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COVERABLE RADON MEASURES IN TOPOLOGICAL SPACES WITH COVERING PROPERTIES

BY

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1. Introduction. Let X be a regular Hausdorff space and let $\mathcal{S}$ be the σ-algebra generated by the collection $\mathcal{G}$ of all open sets in X. Let μ be a Radon measure in X, i.e., for any $E \in \mathcal{S}$,

$$\mu(E) = \sup\{\mu(K) : E \supset K,\ K \text{ is compact}\}.$$

We assume that each point of X has a neighbourhood of μ-finite measure.

We have introduced coverable measures in [Ku1]. A Radon measure μ in X is said to be *coverable* if any subset of X has a measurable cover with respect to μ. By "topological spaces with covering properties" we mean generalized paracompact spaces. We shall assume fairly mild conditions on X and study a not necessarily σ-finite Radon measure μ in X and the support of μ. These measures are localizable and coverable, and play an important role in statistical structures (see the opening of §6). Okada ([O], Theorem 3.1, p. 226) proved that the support of a finite Borel measure in a metacompact space is Lindelöf. We shall prove that the support of any σ-finite Radon measure in a fairly wide class of topological spaces is Lindelöf (Theorem 4.7).

The problem of when a Radon measure is μ^*-semifinite was raised by Schwartz ([Sc], p. 17). Prinz [P] proved that a Radon measure in a metacompact space is μ^*-semifinite. A Radon measure is μ^*-semifinite if and only if each locally negligible set is negligible ([P], Proposition 1, p. 442). We shall generalize this result (Theorem 5.3). Gardner and Pfeffer [GaPf3] studied Radon measures μ in a wide class of topological spaces and proved that μ is localizable and locally determined. We shall prove in addition that μ is μ^*-semifinite and coverable. Coverable Radon measures are localizable, locally determined and μ^*-semifinite (Propositions 3.1, 3.2). Let $\{X_\alpha : \alpha \in A\}$ be a concassage of μ and $X^* = \bigcup_\alpha X_\alpha$. They proved that X^* is a free union of σ-compact subspaces under some conditions. We shall study the support Y of μ instead of X^*, which seems to be natural (to the author), and prove

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that Y is often strongly paracompact and a free sum of Lindelöf subspaces (Theorems 4.2, 4.3), but Y is not always so (Theorem 4.5, Example 4.6). We shall prove that Radon measures in a wide class of topological spaces are coverable (Theorem 5.2). The coverability of measures is essential in applications because we need the Lebesgue decomposition in addition to the Radon–Nikodym theorem.

2. Definitions. Let $(X, \mathcal{S}, \mu)$ be a measure space, where $\mathcal{S}$ is a σ-algebra and μ is a countably additive measure on $\mathcal{S}$. Throughout the paper we assume that μ is *semifinite*, i.e., $\mu(E) = \sup\{\mu(F) : E \supset F,\ \mu(F) < \infty\}$ for any $E \in \mathcal{S}$.

Let $D \subset X$. If $E \in \mathcal{S}$, $E \supset D$ and $\mu(F) = 0$ for any set $F \in \mathcal{S}$ with $F \subset E - D$, then E is said to be a *measurable cover* of D. μ is said to be *coverable* if each subset of X has a measurable cover. If each subset D with $D \cap F \in \mathcal{S}$ for any set F of μ-finite measure is measurable, then μ is said to be *locally determined* [Fr] or *saturated*.

Let μ^* be the outer measure induced by μ. Then μ is said to be μ^*-*semifinite* if $\mu^*(E) = \sup\{\mu^*(F) : E \supset F,\ \mu^*(F) < \infty\}$ for any $E \subset X$. Let $\widetilde{E} = \{F \in \mathcal{S} : \mu(E \triangle F) = 0\}$ for $E \in \mathcal{S}$ and $\widetilde{\mathcal{S}} = \{\widetilde{E} : E \in \mathcal{S}\}$. We write $\widetilde{E} \le \widetilde{F}$ if $\mu(E - F) = 0$. Then $\le$ is a partial order in $\widetilde{\mathcal{S}}$, and μ is said to be *localizable* if any subset of $\widetilde{\mathcal{S}}$ has a supremum. μ is localizable if and only if its measure algebra is complete.

