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CYCLIC APPROXIMATION OF ANALYTIC COCYCLES OVER IRRATIONAL ROTATIONS

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Introduction. We identify the space $C(\mathbb{T})$ of real-valued continuous functions on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the 1-periodic continuous mappings $\mathbb{R} \to \mathbb{R}$. It was shown in [I] that if an irrational number α admits a sufficiently good approximation by rationals then for every $r = 1, 2, \ldots, \infty$ and "most" functions $f \in C^r(\mathbb{T})$, referred to as *cocycles* (more precisely, smooth cocycles of topological degree zero), the *Anzai* [A] *skew product*

$$T_f(x, y) = (x + \alpha, y + f(x)) \bmod 1$$

defined on the 2-torus $\mathbb{T}^2$ admits a good cyclic approximation by periodic transformations without having purely discrete spectrum. In fact, the cocycle is weakly mixing, which means that the only eigenfunctions are of the form

$$h(x, y) = Ce^{2\pi inx}.$$

A similar result was earlier obtained for $C(\mathbb{T})$ [IS].

In the present note we study other classes of smooth cocycles. Instead of approximating by piecewise polynomial functions [I], we use trigonometric polynomials, which seem to be a simpler and more powerful tool. Not only do we recover the results of [I], but we also generalize them to new spaces of cocycles—such as real-analytic or entire functions.

We consider rather general subspaces E of $C^1(\mathbb{T})$. It is proved that for a residual subset of functions $f \in E$, the skew product T_f admits a cyclic approximation with speed $o(\varepsilon(n)/n)$ as soon as α admits a rational approximation with some speed related to $\varepsilon(n)$ (see Theorem 1 for details). The result does not depend on the choice of the space E. The method is based on [I] with some ideas going back to [R]. Theorem 2 shows that in some E's, such as certain subspaces of 1-periodic real-analytic functions

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(or even those extending to entire functions $\mathbb{C} \to \mathbb{C}$), the weakly mixing cocycles form a dense G_δ set. As in [I], the proof is based on Katok's criterion [K], Theorem 12.7. By intersecting the residual sets of Theorems 1 and 2 we get a "large" set of exponentially approximated weakly mixing analytic cocycles (Corollary). In particular, the corresponding Anzai skew products are rank-1, rigid, and have partly continuous spectrum (examples of analytic rank-1 Anzai skew products with partly continuous spectrum have also been obtained by a different method in [KLR], Prop. 3). Finally, we give a simple construction which, for any α with unbounded partial quotients in its continued fraction expansion, produces a weakly mixing smooth cocycle with multiplicity greater than one. The cocycle is analytic for α sufficiently well approximated by rationals.

Throughout, we use notation of [I], where the reader is also referred for some details of the proofs. For other necessary definitions see [CFS].

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1. Cyclic approximation. We fix an irrational number α and a sequence of rationals $p_n/q_n \to \alpha$ with q_n positive and p_n, q_n relatively prime. For a fixed n, if $f : \mathbb{R} \to \mathbb{R}$ and $k = 1, 2, \ldots$, we use the notation

$$f^{(k)}(x) = f(x) + f(x + p_n/q_n) + \ldots + f(x + (k-1)p_n/q_n).$$

Consider an additive subgroup E of $C^1(\mathbb{T})$ endowed with its own topology, stronger than the C^1-convergence, and such that

(1) E is a complete metric group,
(2) E contains the constants with natural topology,
(3) E has a dense subset of trigonometric polynomials.

The following result extends Theorem 1 in [I].

THEOREM 1. *Suppose* $|\alpha - p_n/q_n| = o(\varepsilon(q_n t_n)/q_n^2)$, *where* $\varepsilon(n)$ *is a nonincreasing sequence of positive numbers and* $t_n \to \infty$. *Then the set of cocycles* $f \in E$ *such that* T_f *admits cyclic approximation with speed* $o(\varepsilon(n)/n)$ *is residual in* E.

