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BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS WITH HOLOMORPHIC KERNELS ON STAR-SHAPED CLOSED LIPSCHITZ CURVES

BY

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Dedicated to Professor S. Igari in honour of his 60th birthday

The aim of this paper is to study singular integrals T generated by holomorphic kernels Φ defined on a natural neighbourhood of the set $\{z\zeta^{-1} : z,\zeta \in \Gamma,\ z \neq \zeta\}$, where Γ is a star-shaped Lipschitz curve, $\Gamma = \{\exp(iz) : z = x + iA(x),\ A' \in L^\infty[-\pi,\pi],\ A(-\pi) = A(\pi)\}$. Under suitable conditions on F and z, the operators are given by

$$(1)\qquad TF(z) = \text{p.v.} \int_\Gamma \Phi(z\eta^{-1})F(\eta)\frac{d\eta}{\eta}.$$

We identify a class of kernels of the stated type that give rise to bounded operators on $L^2(\Gamma,|d\Gamma|)$. We establish also transference results relating the boundedness of kernels on closed Lipschitz curves to corresponding results on periodic, unbounded curves.

1. Introduction. We are interested in holomorphic kernels which satisfy locally the standard Calderón–Zygmund size conditions and give rise to L^2-bounded operators on star-shaped closed Lipschitz curves. Our results identify the class of kernels, and the corresponding Fourier multipliers.

We could consider more general closed curves such as those which can be extended to be regular curves in the sense of David [D]. The reason we restrict ourselves to star-shaped ones is as follows. Denote a closed curve by Γ and let $\gamma = \{\zeta = i^{-1}\log z : z \in \Gamma,\ \operatorname{Re}\zeta \in [-\pi,\pi]\}$. A convolution kernel on γ has to be defined on a neighbourhood of the difference set $D = \{z - \eta : z,\eta \in \gamma,\ z \neq \eta\}$. If Γ is not star-shaped, then D may cover an annular region $0 < |z| < a$. However, according to Laurent series

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theory, a 2π-periodic holomorphic kernel on such a region which satisfies the Calderón–Zygmund size conditions at $z = 0$ is of the form $A\cot(z/2)+\psi(z)$, where A is a constant and ψ is a bounded holomorphic function on a bounded neighbourhood of D. The corresponding singular integral theory on Γ can then be deduced, for instance, from David's theory, by using a partition of unity [D].

Section 2 presents notation and terminology, some previously known results that will be referred to, as well as statements of the results of the paper. Sections 3 and 4 give the details of two alternative proofs of the assertion (ii) of Theorem 2.1, our main result. The proof in Section 3 adapts the proof by Coifman and Meyer in [CM1] for the infinite Lipschitz graph case to our situation, and relies on a Littlewood–Paley type result of Jerison and Kenig [JK]. Section 4 contains a proof based on the result in the infinite graph case [McQ1], the Fourier transform result of [Q], and adaptation of multiplier restriction theorems from non-periodic to periodic curves. Section 5 gives a brief indication of how a martingale version of the $T(b)$ theorem could also be used to give a further proof, adapting the ideas of [CJS], [GLQ], and [T].

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2. Preliminaries. Let γ be a Lipschitz curve defined on the interval $[-\pi,\pi]$ with the parameterization $\gamma(x) = x + iA(x)$, $A : [-\pi,\pi] \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, $A(-\pi) = A(\pi)$, $A' \in L^\infty([-\pi,\pi])$, and $\|A'\|_\infty = N < \infty$. Denote by γ_p the periodic extension of γ to $-\infty < x < \infty$, and by Γ the closed curve

$$\Gamma = \{\exp(iz) : z \in \gamma\} = \{\exp(i(x + iA(x))) : -\pi \le x \le \pi\}.$$

We shall call Γ the *star-shaped Lipschitz curve associated with* γ. We shall frequently identify a curve, thought of as a function, with its range. This is legitimate, since our curves are injective.

If f is a function on γ, we shall denote by f_p its periodic extension to γ_p. We shall usually use upper-case letters to denote functions on Γ. The L^p-spaces are taken with respect to arc-length measure.

