C O L L O Q U I U M   M A T H E M A T I C U M

VOL. LXX | 1996 | FASC. 1

CHARACTERIZATION OF THE BOUNDEDNESS FOR A FAMILY OF COMMUTATORS ON L^p

BY

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1. Introduction. Let (X,d,μ) be a space of homogeneous type. Let T_K be a singular integral operator which is bounded on $L^2(X)$ (see [2] for definitions and characterization: $T(1)$-theorem). Let $f\in L^2(X)$. We use M_f to denote the multiplication operator on function spaces on X. Then the commutator of M_f and T_K is defined as $C_f=[M_f,T_K]=M_fT_K-T_KM_f$.

The characterization of f such that C_f is bounded or compact on $L^p(X)$ or belongs to the trace ideal space for some singular integral operators has received considerable attention. When X is $\mathbb{R}^n$, and $T_K=R_j=(-\Delta)^{-1/2}\partial/\partial x_j$ $(j=1,\ldots,n)$ are the Riesz transforms, it was proved by Coifman, Rochberg and Weiss [5] that $[M_f,R_j]$ is bounded on $L^p(\mathbb{R}^n)$ for all $1\le j\le n$ for some $1<p<\infty$ if and only if $f\in \mathrm{BMO}(\mathbb{R}^n)$; and by Uchiyama [18] that C_f is compact on $L^p(\mathbb{R}^n)$ for all $1\le j\le n$ and some $1<p<\infty$ if and only if $f\in \mathrm{VMO}(\mathbb{R}^n)$. The characterization of f such that C_f belongs to the trace ideal space S_p was given by Janson and Wolff [9] and more general results were proved by Rochberg and Semmes [15] and Janson and Peetre [8] (see also the references therein). When X is a space of homogeneous type, it was proved by Krantz and the author [11] that if $f\in\mathrm{BMO}(X)$, then C_f is bounded on $L^p(X)$ for all $1<p<\infty$. If $f\in\mathrm{VMO}(X)$, then C_f is bounded on $L^p(X)$ for all $1<p<\infty$. In [4], Coifman, Lions, Meyer and Semmes proved that the above theorem of Coifman, Rochberg and Weiss is equivalent to the statement that $\{f_jR_jg_j+g_jR_jf_j : f_j\in L^p(\mathbb{R}^n),\ g_j\in L^{p'}(\mathbb{R}^n)\}$ is a subspace of $H^1(\mathbb{R}^n)$ and is dense in weak topology, which was called *compensated compactness* for H^1. Moreover, this fact gives a decomposition theorem for $H^1(\mathbb{R}^n)$. Furthermore, many interesting examples were given in [4] which connect the compensated compactness of H^1 and quantities in PDEs, such as Dir-Curl lemma, etc. In [20], Z. Wu studied a Clifford algebra of functions on $\mathbb{R}^n$ and produced a class of singular integral operators T_j (some

1991 *Mathematics Subject Classification*: Primary 42B20.

combinations of Riesz transforms) which can be used to characterize f such that $[M_f, T_j]$ are bounded on $L^p(\mathbb{R}^n)$.

The main purpose of the present paper is to characterize the boundedness of $[M_f, K_j]$ on $L^p(\mathbb{R}^n)$ for the family of operators K_j $(j = 1, \ldots, m)$ introduced by Uchiyama [19]. As a consequence, we generalize the mentioned results of [5] and [20].

Let $\theta_1(\xi), \ldots, \theta_m(\xi) \in C^\infty(S^{n-1})$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Let

$$K_j f(x) = (\theta_j(\xi/|\xi|)\hat{f}(\xi))^\vee(x), \qquad j = 1, \ldots, m, \tag{1.1}$$

where $\hat{f}$ denotes the Fourier transform of f while $\check{f}$ denotes the inverse Fourier transform of f.

If $\theta_j(\xi/|\xi|) = i\xi_j/|\xi|$, then $K_j = R_j$. According to [19], there exist a number $a_j = a(\theta_j) \in \mathbb{C}$ and function $\Omega_j \in C^\infty(S^{n-1})$ such that

$$\int_{S^{n-1}} \Omega_j(x)\, d\sigma(x) = 0$$

and

$$K_j f(x) = a_j f(x) + \mathrm{P.V.} \int_{\mathbb{R}^n} \Omega_j((x-y)/|x-y|)|x-y|^{-n} f(y)\, dy.$$

So K_j is a family of singular integrals which are bounded on $L^p(\mathbb{R}^n)$.