Let μ be a Radon measure in X. Let $Y = \{y \in X : \mu(U) > 0$ for any neighbourhood U of $y\}$. Then Y is a closed set satisfying $\mu(X - Y) = 0$; it is called the *support* of μ.

A *concassage* of a Radon measure μ is a disjoint collection $\{X_\alpha : \alpha \in A\}$ of compact sets such that

(2.1) If $O \cap X_\alpha \neq \emptyset$ for an open set O, then $\mu(O \cap X_\alpha) > 0$;

(2.2) $\mu(E) = \sum_\alpha \mu(E \cap X_\alpha)$ for any Borel set E.

(See [Sc], Theorem 13, p. 46; [GaPf1], Lemma 3.4, p. 71.)

We assume that the reader is familiar with basic notions of paracompact spaces ([En], Chapter 5). We refer the reader to [Bu] for detailed covering properties. We denote the cardinality of a set E by $|E|$. Let $\mathcal{U}$ be a collection of open sets. We denote $|\{U \in \mathcal{U} : x \in U\}|$ by $\operatorname{ord}(x, \mathcal{U})$ for each x in X. We shall consider only open covers and refinements in a Hausdorff space. We use "metacompact" instead of "weakly paracompact". A topological space X is said to be σ-*metacompact* if for any cover $\mathcal{U}$ of X there exists a refinement $\mathcal{V} = \bigcup_n \mathcal{V}_n$ of $\mathcal{U}$ such that each $\mathcal{V}_n$ $(n < \omega)$ is point-finite. X is said to be *meta-Lindelöf* if for any cover $\mathcal{U}$ of X there exists a point-countable refinement $\mathcal{V}$ of $\mathcal{U}$. Here a collection $\mathcal{O}$ of open sets is said to be *point-countable* if $\mathrm{ord}(x,\mathcal{O}) \le \omega$ for any x in X. X is said to be *weakly θ-refinable* if for any cover $\mathcal{U}$ of X there exists a refinement $\mathcal{V} = \bigcup_n \mathcal{V}_n$ of $\mathcal{U}$ such that there exists $n < \omega$ with $1 \le \mathrm{ord}(x, \mathcal{V}_n) < \omega$ for any x in X. A σ-metacompact space is weakly θ-refinable. Many important examples (a semi-stratifiable space, a strict p-space, a Moore space etc.) in generalized metric spaces are weakly θ-refinable ([Gr1], [Gr2]). We note that these covering properties are hereditary with respect to closed subspaces. Finally, X is said to satisfy the *countable chain condition* (ccc) if each disjoint collection of open sets in X is countable.

(2.3) Under Martin's axiom and the negation of the continuum hypothesis every point-countable cover of a compact space with the ccc is countable ([GaPf4], Theorem 4.8, p. 971).

Martin's axiom and the negation of the continuum hypothesis are denoted by MA and nonCH respectively.

3. Coverable measures. Let $(X, \mathcal{S}, \mu)$ be a measure space. In this section we study coverable, localizable, locally determined and μ^*-semifinite measures from a general point of view.

3.1. Proposition. *If μ is coverable, then it is μ^*-semifinite and locally determined.*

Proof. Let $\{X_\alpha : 0 < \mu(X_\alpha) < \infty \ (\alpha \in A),\ \mu(X_\alpha \cap X_\beta) = 0 \ (\alpha \ne \beta)\}$ be a maximal collection. Then for any set $E \in \mathcal{S}$,

$$\mu(E) = \sum_\alpha \mu(E \cap X_\alpha).$$

Now for any set D in X we have

$$\mu^*(D) = \sum_\alpha \mu^*(D \cap X_\alpha). \tag{3.1.1}$$

Indeed, if E is a measurable cover of D, then

$$\mu^*(D) = \mu(E) = \sum_\alpha \mu(E \cap X_\alpha) = \sum_\alpha \mu^*(D \cap X_\alpha),$$

since $E \cap X_\alpha$ is a measurable cover of $D \cap X_\alpha$. If $\mu^*(D) > 0$, then $0 < \mu^*(D \cap X_\alpha) < \infty$ for some α in A, which implies that μ is μ^*-semifinite.