For $F \in L^2(\Gamma)$ define

$$\widehat{F}_\Gamma(n) = \frac{1}{2\pi i}\int\limits_{\Gamma} z^{-n}F(z)\frac{dz}{z},$$

the nth Fourier coefficient of F with respect to Γ. We shall sometimes suppress the subscript and write $\widehat{F}(n)$, if no confusion occurs when we do so.

Let
$$\sigma = \exp(-\max A(x)), \qquad \tau = \exp(-\min A(x)).$$
As in [CM1], we consider the following subclass of $L^2(\Gamma, |dz|)$:
$$\mathcal{A}(\Gamma) = \{F(z) : F(z) \text{ is holomorphic in } \sigma - \eta < |z| < \tau + \eta \text{ for some } \eta > 0\}.$$
The subclass is dense. Otherwise, there would exist a non-zero function $g(\zeta) \in L^2(\Gamma) = L^2(\Gamma, |dz|)$ orthogonal to all of $\mathcal{A}(\Gamma)$, and so in particular to $f_z(\zeta) = (\zeta - z)^{-1}$ for z outside the annulus $\sigma - \eta < |z| < \tau + \eta$ and $\sigma - \eta < |\zeta| < \tau + \eta$. We would have, therefore,
$$\int_\Gamma h(\zeta) \frac{d\zeta}{\zeta - z} = 0, \tag{2}$$
where
$$h(\zeta) = \overline{g(\zeta)} \frac{\sqrt{1 + A'(x)^2}}{i - A'(x)} \exp(-ix), \qquad \zeta = \exp(ix - A(x)).$$
Since the integral in (2) is absolutely convergent, (2) would remain valid for all $z \notin \Gamma$, by analytic continuation. Taking, in particular, $z = r\exp(ix - A(x))$ and $z^* = z/r^2$, $0 < r < 1$, we would have, as a consequence of [CMcM],
$$0 = h(\exp(ix - A(x))) = \lim_{r\to 1-} \frac{1}{2\pi i} \int_\Gamma h(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} \right) d\zeta$$
for almost all $x \in [0, 2\pi]$, and so $g(\zeta) = 0$ for almost all $\zeta \in \Gamma$. This is a contradiction.

Without loss of generality, we assume that $A(0) = 0$, and that $\min A(x) < 0$, $\max A(x) > 0$. In this case the domains of the functions in $\mathcal{A}(\Gamma)$ contain the unit circle $\mathbb{T}$. In view of the Cauchy theorem, we have $\widehat{F}_\Gamma(n) = \widehat{F}_\mathbb{T}(n)$. If F and G belong to $\mathcal{A}(\Gamma)$, this remark, together with Laurent series theory, implies the Fourier inversion formula
$$F(z) = \sum_{n=-\infty}^{\infty} \widehat{F}_\Gamma(n) z^n, \tag{3}$$
where z is in the annulus in which F is defined. The Cauchy theorem implies also the Parseval formula
$$\frac{1}{2\pi i} \int_\Gamma F(z) G(z) \frac{dz}{z} = \sum_{n=-\infty}^{\infty} \widehat{F}_\Gamma(n) \widehat{G}_\Gamma(-n). \tag{4}$$

Kernels and their Fourier transforms. We shall use $\arg z$ and $\log z$ to denote the principal branches of argument and logarithm.

We define the following single and double sectors in the complex plane $\mathbb{C}$: for $\omega \in (0, \pi/2]$,

$$\begin{aligned}\mathbf{S}^0_{\omega,+} &= \{z \in \mathbb{C} : |\arg(z)| < \omega,\ z \neq 0\},\\ \mathbf{S}^0_{\omega,-} &= -\mathbf{S}^0_{\omega,+}, \quad \mathbf{S}^0_{\omega} = \mathbf{S}^0_{\omega,+} \cup \mathbf{S}^0_{\omega,-},\end{aligned} \tag{5}$$

and the sets

$$\mathbf{C}^0_{\omega,+} = \mathbf{S}^0_{\omega} \cup \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}, \tag{6}$$

and

$$\mathbf{C}^0_{\omega,-} = \mathbf{S}^0_{\omega} \cup \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}. \tag{7}$$