In [19], Uchiyama proved that $K_1, \ldots, K_m$ characterize $H^1(\mathbb{R}^n)$ if and only if

$$\operatorname{rank}\begin{pmatrix} \theta_1(\xi) & \ldots & \theta_m(\xi) \\ \theta_1(-\xi) & \ldots & \theta_m(-\xi) \end{pmatrix} = 2. \tag{1.2}$$

If one considers $\theta_0 = 1$ and $\theta_j(\xi) = i\xi_j/|\xi|$, then $K_j = R_j$. The result of Fefferman and Stein [7] and Stein and Weiss [17] uses $\{I, R_j : j = 1, \ldots, n\}$ to characterize $H^1(\mathbb{R}^n)$, which is a special family of operators given by (1.1) and satisfying (1.2).

From the results of [5] and [11], we know that if $b \in \mathrm{BMO}(\mathbb{R}^n)$ then $[M_b, K_j]$ are bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. We shall show that the converse is also true. It is easy to see that if (1.2) holds, then the function

$$\Theta(\xi) = \sum_{j=1}^m |\theta_j(\xi) - \theta_j(-\xi)|^2$$

nowhere vanishes on S^{n-1}. By compactness of S^{n-1}, and the smoothness of $\Theta(\xi)$ on S^{n-1}, we know that $\Theta(\xi)$ has a positive lower bound. More generally, we shall consider functions $\theta_1, \ldots, \theta_m \in C^\infty(S^{n-1})$ such that there is a continuous map $\psi : S^{n-1} \to S^{n-1}$ and $\delta_0 > 0$ such that

$$\sum_{j=1}^{m} |\theta_j(x) - \theta_j(\psi(x))|^2 \geq \delta_0, \quad x \in S^{n-1}. \tag{1.3}$$

We prove the following theorems, where p' is the conjugate exponent of p, i.e $1/p + 1/p' = 1$ for $1 < p < \infty$.

THEOREM 1.1. *Suppose that* (1.3) *holds. Let* $1 < p < \infty$ *and* $f \in L^2(\mathbb{R}^n)$. *Then the following statements are equivalent.*

(i) $f \in \mathrm{BMO}(\mathbb{R}^n)$;
(ii) $[M_f, K_j]$ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $j = 1, \dots, m$;
(iii) $[M_f, K_j]$ *is bounded on* $L^q(\mathbb{R}^n)$ *for all* $1 \leq j \leq m$ *and all* q *with* $1 < q < \infty$;
(iv) $\sum_{j=1}^{m} \widetilde{K}_j[M_f, K_j] - [M_f, K_j]\widetilde{K}_j$ *is bounded on* $L^p(\mathbb{R}^n)$,

where $\widetilde{K}_j(f)(x) = (\overline{\theta}_j(\xi/|\xi|)\hat{f}(\xi))\check{}(x)$, $j = 1, \dots, m$.

THEOREM 1.2. *If* (1.3) *holds, then* $f \in H^1(\mathbb{R}^n)$ *if and only if there are a sequence* $\{\lambda_j\}$ *of numbers, and sequences* $\{f_k\}$ *of functions in* $L^p(\mathbb{R}^n)$ *and* $\{g_k\}$ *of functions in* $L^{p'}(\mathbb{R}^n)$ *such that* $\|g_k\|_{p'}\|f_k\|_p = C_p > 0$ *for all* k, $\sum_{k=1}^{\infty} |\lambda_k| \approx \|f\|_{H^1}$, *and*

$$f = \sum_{k=1}^{\infty} \lambda_k \sum_{j=1}^{m} [K_j(f_k)\overline{\widetilde{K}_j^*(g_k)} + \widetilde{K}_j(f_k)\overline{K_j^*(g_k)} - f_k\overline{(K_j^*\widetilde{K}_j^*)(g_k)} - (K_j\widetilde{K}_j f_k)\overline{g}_k].$$

THEOREM 1.3. *If* (1.3) *holds, then* $f \in H^1(\mathbb{R}^n)$ *if and only if there are a sequence* $\{\lambda_k\}$ *of numbers, and sequences* $\{f_{j,k}\}$ *of functions in* $L^p(\mathbb{R}^n)$ *and* $\{g_{j,k}\}$ *of functions in* $L^{p'}(\mathbb{R}^n)$ *such that* $\|g_{j,k}\|_{p'}\|f_{j,k}\|_p = C_p$ *and*

$$f = \sum_{k=1}^{\infty} \lambda_k \sum_{j=1}^{m} (f_{j,k}\overline{K_j(g_{j,k})} - \overline{g}_{j,k}K_j^*(f_{j,k})), \qquad \sum_{k=1}^{\infty} |\lambda_k| \approx \|f\|_{H^1}.$$

The paper is organized as follows. In Section 2, we prove Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are given in Section 3. In Section 4, we give some application of the above theorems. As a special case of Theorem 1.2, we obtain the main theorem of [20].