Assume that $D \cap F \in \mathcal{S}$ for any set F with $\mu(F) < \infty$. Let E be a measurable cover of D. Then by (3.1.1),

$$\mu^*(E - D) = \sum_\alpha \mu^*(E \cap X_\alpha - D \cap X_\alpha) = \sum_\alpha \mu(E \cap X_\alpha - D \cap X_\alpha) = 0,$$

since $D \cap X_\alpha \in \mathcal{S}$. Therefore there exists a null set N with $E - D \subset N$, which implies that $E - D = E \cap N - D \cap N \in \mathcal{S}$ and hence $D \in \mathcal{S}$. ∎

3.2. PROPOSITION. *Assume that there exists a disjoint collection* $\{X_\alpha : \alpha \in A\}$ *of sets of finite measure such that for any set* $E \in \mathcal{S}$ *we have*

$$\mu(E) = \sum_{\alpha} \mu(E \cap X_\alpha). \tag{3.2.1}$$

Then μ *is coverable if and only if it is localizable and* μ^**-semifinite.*

Remark. In general μ is coverable if it is localizable and μ^*-semifinite. The converse is not true if there exist two different measurable cardinals.

Proof of Proposition 3.2. *Necessity.* We first show that any $\{\widetilde{E}_\alpha : E_\alpha \subset X_\alpha\ (\alpha \in A)\}$ has a supremum. Let $E = \bigcup_\alpha E_\alpha$ and let E^* be a measurable cover of E. Then $E^* \cap X_\alpha$ is a measurable cover of $E_\alpha \in \mathcal{S}$ and therefore $(E^* \cap X_\alpha)^\sim = \widetilde{E}_\alpha$. By (3.2.1), $\widetilde{X} = \sup_\alpha \widetilde{X}_\alpha$. Hence

$$\widetilde{E^*} = \widetilde{E^*} \cap (\sup \widetilde{X}_\alpha) = \sup_\alpha (\widetilde{E^*} \cap \widetilde{X}_\alpha) = \sup_\alpha \widetilde{E}_\alpha.$$

If now $\{\widetilde{D}_\beta : \beta \in B\}$ is a collection, then for each $\alpha \in A$, $\widetilde{E}_\alpha = \sup\{(D_\beta \cap X_\alpha)^\sim : \beta \in B\}$ exists, because $\mu(X_\alpha) < \infty$. Hence $\widetilde{E^*} = \sup_\alpha \widetilde{E}_\alpha$ exists and is equal to $\sup_\beta \widetilde{D}_\beta$. Together with Proposition 3.1 we get necessity.

Sufficiency. For any set D we have

$$\mu^*(D) = \sum_{\alpha} \mu^*(D \cap X_\alpha). \tag{3.2.2}$$

Indeed, if $\mu^*(D) < \infty$, then there exists a measurable cover of D, and by assumption (3.2.1) we get (3.2.2). Since μ is μ^*-semifinite, we get (3.2.2) in general.

If $X^* = \bigcup_\alpha X_\alpha$, then by (3.2.2), $\mu^*(X - X^*) = \sum_\alpha \mu^*(X_\alpha - X_\alpha) = 0$. Therefore there exists a null set N with $X - X^* \subset N$.

Let now $D \subset X$. Let $E_\alpha \subset X_\alpha$ be a measurable cover of $D_\alpha = D \cap X_\alpha$ for each α in A. We put $E = \bigcup_\alpha E_\alpha$ and $\widetilde{E^*} = \sup_\alpha \widetilde{E}_\alpha$. Then $\mu^*(E \triangle E^*) = \sum_\alpha \mu^*(E_\alpha \triangle (E^* \cap X_\alpha)) = 0$, since $(E^* \cap X_\alpha)^\sim = \widetilde{E}_\alpha$, which implies that E belongs to the completion of μ. Therefore there exists a set $E^0 \in \mathcal{S}$ with $E \subset E^0$ and $\overline{\mu}(E^0 - E) = 0$.