If $\mathbf{X}$ is any subset of $\mathbb{C}$, let $\mathbf{X}(\pi)$ be the truncated set

$$\mathbf{X}(\pi) = \mathbf{X} \cap \{z \in \mathbb{C} : |\mathrm{Re}(z)| \le \pi\}, \tag{8}$$

and

$$\mathbf{X}_p(\pi) = \bigcup_{k=-\infty}^{\infty} \{\mathbf{X}(\pi) + 2k\pi\} \tag{9}$$

be the periodic set associated with $\mathbf{X}(\pi)$. We shall use sets of the form $\exp(i\mathbf{O}) = \{\exp(iz) : z \in \mathbf{O}\}$, where $\mathbf{O}$ is the truncation of one of the sectors, or is a curve lying in such a set. If $\mathbf{Q}$ is an open subset of $\mathbb{C}$, $H^\infty(\mathbf{Q})$ denotes the function space $\{f : \mathbf{Q} \to \mathbb{C} : f$ is holomorphic and bounded on $\mathbf{Q}\}$. We shall usually take $\mathbf{Q}$ to be a double sector or a single sector. The norm $\|\cdot\|_\infty$ is the sup-norm on the corresponding space $H^\infty(\mathbf{Q})$.

The sectors just defined arise because, if z and ζ, with $z \neq \zeta$, are points of Γ, then the point $z\zeta^{-1}$ appearing in (1) lies in $\exp(i\mathbf{S}^0_\omega)(\pi)$. In a similar way, if z and ζ, with $z \neq \zeta$, are points of a non-periodic Lipschitz curve $x + ia(x)$, $x \in (-\infty, \infty)$, $\|a'\|_\infty = \tan\omega$, then $z - \zeta$ lies in $\mathbf{S}^0_\omega$. So convolution kernels for a Lipschitz curve having Lipschitz constant N are in practice defined on (subsets of) $\mathbf{S}^0_\omega$, where $\tan\omega > N$. The intervention of the sets $\mathbf{C}^0_{\omega,\pm}$ is explained below.

The basic theory relating a holomorphic Calderón–Zygmund kernel ϕ on $\mathbf{S}^0_\omega$ with its Fourier transform b is contained in [McQ1] and [Q]. It is summarised as follows.

Assume $b \in H^\infty(\mathbf{S}^0_\omega)$. Decompose b as $b = b^+ + b^-$, where

$$b^+ = b\chi_{\{z:\mathrm{Re}(z)>0\}}, \qquad b^- = b\chi_{\{z:\mathrm{Re}(z)<0\}}$$

and $b^\pm \in H^\infty(\mathbf{S}^0_{\omega,\pm})$, respectively. In each of the following statements where "$\pm$" occurs, the reader should take "$+$" or "$-$" throughout.

If $b^+ \in H^\infty(\mathbf{S}^0_{\omega,+})$, and $z \in \mathbf{C}^0_{\omega,+}$, we may choose a ray $\varrho^+_\theta = s\exp(i\theta)$, $0 < s < \infty$, in $\mathbf{S}^0_{\omega,+}$ in such a way that $\exp(iz\zeta)$ decays exponentially as

$z \to \infty$ on ϱ_θ^+. So the transform

$$G^+(b^+)(z) = \phi^+(z) = \frac{1}{2\pi} \int\limits_{\varrho_\theta^+} \exp(iz\zeta) b^+(\zeta)\, d\zeta$$

converges. Its value is independent of the choice of ϱ_θ^+, subject to the requirements just stated. In a similar way, we may define

$$G^-(b^-)(z) = \phi^-(z) = \frac{1}{2\pi} \int\limits_{\varrho_\theta^-} \exp(iz\zeta) b^-(\zeta)\, d\zeta$$

for each $z \in \mathbf{C}^0_{\omega,-}$, with ϱ_θ^- chosen appropriately in $\mathbf{S}^0_{\omega,-}$. Define

$$\phi_1^{\pm}(z) = \int\limits_{\delta^{\pm}(z)} \phi^{\pm}(\zeta)\, d\zeta, \quad z \in \mathbf{S}^0_{\omega,+}, \tag{10}$$

where the integral is along a path $\delta^{\pm}(z)$, from $-z$ to z, in $\mathbf{C}^0_{\omega,\pm}$.