The author would like to thank Steven Krantz and Richard Rochberg for some useful conversations he has had during the preparation of this work.

2. Proof of Theorem 1.1. To prove Theorem 1.1, we first collect some results from Janson and Peetre [8] and C. Li [13] (a similar idea of the proof was used by Wu [20]). Let $\theta_j \in C^\infty(S^{n-1})$ and let K_j be given by

(1.1) for $j = 1, \ldots, m$. Then we have the following identity due to Janson and Peetre [8]:

$$([M_b, K_j]f)^\wedge(\xi) = \int\limits_{\mathbb{R}^n} \hat{b}(\xi - y)(\theta_j(\xi/|\xi|) - \theta_j(y/|y|))\hat{f}(y)\,dy. \tag{2.1}$$

Let

$$L_1(\xi, \eta) = \sum_{j=1}^{m} \overline{\theta}_j(\eta/|\eta|)(\theta_j(\xi/|\xi|) - \theta_j(\eta/|\eta|)). \tag{2.2}$$

Then it is easy to verify that

$$\int\limits_{\mathbb{R}^n} \hat{b}(\xi - y)L_1(\xi, y)\hat{f}(y)\,dy = \sum_{j=1}^{m}([M_b, K_j]\widetilde{K}_j(f))^\wedge(\xi).$$

It is obvious that L_1 is homogeneous of degree 0. In other words, L_1 satisfies Assumption A_0 in [8].

For convenience, we recall Theorem 10.1 of [8] or Theorem C of [20] proved by C. Li in [13], which we shall use later. First we need to introduce the following function space of Schur multipliers. Let U and V be two subsets of $\mathbb{R}^n$. Let $M(U \times V)$ denote the set of *Schur multipliers* on $U \times V$ consisting of all functions $\phi \in L^\infty(U \times V)$ that admit a representation

$$\phi(\xi, \eta) = \int\limits_{Y} \alpha(\xi, x)\beta(\eta, x)\,d\mu(x) \tag{2.3}$$

for some σ-finite measure space (Y, μ) and measurable functions α on $U \times Y$ and β on $V \times Y$, with the norm

$$\|\phi\|_{M(U\times V)} = \inf\Big\{ \int\limits_{Y} \|\alpha(\cdot, x)\|_{L^\infty(U)}\|\beta(\cdot, x)\|_{L^\infty(V)}\,d\mu(x)\Big\},$$

where the infimum is taken over all α and β such that (2.3) holds. We know (see [8]) that $M(U \times V)$ is a Banach algebra.

Let b be a complex-valued function in $\mathbb{R}^n$. The *paracommutator* with symbol b and kernel $A(\xi, \eta)$ is the operator $T_b(A)$ defined by the following bilinear form on $C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$:

$$\langle T_b(A)f, g\rangle = \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \hat{f}(\eta)\hat{b}(\xi - \eta)A(\xi, \eta)\hat{g}(-\xi)\,d\eta\,d\xi.$$

Then we have the following theorem.

THEOREM 2.1. *If the kernel function A satisfies the following conditions*:

A0: $A(r\xi, r\eta) = A(\xi, \eta)$ *for all* $r \neq 0$ *and* $\xi, \eta \in \mathbb{R}^n$;

A1: $A \in M(\mathbb{R}^n \times \mathbb{R}^n)$;

A3: $A(\xi, \xi) = 0$, *and there are* $\gamma, \delta > 0$ *such that* $\|A\|_{M(B\times B)} \leq C(r/|\xi_0|)^\gamma$ *for* $B = B(\xi_0, r) = \{\xi : |\xi - \xi_0| < r\}$ *and* $0 < r < \delta|\xi_0|$;

A5: *For any* $\xi_0 \neq 0$ *there exist* $\delta > 0$ *and* $\eta_0 \in \mathbb{R}^n$ *such that* $\|1/A\|_{M(U\times V)} \le C$, *where* $U = \{\xi : |\,\xi/|\xi| - \xi_0/|\xi_0|\,| < \delta,\ |\xi| > |\xi_0|\}$ *and* $V = B(\eta_0, \delta|\xi_0|)$.