Proof. We may choose a sequence of positive integers $s_n \to \infty$ such that

$$|\alpha - p_n/q_n| = o(\varepsilon(s_n q_n)/(s_n q_n^2)).$$

The following observation will be useful. If

$$P(x) = \sum_{k=-r}^{r} a_k \exp(2\pi i k x)$$

is a trigonometric polynomial then for every $q_n > r$ we have $P^{(q_n)} = q_n a_0$. Indeed,

$$P^{(q_n)}(x) = \sum_{j=0}^{q_n-1} \sum_{k=-r}^{r} a_k e^{2\pi i k(x+jp_n/q_n)}$$
$$= \sum_{k=-r}^{r} a_k e^{2\pi i k x} \sum_{j=0}^{q_n-1} e^{2\pi i k j p_n/q_n} = q_n a_0,$$

since the inner sum vanishes for $k \neq 0$.

Consequently, there is a constant $0 \le c_n < 1/q_n$ such that $(P+c_n)^{(q_n)} = 1/s_n \bmod 1$. We denote by $E(n)$ the set of all trigonometric polynomials Q in E satisfying the identity $Q^{(q_n)} = 1/s_n \bmod 1$. By (2) and (3) the union $\bigcup_{n\ge N} E(n)$ is dense in E for all $N \ge 1$. We choose a sequence of positive numbers ϱ_n such that if $\operatorname{dist}(f_n, f) < \varrho_n$ in E then

$$\|f_n - f\| = o(\varepsilon(s_n q_n)/(s_n^2 q_n)),$$

where $\| \ \|$ is the uniform norm. Let $E(n)^{\varrho_n}$ denote the ϱ_n-neighborhood of $E(n)$ in E. The union $\bigcup_{n\ge N} E(n)^{\varrho_n}$ is open and dense, so by (1) the intersection

$$\widetilde{E} = \bigcap_N \bigcup_{n\ge N} E(n)^{\varrho_n}$$

is a dense G_δ set.

We have to prove that for every $f \in \widetilde{E}$ the skew product T_f admits cyclic approximation with required speed. By passing to a subsequence we may assume that there is a sequence of trigonometric polynomials $f_n \in E_n$ such that $\operatorname{dist}(f_n, f) < \varrho_n$.

The rest of the proof is as in [I]. We sketch it briefly. Define $T_n(x,y) = (x + p_n/q_n, y + f_n(x))$,

$$C_0 = [0, 1/q_n) \times [0, 1/s_n),$$

and $C_j = T_n^j C_0$ for $j = 1, \ldots, Q_n - 1$, where $Q_n = s_n q_n$. Since $f_n^{(q_n)}(x) = 1/s_n \bmod 1$, we have

$$C_{iq_n} = [0, 1/q_n) \times [i/s_n, (i+1)/s_n)$$

for $i = 0, 1, \ldots, s_n - 1$. Since $T_n C_{Q_n-1} = C_0$, we obtain a cycle of length Q_n. It is clear that $\xi_n = \{C_0, \ldots, C_{Q_n-1}\}$ is a partition. To prove $\xi_n \to \varepsilon$ we use the fact that $E \subset C^1(\mathbb{T})$ and repeat the argument in [I] based on [CFS], 16.3, Lemma 2. The approximation error $\Delta = \Delta_1 + \Delta_2$ is also estimated as in [I] with $\Delta_1 \le 2q_n|\alpha - p_n/q_n| = o(\varepsilon(Q_n)/Q_n)$ by the assumption on α and $\Delta_2 = o(\varepsilon(s_n q_n)/(s_n^2 q_n)) s_n = o(\varepsilon(Q_n)/Q_n)$ by the choice of ϱ_n. ∎

2. Weakly mixing analytic cocycles. We will show that, at least for some spaces E satisfying the conditions (1)–(3) and for α sufficiently well approximated by rationals, most cocycles in E are weakly mixing. As in [I], the proof is based on the following criterion due to Katok [K], Theorem 12.7 (a proof can also be found in [KLR], Theorem 4):

