M_1^3 in X.

If $\mathcal{U}$ is a collection of sets, then $\mathrm{St}(A,\mathcal{U}) = \{U \in \mathcal{U} : U \cap A \neq \emptyset\}$, $\mathcal{U}^*$ denotes the union of $\mathcal{U}$, and $\mathrm{St}^*(A,\mathcal{U}) = (\mathrm{St}(A,\mathcal{U}))^*$.

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Results

THEOREM 1. *Let X be a Peano continuum. Then the following conditions are equivalent.*

(i) *X has no local separating points and no open nonempty subset of X is planar*;

(ii) *Any curve in X is contained in some $M \in \mathbf{M}$*;

(iii) *$\mathbf{M}$ is dense in $C(X)$*;

(iv) *X has the DAP*;

(v) *Any arc in X has empty interior in X and is approximately non-locally-separating.*

Proof. (i)⇒(ii). Let $C \subset X$ be a curve. We are going to construct inductively the following three sequences: $\{C_n\}_{n=1}^{\infty}$ of curves in X, $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of finite families of regions in X and $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers such that

(1_n) $C_1 \supset C$, $C_n \supset C_{n-1}$ for $n > 1$, $\mathcal{U}_n$ is an irreducible covering of C_n and if $U, U' \in \mathcal{U}_n$ and $U \cap U' \neq \emptyset$, then some arc in $C_n \cap (U \cup U')$ intersects both $U \setminus U'$ and $U' \setminus U$,

(2_n) $C_n \cap U$ contains a nonplanar graph for any $U \in \mathcal{U}_n$,

(3_n) $\operatorname{mesh} \mathcal{U}_n < \varepsilon_n < 1/n$ and the family of closures of elements of $\mathcal{U}_n$ is of order 2,

(4_n) $\mathcal{U}_n$ is a star closure refinement of $\mathcal{U}_{n-1}$ for $n > 1$,

(5_n) if $V, V' \in \mathcal{U}_n$, $U \in \mathcal{U}_{n-1}$, $n > 1$, and $\operatorname{cl}(V \cup V') \subset U$, then there is a chain $\mathcal{A}$ in $\mathcal{U}_n$ joining V to V' such that $\operatorname{cl} \mathcal{A}^* \subset U$,

(6_n) if $V_0, V_1, V_2 \in \mathcal{U}_n$, $V_i \cap C_{n-1} \neq \emptyset$ for $i = 0, 1, 2$, $\operatorname{dist}(V_0, V_1) > \varepsilon_{n-1}$, $\operatorname{dist}(V_0, V_2) > \varepsilon_{n-1}$ and $V_0 \cup V_1 \cup V_2 \subset U$, where $U \in \mathcal{U}_i$, $i < n-1$, then there is a chain $\mathcal{A}$ in $\mathcal{U}_n$ joining V_1 to V_2 and such that $\operatorname{cl} \mathcal{A}^* \subset \operatorname{St}^*(U, \mathcal{U}_i) \setminus \operatorname{cl} V_0$.

Let $\varepsilon_1 = 1$ and let $\mathcal{U}_1$ be an irreducible covering of C by regions in X with $\operatorname{mesh} \mathcal{U}_1 < 1$ such that the family of closures of elements of $\mathcal{U}_1$ is of order 2. For any $U \in \mathcal{U}_1$ find a complete five-point graph $K_U \subset U$ and an arc $J_U \subset U$ joining K_U to a point of $U \cap C$. Also, for any $U, U' \in \mathcal{U}_1$ such that $U \cap U' \neq \emptyset$ choose an arc $J_{UU'} \subset U \cup U'$ joining $U \setminus U'$ to $U' \setminus U$ and an arc $L_U \subset U$ intersecting both C and $J_{UU'}$. Define C_1 as the union of C and of all the arcs J_U, $J_{UU'}$, L_U and graphs K_U.