Then

$$\|b\|_{\mathrm{BMO}} \le C\|T_b(A)\|_{(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))}.$$

Theorem 2.1 is due to Janson and Peetre [8] for the case $p = 2$; for the general case $1 < p < \infty$ it was given by C. Li [13]. It is also stated in [20].

We shall prove the following proposition.

PROPOSITION 2.2. $L_1(\xi,\eta)$ *defined by* (2.2) *belongs to* $M(\mathbb{R}^n\times\mathbb{R}^n)$, *i.e.*, L_1 *satisfies* A1.

Proof. Since

$$L_1(\xi,\eta) = \sum_{j=1}^m \theta_j(\xi/|\xi|)\overline{\theta}_j(\eta/|\eta|) - \sum_{j=1}^m \theta_j(\eta/|\eta|)\overline{\theta}_j(\eta/|\eta|),$$

it is obvious that $L_1(\xi,\eta)$ admits a representation (2.3) with $d\mu$ being the Dirac mass concentrated at $x = 0$ and $Y = [-1,1]$. Moreover, we have

$$\|L_1\|_{M(\mathbb{R}^n\times\mathbb{R}^n)} \le \Big\|\sum_{j=1}^m \theta_j(\xi/|\xi|)\overline{\theta}_j(\eta/|\eta|)\Big\|_{M(\mathbb{R}^n\times\mathbb{R}^n)} + \Big\|\sum_{j=1}^m |\theta_j|^2\Big\|_{M(\mathbb{R}^n\times\mathbb{R}^n)}$$
$$\le 2\sum_{j=1}^m \|\theta_j\|_\infty^2.$$

This completes the proof of the proposition. ■

PROPOSITION 2.3. *There exists* $\delta > 0$ *such that if* $B_0 = B(\xi_0, r)$ *and* $r/|\xi_0| < \delta$, *then* L_1 *satisfies* A3 *and*

$$\|L_1\|_{M(B_0\times B_0)} \le Cr/|\xi_0|.$$

Proof. Since $\theta_j \in C^1(S^{n-1})$, we have

$$\sum_{j=1}^m |\theta_j(\xi) - \theta_j(\xi_0)| \le C_n \sum_{j=1}^m \|\theta_j\|_{C^1(S^{n-1})}|\xi - \xi_0|$$

for all $\xi, \xi_0 \in S^{n-1}$.

Now we choose $0 < \delta < 1/2$. For any $r > 0$, we consider $\xi_0 \in \mathbb{R}^n$ so that $|\xi_0|\delta > r$. We claim that

$$\left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| \le 4\frac{r}{|\xi_0|}$$

for all $\xi \in B_0$. In fact,

$$\left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| = \frac{|\,\xi|\xi_0| - \xi_0|\xi|\,|}{|\xi|\cdot|\xi_0|} = \frac{|(\xi-\xi_0)|\xi_0| + \xi_0(|\xi| - |\xi_0|)|}{|\xi|\cdot|\xi_0|}$$
$$\le \frac{|\xi-\xi_0|\cdot|\xi_0| + |\xi_0|\cdot|\xi-\xi_0|}{|\xi|\cdot|\xi_0|} \le \frac{2r|\xi_0|}{(|\xi_0|-r)|\xi_0|} \le \frac{4r}{|\xi_0|},$$

so the claim is proved. Now we have

$$\|L_1\|_{M(B_0\times B_0)} \le \Big\|\sum_{j=1}^m (\theta_j(\xi/|\xi|) - \theta_j(\xi_0/|\xi_0|))\overline{\theta}_j(\eta/|\eta|)\Big\|_{M(B_0\times B_0)}$$
$$+ \Big\|\sum_{j=1}^m (\theta_j(\eta/|\eta|) - \theta_j(\xi_0/|\xi_0|))\overline{\theta}_j(\eta/|\eta|)\Big\|_{M(B_0\times B_0)}$$
$$\le 2C_n \sum_{j=1}^m \|\theta_j\|_{C^1(S^{n-1})}(4r/|\xi_0|) = Cr/|\xi_0|.$$

This completes the proof of the proposition. ∎

Now if we let

$$L_2(\xi,\eta) = \sum_{j=1}^m [\overline{\theta}_j(\xi/|\xi|)(\theta_j(\xi/|\xi|) - \theta_j(\eta/|\eta|))]$$

then

$$\int_{\mathbb{R}^n} \hat{b}(\xi - y)L_2(\xi,y)\hat{f}(y)\,dy = \sum_{j=1}^m (\widetilde{K}_j[M_b,K_j](f))^\wedge(\xi).$$

It is clear that L_2 is homogeneous of degree zero. With the same arguments as above, we find that the conclusions of Propositions 2.2 and 2.3 hold for $L_2(\xi,\eta)$. Now we let

$$L(\xi,\eta) = L_2(\xi,\eta) - L_1(\xi,\eta).$$

Then

$$L(\xi,\eta) = \sum_{j=1}^m |\theta_j(\xi/|\xi|) - \theta_j(\eta/|\eta|)|^2. \tag{2.4}$$

Thus L is a homogeneous kernel of degree 0 and Propositions 2.2 and 2.3 hold for L.