Now $E^0 \cup N$ is a measurable cover of D. Indeed, we have $E^0 \cup N \supset E \cup N \supset (D \cap X^*) \cup (D - X^*) = D$. If $F \subset (E^0 \cup N) - D$ and $F \in \mathcal{S}$, then $E^0 - D \supset F - N$. We put $F' = F - N$ and $E^0_\alpha = E^0 \cap X_\alpha$, $F'_\alpha = F' \cap X_\alpha$ for each α in A. Then $F'_\alpha \subset (E^0_\alpha - E_\alpha) \cup (E_\alpha - D_\alpha)$ and therefore $F'_\alpha - (E^0_\alpha - E_\alpha) \subset E_\alpha - D_\alpha$. Since E_α is a measurable cover of D_α, $\mu(F'_\alpha - (E^0_\alpha - E_\alpha)) = 0$, which implies $\mu(F'_\alpha) = 0$. Hence $\mu(F') = 0$ by (3.2.1), which implies $\mu(F) = 0$. ∎

4. Supports of Radon measures. We prove that the support of a Radon measure in a σ-metacompact space or in a meta-Lindelöf space under MA + nonCH is strongly paracompact and a free sum of Lindelöf spaces. This is not true in general for a weakly θ-refinable space.

4.1. PROPOSITION. *The support Y of a Radon measure μ in a strongly paracompact space X is a free sum of Lindelöf spaces Y_β $(\beta \in B)$, i.e., there exists a disjoint collection $\{Y_\beta : \beta \in B\}$ of open Lindelöf subspaces of Y such that $Y = \bigcup_\beta Y_\beta$ (disjoint).*

Proof. Since strong paracompactness is hereditary with respect to closed subspaces, we may assume that Y is strongly paracompact. For each point y in Y there exists an open set O in Y with $y \in O$ and $0 < \mu(O) < \infty$. Since O satisfies the ccc, so does $\mathrm{Cl}(O)$, and it is strongly paracompact. Hence it is Lindelöf. Let $\mathcal{O}$ be a cover of Y. There exists a star-finite refinement $\mathcal{U} = \{U\}$ of $\mathcal{O}$. By ([En], Lemmas 5.3.8, 5.3.9, p. 404), $Y = \bigcup_\beta \{Y_\beta : \beta \in B\}$ (disjoint) and each Y_β is a countable union of elements of $\mathcal{U}$. Since each Y_β is a closed subspace in a countable union of Lindelöf subspaces of the form $\mathrm{Cl}(U)$, it is Lindelöf.

4.2. THEOREM. *The support Y of a Radon measure μ in a regular σ-metacompact space is strongly paracompact.*

Proof. We may assume that Y is σ-metacompact. Let $\mathcal{O}$ be any cover of Y. There exists a refinement $\mathcal{U}$ of $\mathcal{O}$ consisting of sets of finite μ-measure. Then there exists a refinement $\mathcal{V} = \bigcup_n \mathcal{V}_n$ of $\mathcal{U}$ such that each $\mathcal{V}_n$ is point-finite. Let $\mathcal{V}^n = \bigcup_{i \le n} \mathcal{V}_i$ for each $n < \omega$.

By ([En], Theorem 5.3.10, pp. 404–405) we need only prove that $\mathcal{V}$ is star-countable. Indeed, let $\{X_\alpha : \alpha \in A\}$ be a concassage of μ. If $V \in \mathcal{V}$, then $0 < \mu(V) < \infty$ and therefore $A_0 = \{\alpha \in A : V \cap X_\alpha \neq \emptyset\}$ is countable. If $V' \cap V \neq \emptyset$ for $V' \in \mathcal{V}$, then $\mu(V' \cap V) > 0$ and hence $V' \cap X_\alpha \neq \emptyset$ and $\mu(V' \cap X_\alpha) > 0$ for some α in A_0. It remains to show that $\{V' \in \mathcal{V}^n : V' \cap X_\alpha \neq \emptyset\}$ is countable for each α in A_0. Since $\mathcal{V}^n$ is point-finite, by ([GaPf4], Lemma 12.1, pp. 1014–1015), $\{V' \cap X_\alpha : V' \cap X_\alpha \neq \emptyset,\ V' \in \mathcal{V}^n\}$ is countable and the conclusion follows since $\mathcal{V}^n$ is point-finite. ∎