Suppose $\sum |na_n| < \infty$ *and* $a_{-n} = \overline{a}_n$. *If* $|\alpha - p_n/q_n|q_n = o(|a_{q_n}|)$ *and* $\inf_n(|a_{q_n}|/\sum_{k\geq 1}|a_{kq_n}|) > 0$ *then the cocycle* $f(x) = \sum a_n \exp(2\pi inx)$ *is weakly mixing.*

Obvious examples of spaces satisfying (1)–(3) are $E = C^r(\mathbb{T})$ studied in [I]. Now we consider another family of E's defined in terms of Fourier coefficients.

Fix a sequence $\lambda(0), \lambda(1), \ldots$ of nonnegative numbers such that $\lambda(n) > 0$ infinitely often, $\lambda(0) > 0$, and $\sum n\lambda(n) < \infty$. If $\overline{a}_{-n} = a_n = o(\lambda(n))$ then clearly $\sum a_n \exp(2\pi inx)$ is in $C^1(\mathbb{T})$. We define

$$E^\lambda = \Big\{f \in C^1(\mathbb{T}) : f(x) = \sum a_n e^{2\pi inx},\ \overline{a}_{-n} = a_n = o(\lambda(n))\Big\},$$

where the summation is over all $n \in \mathbb{Z}$ such that $\lambda(|n|) > 0$. Endowed with the norm

$$\|f\|_\lambda = \sup_{n\geq 0} |a_n/\lambda(n)|,$$

it becomes a Banach space isometrically isomorphic with $c_0(\mathbb{Z})$. The identity imbedding $E \to C^1(\mathbb{T})$ is continuous; the condition (1) is clear, (2) is trivially satisfied, and (3) is easy to verify.

It should be noted that the functions in E^λ are real-analytic iff

$$\limsup \sqrt[n]{\lambda(n)} < \infty$$

and they extend to entire functions on the complex plane iff $\sqrt[n]{\lambda(n)} \to 0$.

THEOREM 2. *Let* λ *be as above and suppose* $|\alpha - p_n/q_n| = o(\lambda(q_n)/q_n)$. *Then the weakly mixing cocycles form a dense* G_δ *set in* E^λ.

Proof. The proof is a modification of [I], Theorem 2. It suffices to produce at least one weakly mixing cocycle in E^λ. By passing to a subsequence we may assume $|\alpha - p_n/q_n| = \varepsilon_n^2\lambda(q_n)/q_n$, where $\varepsilon_n \to 0$ and $\varepsilon_{n+1}\lambda(q_{n+1}) \leq \varepsilon_n\lambda(q_n)/2$. Now let $a_{q_n} = \varepsilon_n\lambda(q_n)$ and

$$f(x) = \sum_{n=1}^{\infty} a_{q_n} \cos 2\pi q_n x.$$

Clearly $f \in E^\lambda$ and it is easy to check that f satisfies Katok's criterion. ∎

Let λ be as above and $q_n \to \infty$. The set A of irrational numbers α which admit a rational approximation specified in the assumption of Theorem 2 (along a subsequence) is residual in $\mathbb{R}$. We may apply Theorems 1 and 2 to

E^λ, where $\lambda(n) = 1/n^{n+1}$, $\varepsilon(n) = e^{-n}$, $t_n = [\log q_n]$ $(n \geq 1)$, and $\alpha \in A$. By intersecting two dense G_δ subsets of E^λ we obtain the following corollary.