The main lemma of this section is:

LEMMA 2.4. *If* θ_j $(j = 1,\ldots,m)$ *satisfy* (1.3), *then* $L(\xi,\eta)$ *defined above satisfies* A5.

P r o o f. For each $\xi_0 \in \mathbb{R}^n \setminus \{0\}$, since θ_j satisfy (1.3), there is $\delta = \delta(\xi_0) \ll 1$ such that

$$\sum_{j=1}^{m} |\theta_j(\xi/|\xi|) - \theta_j(\psi(\xi_0/|\xi_0|))|^2 \ge \delta_0/2.$$

Since the map ψ involved in (1.3) is continuous, we may choose η_0 with norm large enough such that $\eta_0/|\eta_0| = \psi(\xi_0/|\xi_0|)$, and

$$\sum_{j=1}^{m} |\theta_j(\xi/|\xi|) - \theta_j(\eta/|\eta|)|^2 \ge \delta_0/4 \tag{2.5}$$

for all $\xi \in U$ and $\eta \in V = B(\eta_0, \delta|\xi_0|)$. Thus $1/L$ is bounded by $4/\delta_0$ on $U \times V$.

Next we show that

$$\|1/L\|_{M(U\times V)} \le C\delta_0. \tag{2.6}$$

Since

$$\begin{aligned}\frac{1}{L(\xi,\eta)} &= \frac{1}{L(\xi,\eta_0) + L(\xi,\eta) - L(\xi,\eta_0)}\\ &= \frac{1}{L(\xi,\eta_0)} \cdot \frac{1}{1 + L(\xi,\eta_0)^{-1}(L(\xi,\eta) - L(\xi,\eta_0))}\\ &= \frac{1}{L(\xi,\eta_0)} \sum_{k=0}^{\infty} \left(\frac{L(\xi,\eta) - L(\xi,\eta_0)}{L(\xi,\eta_0)}\right)^k\\ &= \sum_{k=0}^{\infty} \frac{(L(\xi,\eta) - L(\xi,\eta_0))^k}{L(\xi,\eta_0)^{k+1}},\end{aligned}$$

to prove (2.6), it suffices to show that there is a sequence $\{d_k\}$ of positive numbers such that

$$\sum_{k=0}^{\infty} d_k \le C\delta_0 \tag{2.7}$$

and

$$\left\| \frac{(L(\xi,\eta) - L(\xi,\eta_0))^k}{L(\xi,\eta_0)^{k+1}} \right\|_{M(U\times V)} \le d_k \tag{2.8}$$

for all $k = 0, 1, \ldots$

In order to prove (2.8), we introduce the following notation. For each $1 \le j \le m$, we let

$$b_j(\xi,\eta) = \theta_j(\xi/|\xi|) - \theta_j(\eta/|\eta|). \tag{2.9}$$

Then

$$
\begin{aligned}
&L(\xi,\eta) - L(\xi,\eta_0)\\
&\quad= \sum_{j=1}^m |b_j(\xi,\eta)|^2 - \sum_{j=1}^m |b_j(\xi,\eta_0)|^2\\
&\quad= \sum_{j=1}^m [|\theta_j(\xi/|\xi|)|^2 + |\theta_j(\eta/|\eta|)|^2 - 2\operatorname{Re}(\theta_j(\xi/|\xi|)\overline{\theta}_j(\eta/|\eta|))\\
&\qquad - |\theta_j(\xi/|\xi|)|^2 - |\theta_j(\eta_0/|\eta_0|)|^2 + 2\operatorname{Re}(\theta_j(\xi/|\xi|)\overline{\theta}_j(\eta_0/|\eta_0|))]\\
&\quad= \sum_{j=1}^m [\theta_j(\eta/|\eta|)(\overline{\theta}_j(\eta/|\eta|) - \overline{\theta}_j(\eta_0/|\eta_0|)) + (\theta_j(\eta/|\eta|)\\
&\qquad - \theta_j(\eta_0/|\eta_0|))\overline{\theta}_j(\eta_0/|\eta_0|) - 2\operatorname{Re}(\theta_j(\xi/|\xi|)(\overline{\theta}_j(\eta/|\eta|) - \overline{\theta}_j(\eta_0/|\eta_0|))]\\
&\quad= \sum_{j=1}^m [|b_j(\eta,\eta_0)|^2 - 2\operatorname{Re}(\theta_j(\eta_0/|\eta_0|)\overline{b}_j(\eta,\eta_0)) - 2\operatorname{Re}(\theta_j(\xi/|\xi|)\overline{b}_j(\eta,\eta_0))]\\
&\quad= \sum_{j=1}^m [|b_j(\eta,\eta_0)|^2 - 2\operatorname{Re}(b_j(\xi,\eta_0)\overline{b}_j(\eta,\eta_0))].
\end{aligned}
$$