Assume now that C_i, $\mathcal{U}_i = \{U_1^i, \ldots, U_{k_i}^i\}$ and ε_i have been constructed for $i \le n$. In order to find ε_{n+1} and $\mathcal{U}_{n+1}$ choose first a family $\mathcal{D}_n = \{D_1^n, \ldots, D_{k_n}^n\}$ of closed subsets of C_n such that $D_j^n \subset U_j^n$ and $\mathcal{D}_n^* = C_n$. Let

$$F_j^i = \operatorname{St}^*(U_j^i, \mathcal{D}_n) \quad \text{and} \quad \mathcal{F}_i = \{F_1^i, \ldots, F_{k_i}^i\} \quad \text{for } i \le n.$$

Observe that

$$C_n \cap U^i_j \subset F^i_j \subset C_n \cap \mathcal{U}^*_n \cap \mathrm{St}^*(U^i_j, \mathcal{U}_i). \tag{1}$$

Next, we construct, for $i \le n$, a family $\mathcal{G}_i = \{G^i_1, \dots, G^i_{k_i}\}$ of regions in X such that

$$F^i_j \subset G^i_j \subset \mathcal{U}^*_n \cap \mathrm{St}^*(U^i_j, \mathcal{U}_i). \tag{2}$$

Define $E^i_j = \mathrm{St}^*(U^i_j, \mathcal{U}_n)$. The set E^i_j is an open subset of X satisfying (2) with G^i_j replaced by E^i_j. The sets E^n_j are also connected, so we put $G^n_j = E^n_j$. However, E^i_j need not be connected for $i < n$. Therefore, to obtain a region $G^i_j \supset E^i_j$ satisfying (2) observe that

(3) if $V, V' \in \mathrm{St}(U^i_j, \mathcal{U}_n)$, $i < n$, then there is a chain $\mathcal{A}$ in $\mathcal{U}_n$ joining V to V' such that $\mathrm{cl}\,\mathcal{A}^* \subset \mathrm{St}^*(U^i_j, \mathcal{U}_i)$.

Indeed, because of (4_n), $(4_{n-1}), \dots, (4_{i+1})$, there are $V_n = V, V_{n-1}, \dots, V_i$ and $V'_n = V', V'_{n-1}, \dots, V'_i$ such that

$$V_{k-1}, V'_{k-1} \in \mathcal{U}_{k-1}, \quad \mathrm{cl}\,\mathrm{St}^*(V_k, \mathcal{U}_k) \subset V_{k-1} \quad \text{and} \quad \mathrm{cl}\,\mathrm{St}^*(V'_k, \mathcal{U}_k) \subset V'_{k-1} \quad \text{for } i+1 \le k \le n.$$

Since $V \cap U^i_j \ne \emptyset \ne V' \cap U^i_j$, we have $V_i, V'_i \in \mathrm{St}(U^i_j, \mathcal{U}_i)$. By (1_i) and (4_{i+1}), there exist $W, W' \in \mathcal{U}_{i+1}$ such that $W \subset V_i \cap U^i_j$ and $W' \subset V'_i \cap U^i_j$. It follows from (5_{i+1}) that there is a chain $\mathcal{A}_{i+1}$ in $\mathcal{U}_{i+1}$ from V_{i+1} to V'_{i+1} (through W and W') such that $\mathrm{cl}\,\mathcal{A}^*_{i+1} \subset \mathrm{St}^*(U^i_j, \mathcal{U}_i)$. Observe further that $V_{i+1}, V'_{i+1} \in \mathrm{St}(U^i_j, \mathcal{U}_{i+1})$. Again, using conditions (1_{i+1}), (4_{i+2}) and (5_{i+2}) we get a chain $\mathcal{A}_{i+2}$ in $\mathcal{U}_{i+2}$ from V_{i+2} to V'_{i+2} (through some elements of $\mathcal{U}_{i+2}$ lying in the intersections of elements of $\mathcal{A}_{i+1}$) such that $\mathrm{cl}\,\mathcal{A}^*_{i+2} \subset \mathrm{St}^*(U^i_j, \mathcal{U}_i)$. Proceeding that way we finally get the required chain $\mathcal{A}$ in $\mathcal{U}_n$, so that (3) is satisfied.

The existence of G^i_j immediately follows from (3).