Corollary. *For any α from a residual set A, there exists a weakly mixing cocycle f extending to an entire function $\mathbb{C} \to \mathbb{C}$ such that $T_f : \mathbb{T}^2 \to \mathbb{T}^2$ admits a cyclic approximation with exponential speed. In particular*, T_f *is rank*-1, *rigid*, *and has a partly continuous singular simple spectrum of Hausdorff dimension* 0. ∎

3. Cocycles with multiplicity. Although our generic construction produces skew products with a good cyclic approximation, hence of simple spectrum, there also exist ergodic real-analytic cocycles (of topological degree zero) with infinite maximal spectral multiplicity [KLR]. Moreover, there exist ergodic real-analytic cocycles with mutiplicity 2 over any rotation from a residual set of irrational numbers. (This follows from a modification of [BL]: it suffices to choose $v(x) = -x$ as an automorphism of $\mathbb{T}$ and proceed along the lines of the argument in [BL] with $n = 1$ if n denotes a dimension and $n = 2$ whenever n occurs as the order of the automorphism; Corollary 5 remains valid in the 2-torus with maximal spectral multiplicity equal to 2.) In both [KLR] and [BL], a measurable cocycle is constructed and is subsequently shown to be cohomologous to a real-analytic function via an "almost analytic cocycle construction procedure". Below, in a more direct way, and for every α with unbounded partial quotients, we construct a weakly mixing smooth cocycle f with multiplicity greater than or equal to 2. The cocycle will be at least C^1, with more regularity (including analyticity) for more special α's, and with rigid T_f.

Example. Let $\alpha, \lambda(n), \varepsilon_n$ be as in the proof of Theorem 2 and let $\delta_n > 0$ with $\sum(\delta_n q_n/\varepsilon_n)^2 < \infty$. The set of numbers β such that $\|q_n\beta - 1/2\| < \delta_n$ for infinitely many n's is residual in $\mathbb{R}$ (here $\|\ \|$ denotes the distance from the nearest integer). We choose one such β and fix a subsequence q_{n_k} with $\|q_{n_k}\beta - 1/2\| < \delta_{n_k}$. As before, we write $q_{n_k} = q_n$ and define a weakly mixing $f \in E^\lambda$ as before. Note that for any α with unbounded partial quotients a suitable sequence $\lambda(n)$ can be found according to a subsequence of p_n/q_n, so $f \in C^1(\mathbb{T})$. Now let

$$b_{q_n} = a_{q_n} \frac{e^{2\pi i q_n \beta} + 1}{e^{2\pi i q_n \alpha} - 1}, \qquad b_{-q_n} = \overline{b}_{q_n},$$

with $b_k = 0$ if k is not one of the numbers $\pm q_n$. We have $\sum |b_{q_n}|^2 < \infty$ since $\sum(\varepsilon_n \lambda(q_n)\|q_n\beta - 1/2\|/\|q_n\alpha\|)^2 < \sum(\delta_n q_n/\varepsilon_n)^2 < \infty$. Therefore the real-valued function

$$g(x) = \frac{1}{2}\sum_{k\in\mathbb{Z}} b_k e^{2\pi i k x}$$

belongs to $L^2(\mathbb{T})$ and it is easy to verify that $g(x+\alpha)-g(x) = f(x+\beta)+f(x)$ a.e. on $\mathbb{T}$. It is now clear that the measure-preserving transformation

$$S(x,y) = (x+\beta, -y+g(x)) \bmod 1$$

commutes with T_f and conjugates the invariant subspaces H_N and H_{-N}, where $H_N = \{h(x)e^{2\pi iNy} : h \in L^2(\mathbb{T})\}$ for $N \in \mathbb{Z}$. This implies that the multiplicity of T_f on $L^2(\mathbb{T}^2)$ is greater than one. The rigidity of T_f along q_n follows from the uniform convergence of the sum $\sum_{j=0}^{q_n-1} f(x+j\alpha)$ to 0, which is a well-known consequence of $\int_0^1 f(x)\,dx = 0$ for $f \in C^1(\mathbb{T})$.

We also note that if α admits approximation with odd q_n's then the argument can be shortened by simply letting $\beta = 1/2$ and $S(x,y) = (x + 1/2, -y)$.

It would be interesting to find all the possible spectral multiplicities for (smooth, analytic, etc.) ergodic Anzai skew products on the 2-torus.

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