We may choose our η_0 with $|\eta_0|$ large enough so that

$$\sum_{j=1}^m |b_j(\eta,\eta_0)| \le \delta_0^2/(32mM^2), \tag{2.10}$$

where

$$M = \sum_{j=1}^m \|b_j(\cdot,\cdot)\|_{L^\infty(S^{n-1}\times S^{n-1})}. \tag{2.11}$$

Thus we only need to show

$$\left\| \frac{(\sum_{j=1}^m [|b_j(\eta,\eta_0)|^2 - 2\operatorname{Re}(b_j(\xi,\eta_0)\overline{b}_j(\eta,\eta_0))])^k}{L(\xi,\eta_0)^{k+1}} \right\|_{M(U\times V)} \le d_k. \tag{2.12}$$

To prove (2.12), we use the notation

$$b(\xi,\eta) = (b_1(\xi,\eta),\ldots,b_m(\xi,\eta)),$$

and let $\alpha = (\alpha_1,\ldots,\alpha_m)$ be a multiindex with non-negative integers. Thus

$$\frac{(\sum_{j=1}^m [|b_j(\eta,\eta_0)|^2 - 2\operatorname{Re}(b_j(\xi,\eta_0)\overline{b}_j(\eta,\eta_0))])^k}{L(\xi,\eta_0)^{k+1}}$$

can be written as at most $(4m)^k$ terms of the form

$$|b(\eta,\eta_0)^\gamma|^2 b(\xi,\eta_0)^\alpha \overline{b(\xi,\eta_0)}^\beta b(\eta,\eta_0)^\alpha \overline{b(\eta,\eta_0)}^\beta L(\xi,\eta_0)^{-k-1},$$

where $|\alpha|+|\beta|+|\gamma|=k$ and $|\gamma|+|\beta|\geq k/2$. It is obvious that

$$\begin{aligned}\||b(\eta,\eta_0)^\gamma|^2 b(\xi,\eta_0)^\alpha\overline{b(\xi,\eta_0)}^\beta b(\eta,\eta_0)^\alpha\overline{b(\eta,\eta_0)}^\beta L(\xi,\eta_0)^{-k-1}\|_{M(U\times V)}\\ \leq \||b(\xi,\eta_0)|^{2(|\alpha|+|\beta|)}L(\xi,\eta_0)^{-k-1}\|_{L^\infty(U)}\||b(\eta,\eta_0)|^{2|\gamma|+|\alpha|+|\beta|}\|_{L^\infty(V)}\\ \leq \frac{4}{\delta_0}\left(\frac{1}{8m}\right)^k\end{aligned}$$

for all $k\geq 0$. Therefore, if we choose $d_k=4\delta_0^{-1}2^{-k}$ then (2.12) holds, and $\sum_{k=0}^\infty d_k\leq 8/\delta_0$. Therefore, the proof of Lemma 2.4 is complete. ■

By Propositions 2.2, 2.3 and Lemma 2.4, we see that the kernel L satisfies A0, A1, A3 and A5 of Theorem 2.1. Therefore, by Theorem 2.1, we have the following theorem.

THEOREM 2.5. *Suppose that* (1.3) *holds and* $f\in L^2(\mathbb{R}^n)$. *If the operator* $\sum_{j=1}^m[\widetilde{K}_j[M_f,K_j]-[M_f,K_j]\widetilde{K}_j]$ *is bounded on* $L^p(\mathbb{R}^n)$ *for some* $1<p<\infty$, *then* $f\in\mathrm{BMO}(\mathbb{R}^n)$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By a theorem of [5] or [11], we know that (i) implies (iii). It is obvious that (iii) implies (ii). Since K_j is bounded on $L^q(\mathbb{R}^n)$ (see [16]) for all $1<q<\infty$, (ii) implies (iv). Now, by Theorem 2.5, (iv) implies (i). Therefore, (i)–(iv) are equivalent, and the proof of Theorem 1.1 is complete. ■

3. Proof of Theorems 1.2 and 1.3. We need the following theorem of C. Fefferman and Stein [7], and Coifman and Weiss [6].