Let $0 < \varepsilon'_n < \varepsilon_n$. For each $x \in F^i_j$ there exists a neighborhood V_x of x in G^i_j of diameter less than $(\varepsilon_n - \varepsilon'_n)/2$ such that if $y, z \in F^i_j$ and $\varrho(x, y) \ge \varepsilon'_n$, $\varrho(x, z) \ge \varepsilon'_n$, then there is an arc $J = yz \subset G^i_j \setminus \mathrm{cl}\, V_x$.

In fact, cover the compact set $\{p \in F^i_j : \varrho(p, x) \ge \varepsilon'_n\}$ by a finite number of regions whose closures are contained in $G^i_j \setminus \{x\}$. Since x does not separate G^i_j, one can join these regions by arcs in $G^i_j \setminus \{x\}$ and then find a suitable V_x.

Let $\lambda^i_j > 0$ be a Lebesgue number for the covering $\{\, V_x : x \in F^i_j \,\}$ of F^i_j. Let $\mathcal{W}$ be an open covering of C_n which is a star closure refinement of $\mathcal{U}_n$ and each element of which intersects at most two elements of $\mathcal{U}_n$; denote by $\lambda > 0$ its Lebesgue number. Put

$$\varepsilon_{n+1} = \min\left(\frac{1}{n+1}, \frac{\varepsilon_n - \varepsilon'_n}{2}, \lambda, \lambda^i_j : i = 1, \dots, n,\ j = 1, \dots, k_i\right).$$

Find an irreducible covering $\mathcal{V}$ of C_n by regions in X such that the family of closures of elements of $\mathcal{V}$ is of order 2 and $\operatorname{mesh}\mathcal{V} < \varepsilon_{n+1}$. Consider $V_0, V_1, V_2 \in \mathcal{V}$ such that

$$\text{(4)} \qquad \operatorname{dist}(V_0, V_1) > \varepsilon_n, \quad \operatorname{dist}(V_0, V_2) > \varepsilon_n \quad \text{and} \quad V_0 \cup V_1 \cup V_2 \subset U^i_j, \quad \text{where } i < n.$$

Since $\operatorname{mesh}\mathcal{V} < \lambda^i_j$, there is a $V_x \in \{V_p : p \in F^i_j\}$ such that $V_x \supset V_0$. Let $y \in C_n \cap V_1 \subset F^i_j$ and $z \in C_n \cap V_2 \subset F^i_j$ (see (1)). Then $\varrho(x, y) > \varepsilon_n - 2\operatorname{mesh}\mathcal{V} \ge \varepsilon'_n$ and $\varrho(x, z) \ge \varepsilon'_n$, so there is an arc

$$J_{V_0V_1V_2} = yz \subset G^i_j \setminus \operatorname{cl} V_x \subset \mathcal{U}^*_n \cap \operatorname{St}^*(U^i_j, \mathcal{U}_i) \setminus \operatorname{cl} V_0 \qquad \text{(cf. (2))}.$$

Define a curve $C'_n \subset \mathcal{U}^*_n$ as the union of C_n and the arcs $J_{V_0V_1V_2}$ for all triples (V_0, V_1, V_2) satisfying (4). Now, one can easily construct an irreducible covering $\mathcal{V}'$ of C'_n by regions in X whose closures form an order two family such that

- $\operatorname{mesh}\mathcal{V}' < \varepsilon_{n+1}$, $\operatorname{cl}(\mathcal{V}')^* \subset \mathcal{U}^*_n$,
- each element of $\mathcal{V}$ is contained in exactly one element of $\mathcal{V}'$,
- elements of $\mathcal{V}'$ not containing elements of $\mathcal{V}$ are disjoint from C_n,
- for any $V_0, V_1, V_2 \in \mathcal{V}'$ intersecting C_n with $\operatorname{dist}(V_0, V_1) > \varepsilon_n$, $\operatorname{dist}(V_0, V_2) > \varepsilon_n$ and $V_0 \cup V_1 \cup V_2 \subset U^i_j$, $i < n$, there is a chain $\mathcal{A}'$ in $\mathcal{V}'$ from V_1 to V_2 such that $\operatorname{cl}(\mathcal{A}')^* \subset \mathcal{U}^*_n \cap \operatorname{St}^*(U^i_j, \mathcal{U}_i) \setminus \operatorname{cl} V_0$.