4.3. THEOREM. *Under* MA *and* nonCH, *the support Y of a Radon measure μ in a regular meta-Lindelöf space X is strongly paracompact.*

Remark. Under CH there exist a locally compact meta-Lindelöf space X and a σ-finite Radon measure μ in X such that the support of μ is not strongly paracompact ([GaPf1], Theorem 3.7, pp. 72–73).

Proof of Theorem 4.3. Y is a regular meta-Lindelöf space. Let $\mathcal{O}$ be any cover of Y. There exists a refinement $\mathcal{U}$ of $\mathcal{O}$ consisting of sets of finite μ-measure. Then there exists a point-countable refinement $\mathcal{V}$ of $\mathcal{U}$.

We show that $\mathcal{V}$ is star-countable. Indeed, let $\{X_\alpha : \alpha \in A\}$ be a concassage of μ. If $V \in \mathcal{V}$, then $A_0 = \{\alpha \in A : V \cap X_\alpha \neq \emptyset\}$ is countable.

If $V' \cap V \neq \emptyset$ for $V' \in \mathcal{V}$, then $V' \cap X_\alpha \neq \emptyset$ for some α in A_0. We consider the point-countable cover $\{V' \cap X_\alpha : V' \cap X_\alpha \neq \emptyset,\ V' \in \mathcal{V}\}$ of X_α. Then it is countable by (2.3) since X_α satisfies the ccc. Since $\mathcal{V}$ is point-countable, $\{V' \in \mathcal{V} : V' \cap X_\alpha \neq \emptyset\}$ is countable for each α in A_0. ■

4.4. PROPOSITION. *Let μ be a Radon measure in a regular space X. If there exists a star-countable cover $\mathcal{U}$ of X consisting of sets of finite μ-measure, then there exists a null G_δ-set N such that the subspace $X - N$ is strongly paracompact.*

Proof. By ([En], Lemmas 5.3.8, 5.3.9, p. 404) there exists a disjoint collection $\{Y_\beta : \beta \in B\}$ of open sets such that each Y_β is a countable union of elements of $\mathcal{U}$ and $X = \bigcup\{Y_\beta : \beta \in B\}$. Hence $\mu_\beta = \mu|Y_\beta$ is a σ-finite Radon measure and therefore the union Z_β of a concassage of μ_β is σ-compact and $\mu(Y_\beta - Z_\beta) = 0$ for each β in B. Since Z_β is regular, it is strongly paracompact. Therefore $Z = \bigcup_\beta\{Z_\beta : \beta \in B\}$ is strongly paracompact ($Z_\beta \subset Y_\beta$ and Y_β is clopen). We put $N_\beta = Y_\beta - Z_\beta$ for each β in B and $N = \bigcup_\beta\{N_\beta : \beta \in B\}$.

Then $\mu(N) = 0$. Indeed, for each β in B, N_β is a G_δ-set, $N_\beta = \bigcap_n G_{\beta n}$, where $G_{\beta n} \subset Y_\beta$ is open. Therefore $N = \bigcup_\beta \bigcap_n G_{\beta n} = \bigcap_n(\bigcup_\beta G_{\beta n}) \in \mathcal{S}$ ($G_{\beta n} \subset Y_\beta$, $Y_\beta \cap Y_\gamma = \emptyset$ ($\beta \neq \gamma$)). Now, for any Borel set E we have

$$\mu(E) = \sum_\beta (E \cap Y_\beta). \tag{4.4.1}$$

The equality is clear for a compact set E, since then $|\{\beta \in B : E \cap Y_\beta \neq \emptyset\}| < \omega$ and $X = \bigcup_\beta Y_\beta$. In the general case the equality follows because μ is Radon. Finally, this equality yields $\mu(N) = 0$. ■

4.5. THEOREM. *If μ is a Radon measure in a regular weakly θ-refinable space X, then there exists a disjoint collection $\{Y_n : n < \omega\}$ of measurable sets such that each subspace Y_n is strongly paracompact and $Y = \bigcup_n Y_n$ satisfies $\mu(X - Y) = 0$.*

Remark. Under MA and nonCH the conclusion is valid for a regular weakly $\delta\theta$-refinable space with countable tightness. Gardner and Pfeffer ([GaPf3], Proposition 3.7) gave the similar result for X^* in the case where X^* is the union of a concassage of μ.