THEOREM 3.1. *Let* X *be a space of homogeneous type. Then*

(i) $[H^1(X)]^*=\mathrm{BMO}(X)$;
(ii) $[\mathrm{VMO}(X)]^*=H^1(X)$.

We first prove the following proposition.

PROPOSITION 3.2. *Suppose* (1.3) *holds. Let* $1<p<\infty$ *and* p' *be the conjugate exponent of* p. *For any* $f\in L^p(\mathbb{R}^n)$ *and* $g\in L^{p'}(\mathbb{R}^n)$, *we have* $K_j(f)\overline{g}-f\overline{K_j^*(g)}\in H^1(\mathbb{R}^n)$ *and*

$$\|K_j(f)\overline{g}-f\overline{K_j^*(g)}\|_{H^1}\leq C\|f\|_p\|g\|_{p'}. \tag{3.1}$$

Proof. Since $\mathrm{VMO}(\mathbb{R}^n)^*=H^1(\mathbb{R}^n)$, it suffices to prove

$$\Big|\int_{\mathbb{R}^n} b(x)(K_j(f)(x)\overline{g}(x)-f(x)\overline{K_j^*(g)}(x))\,dx\Big|\leq C_p\|b\|_{\mathrm{BMO}}\|f\|_p\|g\|_{p'}. \tag{3.2}$$

This is a direct consequence of Theorems 1.1, 3.1 and the identity

$$\int_{\mathbb{R}^n} b(x)(K_j(f)(x)\overline{g}(x) - f(x)\overline{K_j^*(g)}(x))\,dx = \int_{\mathbb{R}^n} [M_b, K_j](f)(x)\overline{g(x)}\,dx. \tag{3.3}$$

Therefore, the proof of the proposition is complete. ∎

PROPOSITION 3.3. *Suppose* (1.3) *holds. Let* $1 < p < \infty$ *and* p' *be the conjugate exponent. For any* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^{p'}(\mathbb{R}^n)$, *the set*

$$\Big\{ \sum_{j=1}^{m} (K_j(f)\overline{\widetilde{K}_j^*(g)} + \widetilde{K}_j(f)\overline{K_j^*(g)} - f\overline{(K_j^*\widetilde{K}_j^*)(g)} - (K_j\widetilde{K}_j f)\overline{g} : f \in L^p, g \in L^{p'} \Big\}$$

is dense in $H^1(\mathbb{R}^n)$.

Proof. Since

$$\begin{aligned}\int_{\mathbb{R}^n} (\widetilde{K}_j[M_b, K_j])(f)(x)\overline{g(x)}\,dx &= \int_{\mathbb{R}^n} ([M_b, K_j])(f)(x)\overline{\widetilde{K}_j^*(g)}(x)\,dx \\ &= \int_{\mathbb{R}^n} b(x)(K_j(f)\overline{\widetilde{K}_j^*(g)}(x) - f(x)\overline{(K_j^*\widetilde{K}_j^*)(g)}(x))\,dx\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} ([M_b, K_j]\widetilde{K}_j)(f)(x)\overline{g(x)}\,dx &= \int_{\mathbb{R}^n} \widetilde{K}_j(f)(x)\overline{([M_b, K_j])^* g}(x)\,dx \\ &= \int_{\mathbb{R}^n} \widetilde{K}_j(f)(x)\overline{K_j^*(\overline{b}g) - \overline{b}K_j^*(g)}(x)\,dx \\ &= \int_{\mathbb{R}^n} b(x)(K_j\widetilde{K}_j(f)\overline{g} - \widetilde{K}_j(f)\overline{K_j^*(g)})(x)\,dx.\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^n} \sum_{j=1}^{m} (\widetilde{K}_j M_b K_j - \widetilde{K}_j K_j M_b - M_b K_j \widetilde{K}_j + K_j M_b \widetilde{K}_j)(f)(x)\overline{g(x)}\,dx$$

$$= \int_{\mathbb{R}^n} b(x) \sum_{j=1}^{m} [K_j(f)\overline{\widetilde{K}_j^*(g)} + \widetilde{K}_j(f)\overline{K_j^*(g)} - f\overline{(K_j^*\widetilde{K}_j^*)(g)} - (K_j\widetilde{K}_j f)\overline{g}]\,dx.$$

Therefore, the proposition follows from Theorem 1.1. ∎

Now we are ready to prove Theorems 1.2 and 1.3.