Next, we define a new curve C''_n by adding some arcs to C'_n. Namely, for any $V, V' \in \mathcal{V}'$ contained in $U \in \mathcal{U}_n$ choose an arc $A_{VV'} \subset U$ joining $C'_n \cap V$ to $C'_n \cap V'$. Then C''_n is the union of C'_n and all such arcs $A_{VV'}$.

Now, one can construct (similarly to $\mathcal{V}'$) an irreducible covering $\mathcal{U}_{n+1}$ of C''_n by regions in X such that

- the family of closures of elements of $\mathcal{U}_n$ is of order 2,
- $\operatorname{mesh}\mathcal{U}_{n+1} < \varepsilon_{n+1}$, $\operatorname{cl}\mathcal{U}^*_{n+1} \subset \mathcal{U}^*_n$,
- each element of $\mathcal{V}'$ is contained in exactly one element of $\mathcal{U}_{n+1}$,
- elements of $\mathcal{U}_{n+1}$ not containing elements of $\mathcal{V}'$ are disjoint from C'_n.

It is easily seen that $\mathcal{U}_{n+1}$ satisfies conditions (3_{n+1})–(6_{n+1}); in particular, (6_{n+1}) follows from the properties of $\mathcal{V}'$, because elements of $\mathcal{U}_{n+1}$ intersecting C_n contain elements of $\mathcal{V}$.

Finally, the desired curve $C_{n+1} \subset \mathcal{U}^*_{n+1}$ can be constructed (similarly to the case $n = 1$) by adding to C''_n complete five-point graphs and some arcs to fulfil all the conditions (1_{n+1})–(6_{n+1}).

Define

$$M = \bigcap_{n=1}^{\infty} \mathcal{U}^*_n = \bigcap_{n=1}^{\infty} \operatorname{cl}\mathcal{U}^*_n = \operatorname{cl}\Big(\bigcup_{n=1}^{\infty} C_n\Big).$$

It is clear that M is a curve (by (3_n) and (4_n)) and M has no nonempty open planar subsets (by (2_n)). Conditions (1_n) and (5_n), for $n = 1, 2, \ldots$, imply that M is locally connected (cf. (3)). To see that M has no local separating points, suppose $x, y, z \in U \in \mathcal{U}_i$ are distinct points of M. Let

$$y = \lim y_n, \quad z = \lim z_n, \quad \text{where } y_n, z_n \in C_n.$$

It follows from (1_n), (6_n) and the local connectedness of M that there exists a continuum

$$F \subset (M \setminus \{x\}) \cap \mathrm{St}^*(U, \mathcal{U}_i)$$

containing y and z. Such an F can be constructed as the union of three continua joining, respectively, y to y_n, y_n to z_n and z_n to z, for sufficiently great n. Thus, no point $x \in M$ locally separates M. Consequently, $M \in \mathbf{M}$ and $C \subset M$ as required.

(ii)⇒(iii). Any subcontinuum K of X can be approximated by a connected finite union of arcs in X. To see this, consider an arbitrary finite irreducible cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of K by regions in X. Choose a point $x_i \in U_i$ for $i = 1, \ldots, n$. For each pair (i, j) such that $U_i \cap U_j \neq \emptyset$ there is an arc $x_i x_j \subset U_i \cup U_j$. If $A_{\mathcal{U}}$ is the union of such arcs and $\operatorname{mesh} \mathcal{U} \to 0$, then $\operatorname{dist}(A_{\mathcal{U}}, K) \to 0$. Now, $A_{\mathcal{U}}$ is contained in some $M_{\mathcal{U}} \in \mathbf{M}$. It easily follows from the properties of M_1^3 (it is a fractal!) that $M_{\mathcal{U}}$ can be chosen so that $M_{\mathcal{U}} \subset \mathcal{U}^*$. Thus, $\operatorname{dist}(K, M_{\mathcal{U}}) \to 0$.