Proof of Theorem 4.5. There exists a refinement $\mathcal{U} = \bigcup_n \mathcal{U}_n$ of the cover $\mathcal{O} = \{O \in \mathcal{G} : \mu(O) < \infty\}$ of X such that for each x in X there exists $n < \omega$ with $1 \le \operatorname{ord}(x, \mathcal{U}_n) < \omega$. Then $X^n = \{x \in X : 1 \le \operatorname{ord}(x, \mathcal{U}_n) < \omega\}$ is a Borel set for each $n < \omega$. We put $X_n = X^n - \bigcup_{k=1}^{n-1} X^k$. Then $X = \bigcup_n X_n$ (disjoint) and $\mu_n = \mu|X_n$ is a Radon measure in X_n. If X_n^0 is the support of μ_n, then $\mu_n(X_n - X_n^0) = 0$. Let $\mathcal{U}_n^0 = \{U \cap X_n^0 : U \cap X_n^0 \neq \emptyset,\ U \in \mathcal{U}_n\}$.

Then $\mathcal{U}_n^0$ is star-countable: Let $U \cap X_n^0 \in \mathcal{U}_n^0$. Let $\{X_\alpha : \alpha \in A\}$ be a concassage of μ_n. Then $A_0 = \{\alpha \in A : U \cap X_\alpha \neq \emptyset\}$ is countable. If $(U \cap X_n^0) \cap (V \cap X_n^0) \neq \emptyset$ for $V \cap X_n^0 \in \mathcal{U}_n^0$, then $\mu_n(U \cap V) > 0$ and therefore $V \cap X_\alpha \neq \emptyset$ for some α in A_0. Since the collection $\{V \cap X_\alpha : V \cap X_\alpha \neq \emptyset,\ V \in \mathcal{U}_n\}$ is a point-finite cover of the compact set X_α and $\mu(V \cap X_\alpha) > 0$, it is countable by ([GaPf4], Lemma 12.1, pp. 1014–1015). Together with the point-finiteness of $\mathcal{U}_n$ in X_α, $\{V \in \mathcal{U}_n : V \cap X_\alpha \neq \emptyset\}$ is countable. Since A_0 is countable, $\mathcal{U}_n^0$ is star-countable.

We consider the measure space $(X_n^0, \mathcal{S}|X_n^0, \mu|X_n^0)$. By Proposition 4.4 there exists a strongly paracompact subset $Y_n \subset X_n^0$ with $\mu(X_n^0 - Y_n) = 0$. If $Y = \bigcup_n Y_n$, then $\mu(X - Y) = 0$. ∎

4.6. Example. There exists a locally compact weakly θ-refinable space X which is not meta-Lindelöf such that there exists a Radon measure μ in X whose support is not strongly paracompact. Actually, X can be chosen to be subparacompact. The relevant example is due to Gruenhage and Pfeffer ([GrPf], Example 7, pp. 170–171). For the reader's convenience we state it here. Let $I = [0,1]$ and let $X = \{(k/2^n, 1/2^n) \in I \times I : 0 \leq k \leq 2^n,\ n \geq 0\} \cup (I \times \{0\})$. We define a topology in X as follows: the points $(k/2^n, 1/2^n)$ are open and a neighbourhood base at $(t, 0)$ is given by the sets

$$U(t, \varepsilon) = \{(u, v) \in X : 2|u - t| < v < \varepsilon\} \cup \{(t, 0)\},$$

where $\varepsilon > 0$. Then X has the topological properties mentioned above.