1) Theorem 1.2 is a direct consequence of Lemmas III.1 and III.2 of [4] and of Proposition 3.3. ∎

2) Theorem 1.3 is a direct consequence of Lemmas III.1 and III.2 of [4] and of Proposition 3.2. ∎

4. Application of Theorems 1.2 and 1.3. We apply Theorems 1.2 and 1.3 to prove several theorems concerning the compensated compactness on Hardy spaces.

The following theorem is due to Wu [20].

THEOREM 4.1. *Let l be a positive integer, and let $1 < p, p' < \infty$ and $1/p + 1/p' = 1$. Then the bilinear form*

$$\Big\langle fg - \sum_{j_1,\ldots,j_l=1}^{n} R_{j_1}\ldots R_{j_l}(f)R_{j_1}\ldots R_{j_l}(g), b\Big\rangle \tag{4.1}$$

is bounded on $L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Proof. We claim Theorem 4.1 is a special case of Theorem 1.2. In fact, since $\sum_{j=1}^n R_jR_j = -I$ and $R_j^* = -R_j$, if we let

$$K_{j_1\ldots j_l} = R_{j_1}\ldots R_{j_l} \quad \text{and} \quad \theta_{j_1\ldots j_l}(\xi) = i^l \xi_{j_1}\ldots\xi_{j_l}/|\xi|^l,$$

then

$$\widetilde{K}_{j_1\ldots j_l} = (-1)^l R_{j_1}\ldots R_{j_l},$$

$$\sum_{j_1,\ldots,j_l=1}^{n} K_{j_1\ldots j_l}\widetilde{K}_{j_1\ldots j_l} = (-1)^l(-I)^l = I,$$

and

$$\sum_{j_1,\ldots,j_l=1}^{n} K_{j_1\ldots j_l}^*\widetilde{K}_{j_1\ldots j_l}^* = I.$$

Thus

$$\sum_{j_1,\ldots,j_l=1}^{n} |\theta_{j_1\ldots j_l}(\xi/|\xi|) - \theta_{j_1\ldots j_l}(\psi(\xi/|\xi|))|^2 = c_{l,n} > 0, \tag{4.2}$$

where $\psi : S^{n-1} \to S^{n-1}$ is defined as follows: If l is odd, we let $\psi(x) = -x$ for $x \in S^{n-1}$. If l is even, we may choose an orthonormal matrix O such that if we let $\psi(x) = Ox$ for all $x \in S^{n-1}$, then (4.2) holds for some constant $c_{l,n} > 0$. Therefore, by Theorem 1.2,

$$\sum_{j_1,\ldots,j_l} R_{j_1} \ldots R_{j_l}(f) R_{j_1} \ldots R_{j_l}(g) - fg$$

is in H^1 for all $f \in L^p$ and $g \in L^{p'}$ (here we consider real-valued L^p and H^1 functions). Moreover, the set of such forms is dense in H^1 and the proof of Theorem 4.1 is complete. ∎

Finally, we make the following remarks.

Remark 1. In [4], Coifman, Lions, Meyer and Semmes gave many examples in PDE related to the theorems of Coifman, Rochberg and Weiss [5]. We believe that the family of integral operators in Theorems 1.1–1.3 will give some more information on some useful quantities in PDE, harmonic analysis and operator theory (for examples, see [1], [4], [10], [12] and [20]).

Remark 2. By using a theorem in Section 13 of [8], one can prove a similar result to Theorem 1.1 for compactness of commutators; we leave it to the reader.

Remark 3. Theorem 1.1 partially answers the following question: Let X be a space of homogeneous type. Suppose that $K_1, \ldots, K_m$ is a family of singular integral operators which characterize $H^1(X)$. Can one prove that $[M_b, K_j]$ is bounded on $L^p(X)$ $(1 < p < \infty)$ for all $1 \le j \le m$ if and only if $b \in \mathrm{BMO}(X)$?

Theorem 1.1 gives an affirmative answer for $X = \mathbb{R}^n$. More detailed information on families of singular integral operators which characterize $H^1(X)$ can be found in [3].

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Reçu par la Rédaction le 27.2.1995;
en version modifiée le 3.4.1995