(iii)⇒(v). Clearly, any arc in X has empty interior. Assume $A = ab \subset X$ is an arc and $V \subset X$ is a region such that $V \cap A = A \setminus \{a, b\}$. It is easy to find an $M \in \mathbf{M}$ and two disjoint arcs aa' and bb' such that

$$(aa' \cup bb' \cup M) \setminus \{a, b\} \subset V$$

and

$$aa' \cap M = \{a'\}, \qquad bb' \cap M = \{b'\}.$$

Indeed, let c, d be two different points of $A \setminus \{a, b\}$ such that $c \in ad \subset A$. There are two regions C, D containing the points c, d, respectively, such that $\operatorname{cl} C \cap \operatorname{cl} D = \emptyset$ and $\operatorname{cl} C \cup \operatorname{cl} D \subset V$. It follows from (3) that there exists an $M \in \mathbf{M}$ so close to the arc $cd \subset A$ that $M \subset V$, $M \cap C \neq \emptyset \neq M \cap D$, $ac \cap M \subset C$ and $bd \cap M \subset D$. If $ac \cap M \neq \emptyset$, then let a' be the first point of the arc ac (in its order from a to c) that belongs to M. Similarly define $b' \in bd$ in case $bd \cap M \neq \emptyset$. If $ac \cap M = \emptyset$ ($bd \cap M = \emptyset$), then take an arc $ca' \subset C$ ($db' \subset D$) such that $ca' \cap M = \{a'\}$ ($db' \cap M = \{b'\}$). Thus the required arc $aa' \subset ac \cup ca'$ ($bb' \subset bd \cup db'$) exists.

There are a simple closed curve $S \subset M \setminus \{a', b'\}$ (contained in the "irrational" part of M) and an uncountable family $\{L_t\}_{t \in T}$ of arcs in M with end-points a', b' such that $L_t \cap L_{t'} = \{a', b'\}$ for $t \neq t'$ and with the one-point intersection $L_t \cap S$ for each $t \in T$. Then there is a $t_0 \in T$ such that $V \setminus (aa' \cup L_{t_0} \cup bb')$ is connected. In fact, suppose $V \setminus (aa' \cup L_t \cup bb')$ is not connected for all $t \in T$. There is a component C_t of $V \setminus (aa' \cup L_t \cup bb')$ disjoint from the connected set

$$(S \setminus L_t) \cup \bigcup_{t' \neq t} \{aa' \cup L_{t'} \cup bb'\} \setminus \{a, b\}.$$

Observe that since X is locally connected, each component C_t is an open subset of X and $C_t \cap C_{t'} = \emptyset$ for $t \neq t'$. This is impossible in a separable space.

Thus the arc $B = aa' \cup L_{t_0} \cup bb'$ satisfies

$$V \cap B = B \setminus \partial B, \quad \partial A = \partial B \quad \text{and} \quad V \setminus B \text{ is connected.}$$

(v)⇒(i). Suppose a point $p \in U \subset X$ separates a region U. Let C, D be two different components of $U \setminus \{p\}$ and let cd be an arc in U from a point $c \in C$ to some $d \in D$. We have $p \in cd$. Take a region V such that $p \in V \subset \operatorname{cl} V \subset U \setminus \{c, d\}$. Then there is a subarc $ab \subset cd$ such that

$$a \in \operatorname{bd} V \cap C, \quad b \in \operatorname{bd} V \cap D \quad \text{and} \quad p \in ab \setminus \{a, b\} \subset V.$$

By (v), there exists an arc $B \subset \operatorname{cl} V$ such that

$$\partial B = \{a, b\}, \quad B \setminus \partial B \subset V \quad \text{and} \quad V \setminus B \text{ is connected nonempty.}$$

Observe that $p \in B$. Since C is open in X, we have $C \cap (V \setminus B) \neq \emptyset$; otherwise $C \subset (X \setminus V) \cup B$ and $\operatorname{int} B \neq \emptyset$, which contradicts (v). Similarly, $D \cap (V \setminus B) \neq \emptyset$, hence $C \cup (V \setminus B) \cup D$ is a connected subset of U omitting p, a contradiction. Thus X is a Peano continuum with no local separating points.