We define a Radon measure μ in X different from [GrPf]. The collection $\{(t, 0) : 0 \leq t \leq 1\}$ is discrete. Let $\mu|(I \times \{0\})$ be the counting measure. For any subset $E \subset X - (I \times \{0\})$ we define $\mu(E)$ by

$$\sum_{n=0}^{\infty} |\{k : (k/2^n, 1/2^n) \in E\}|/2^n.$$

The support of μ is X. Now, X is separable. If X were strongly paracompact, then it would be Lindelöf. But it is clear that X is not Lindelöf. If $Y_1 = \{(k/2^n, 1/2^n) : 0 \leq k \leq 2^n,\ n \geq 0\}$ and $Y_2 = I \times \{0\}$, then Y_i $(i = 1, 2)$ is strongly paracompact and $X = Y_1 \cup Y_2$ (disjoint).

4.7. Theorem. *If μ is a σ-finite Radon measure in a regular σ-metacompact space, then the support of μ is Lindelöf. The same conclusion is valid for μ in a regular meta-Lindelöf space if* MA+nonCH *holds true. These measures are regular.*

Proof. In both cases the support Y is strongly paracompact by Theorems 4.2 and 4.3. Since Y satisfies the ccc, it is Lindelöf. ∎

Remark. Gruenhage and Pfeffer ([GrPf], Theorem 1, p. 167) proved that a σ-finite Radon measure in a metacompact space is regular. The support of a σ-finite Radon measure μ in a σ-para-Lindelöf space satisfies the ccc and is σ-para-Lindelöf. Hence it is Lindelöf. Moreover, μ is regular.

5. Coverable Radon measures. We prove that a Radon measure in a weakly θ-refinable space or in a meta-Lindelöf space under MA+nonCH is coverable.

5.1. LEMMA. *A Radon measure μ in a strongly paracompact space X is coverable.*

Proof. Let Y be the support of μ. By Proposition 4.1, Y is a free sum of Lindelöf subspaces Y_β ($\beta \in B$). The Radon measure $\mu_\beta = \mu|Y_\beta$ is (outer) regular for each β in B.

Let D be any subset of X and let $D_\beta = D \cap Y_\beta$. Since μ_β is σ-finite there exists a measurable cover E_β of D_β. By regularity of μ_β, there exists a G_δ-set $G^*_\beta \supset E_\beta$ with $\mu(G^*_\beta - E_\beta) = 0$. Let $G^*_\beta = \bigcap_n G_{\beta n}$, where $G_{\beta n} \subset Y_\beta$ is open for each $n < \omega$. If $G^0 = \bigcap_n \bigcup_\beta G_{\beta n}$, then $G^0 \cup (X - Y)$ is a measurable cover of D:

We have $G^0 = \bigcup_\beta \bigcap_n G_{\beta n} = \bigcup_\beta G^*_\beta$ and $G^0 \cup (X - Y) \supset D$. Let $[G^0 \cup (X-Y)] - D \supset F \in \mathcal{S}$. Then we have $F \cap Y_\beta \subset (G^*_\beta - E_\beta) \cup (E_\beta - D_\beta)$. Since E_β is a measurable cover of D_β, $\mu[(F \cap Y_\beta) - (G^*_\beta - E_\beta)] = 0$, which implies $\mu(F \cap Y_\beta) = 0$. We have $\mu(F) = \sum_\beta \mu(F \cap Y_\beta)$ similarly to (4.4.1), which implies $\mu(F) = 0$. ∎

5.2. THEOREM. *Let μ be a Radon measure in a regular space. If*

(a) *X is weakly θ-refinable*, *or*

(b) *X is meta-Lindelöf and* MA+nonCH *holds true*,

then μ is coverable and, *consequently*, *localizable*, *μ^*-semifinite and locally determined.*

Remark 1. A Radon measure in a regular σ-para-Lindelöf space is coverable.

Remark 2. Under CH there exists a Radon measure in a meta-Lindelöf space which is not coverable by ([GaPf3], Example 4.5, pp. 290–291).

Remark 3. A Radon measure μ in a regular σ-para-Lindelöf space is localizable, μ^*-semifinite and locally determined (see Remark 1). Gardner and Pfeffer proved that a Radon measure in a space with the same properties as in the theorem is localizable and locally determined ([GaPf3], Theorem 3.4, pp. 286–287).