Suppose X contains an open nonempty planar subset U. Then either U contains a disk or U is one-dimensional. In the latter case it is well known that U contains an open nonempty subset homeomorphic to an open subset of the Sierpiński universal planar curve (see, e.g., [6, Lemma 1.1]). In both cases U contains an arc which is not approximately non-locally-separating.

(ii)⇒(iv). Assume two mappings $f, g : I \to X$ are given. One can easily approximate f and g by f' and g' such that $f'(I)$ and $g'(I)$ are connected finite unions of arcs. If the images $f'(I)$ and $g'(I)$ intersect, their union, by (ii), embeds in some $M \in \mathbf{M}$ and we use the DAP for M [3] to get mappings $f'', g'' : I \to M$ that approximate f' and g' and have disjoint images.

(iv)⇒(i). Suppose a point p separates a region $U \subset X$ and let C, D be two distinct components of $U \setminus \{p\}$. Choose points $c \in C$ and $d \in D$ and join them by an arc $cd \subset U$ parametrized by a homeomorphism $f : I \to cd$. Since p belongs to each continuum in U that meets both C and D which are open subsets of X, it is impossible to approximate f, arbitrarily closely, by two mappings with disjoint images. Thus, X is a Peano continuum without local separating points.

An argument that X contains no open nonempty planar subsets is similar to that of the proof of (v)⇒(i) (both a planar disk and an open nonempty subset homeomorphic to an open subset of the Sierpiński curve exclude the DAP). ∎

There is yet another property of Peano continua, the so-called *cross-connectedness*, which is equivalent to (i) and studied in [2] and [8, 3.11–3.13]. The equivalence of conditions (i) and (iv) can also be derived from that property.

Implicitly contained in [3] is the fact that an LC^{n-1} compactum X has the disjoint n-disks property (DD^nP) if and only if any continuous mapping from an arbitrary at most n-dimensional compactum into X can be approximated by embeddings (see [4, p. 40]). It follows (similarly to the proof of (ii)⇒(iii)) that for any LC^{n-1} compactum X satisfying the DD^nP the space of all topological copies of the universal n-dimensional Menger compactum is dense in $C(X)$. For $n = 1$ this gives the implication (iv)⇒(iii). Yet, Theorem 1 does not require such an elaborate theory; in its proof we only use the classical Anderson characterization of M_1^3 and standard point-set topology methods.

THEOREM 2. *If X is a homogeneous Peano continuum*, *then X is not an n-manifold for $n \le 2$ if and only if X satisfies either of the conditions* (i)–(v).

Proof. Assume X is not an n-manifold, $n \le 2$. The easiest condition to show is (i). To this end, suppose X contains a local separating point. Then each point of X has this property and it follows from [9, (9.2), p. 61] that all points of X are of order two, so X is a simple closed curve [7, p. 294], contrary to the assumption on X. Hence, X is a Peano continuum without local separating points and we can further argue as in the proofs of (v)⇒(i) and (iv)⇒(i) of Theorem 1. If X contains a planar open disk which is open in X, then X is a 2-manifold; if X contains an open nonempty subset homeomorphic to an open subset of the Sierpiński curve, then X cannot be homogeneous. So, condition (i) is satisfied.

The converse implication is clear. ∎

Theorem 2 is particularly welcome if $\dim X = 2$, when it contributes to understanding homogeneous 2-dimensional Peano continua. Higher-dimensional cases were known to be local Cantor manifolds; in such spaces arcs cannot separate regions (hence, arcs are approximately non-locally-separating) and the DAP holds [5].

As another consequence we get the following topological characterization of M_1^3.

THEOREM 3. *A Peano curve X is homeomorphic to M_1^3 if and only if each arc in X is approximately non-locally-separating and has empty interior in X.*

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MATHEMATICAL INSTITUTE
WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND
E-mail: KRUPSKI@MATH.UNI.WROC.PL

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
BANACHA 2
02-097 WARSZAWA, POLAND

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