Proof of Theorem 5.2. If X is weakly θ-refinable, then by Theorem 4.5 there exists a disjoint collection $\{Y_n : n < \omega\}$ of strongly paracompact subspaces with $\mu(X - Y) = 0$, where $Y = \bigcup_n Y_n$.

Let $D \subset X$. Then by Lemma 5.1 there exists a measurable cover $E_n \subset Y_n$ of $D \cap Y_n$ and $(\bigcup_n E_n) \cup (X - Y)$ is a measurable cover of D. (A Borel set in Y_n is a Borel set in X.)

If X is meta-Lindelöf, then the support of μ is strongly paracompact by Theorem 4.3. And therefore μ is coverable by Lemma 5.1 since $\mu(X-Y) = 0$. By Propositions 3.1 and 3.2 we get the conclusion. ∎

A subset E of X is said to be *negligible* if $\mu^*(E) = 0$. A subset E of X is said to be *locally negligible* if each point of X has a neighbourhood U such that $\mu^*(E \cap U) = 0$. By Prinz ([P], Proposition 1, p. 442) a Radon measure μ is μ^*-semifinite if and only if each locally negligible set is negligible. Prinz proved that each locally negligible set is negligible for any Radon measure in a metacompact space ([P], Theorem, p. 443). Together with Theorem 5.2 we get the following:

5.3. Theorem. *If*

(a) *X is a regular weakly θ-refinable space*, *or*
(b) *X is a regular meta-Lindelöf space and* MA+nonCH *holds true*,

then each locally negligible set is negligible for any Radon measure in X.

6. Applications. Localizable measures and coverable measures play an important role in statistical structures (see [Ma] for background, [LuMu], [RaYa]). A topological space X is said to be *Radon* if each finite Borel measure in X is Radon. Let X be a Radon space. Let $\{\mu_\alpha : \alpha \in A\}$ be a collection of probability Borel measures in X. If there exists a semifinite Borel measure μ such that the Radon–Nikodym derivative $d\mu_\alpha/d\mu$ exists for each α in A, then we have a satisfactory theory of statistical structure ([Ma], 293F, Dominated statistical structure, pp. 873–874). We cannot expect in general that μ is σ-finite and we cannot avoid not necessarily σ-finite localizable measures. The Lebesgue decomposition as well as the Radon–Nikodym theorem for semifinite measures are necessary for statistical structures, and coverable Radon measures play an important role. We note that μ is Radon if μ is semifinite since X is Radon.

We can find the Radon–Nikodym theorem for Radon measures in ([Sc], Theorem 14, p. 47). But it is incomplete since a "Radon–Nikodym derivative" is not measurable. We can give the Radon–Nikodym theorem for a fairly wide class of Radon measures. Under CH the theorem does not hold in general. Together with localizability mentioned above and ([Ku2], Corollary 3.2) we get the following:

6.1. Theorem. *Assume that*

(a) *X is a regular weakly θ-refinable space*, *or*
(b) *X is a regular meta-Lindelöf space and* MA+nonCH *holds true.*

Let ν and μ be Radon measures in X. If ν is absolutely continuous with respect to μ, then there exists a Borel measurable function f such that

$$\nu(E) = \int_E f\,d\mu$$

for any Borel measurable set E.

The Lebesgue decomposition for semifinite measures which are not necessarily σ-finite plays an important role in statistical structures ([RaYa], Theorem, pp. 259–261). For the Lebesgue decomposition, coverability of Radon measures is essential and localizability is not sufficient ([Ku1], Theorem 4.4).

6.2. THEOREM. *Let ν be a Radon measure in a regular space X. If*

(a) *X is weakly θ-refinable*, *or*

(b) *X is meta-Lindelöf and* MA+nonCH *holds true*,

then there exist Radon measures ν_1, ν_2 such that

$$\nu = \nu_1 + \nu_2, \quad \nu_1 \ll \mu, \quad \nu_2 \perp \mu$$

for any Radon measure μ in X.

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