The main results in [McQ1] and [Q] are as follows. The constants c_0, c_1, and C are universal constants and $C_{\omega,\mu}$ are constants that depend on ω, μ, and so on. Each may vary from line to line, and even in the same line.

THEOREM A. *Let $\omega \in (0, \pi/2]$ and $b^{\pm} \in H^\infty(\mathbf{S}^0_{\omega,\pm})$. Then $\phi^{\pm} = G^{\pm}(b^{\pm})$ and $\phi_1^{\pm}$ defined as above are holomorphic functions in their domains, and for every $\mu \in (0,\omega)$,*

(i) *we have*

$$|\phi^{\pm}(z)| \le C_{\omega,\mu} \|b^{\pm}\|_\infty / |z|, \quad z \in \mathbf{C}^0_{\mu,\pm}; \tag{11}$$

(ii) $\phi_1^{\pm} \in H^\infty(\mathbf{S}^0_{\mu,+})$, $\|\phi_1^{\pm}\|_{H^\infty(\mathbf{S}^0_{\mu,+})} \le C_{\omega,\mu}\|b^{\pm}\|_\infty$, *and*

$$\phi_1^{\pm\prime}(z) = \phi^{\pm}(z) + \phi^{\pm}(-z), \quad z \in \mathbf{S}^0_{\omega,+}; \quad \text{and}$$

(iii) *we have*

$$(2\pi)^{-1} \int\limits_{-\infty}^{\infty} b^{\pm}(\zeta) \widehat{f}(-\zeta)\, d\zeta = \lim_{\varepsilon \to 0} \Big\{ \int\limits_{|x| \ge \varepsilon} \phi^{\pm}(x) f(x)\, dx + \phi_1^{\pm}(\varepsilon) f(0) \Big\}$$

for all f in the Schwartz class $\mathcal{S}(\mathbb{R})$, where $\widehat{f}$ stands for the standard Fourier transform of f.

THEOREM B. *Let $\omega \in (0, \pi/2]$ and (ϕ, ϕ_1) be a pair of holomorphic functions defined on $\mathbf{S}^0_\omega$ and $\mathbf{S}^0_{\omega,+}$, respectively, which satisfy*

(i) *there is a constant c_0 such that*

$$|\phi(z)| \le c_0/|z|, \quad z \in \mathbf{S}^0_\omega; \tag{12}$$

(ii) *there is a constant c_1 such that $\|\phi_1\|_{H^\infty(\mathbf{S}^0_{\omega,+})} \le c_1$, and*

$$\phi_1'(z) = \phi(z) + \phi(-z), \quad z \in \mathbf{S}^0_{\omega,+}.$$

Then there is a unique function b such that $b \in H^\infty(\mathbf{S}^0_\mu)$ for every $\mu \in (0,\omega)$,

$$\|b\|_{H^\infty(\mathbf{S}^0_\mu)} \le C_{\omega,\mu}(c_0 + c_1),$$

and the function pair determined by b according to Theorem A is identical to (ϕ,ϕ_1). Moreover, for all complex numbers $\zeta \in \mathbf{S}^0_\omega$, the function b is given by

$$b(\zeta) = \lim_{\varepsilon\to 0}\lim_{N\to 0}\Big\{\int\limits_{\varepsilon<|x|<N} \exp(-i\zeta x)\phi(x)\,dx + \phi_1(\varepsilon)\Big\}. \tag{13}$$

Remark. If $\phi|_{\mathbb{R}}$, the restriction of ϕ to $\mathbb{R}$, is a good enough function, for instance, if $\phi|_{\mathbb{R}}$ is in $L^2(\mathbb{R})\cap L^1(\mathbb{R})$, then $b|_{\mathbb{R}}$ is the standard Fourier transform of $\phi|_{\mathbb{R}}$, $\lim_{\varepsilon\to 0}\phi_1(\varepsilon z) = 0$ for $z \in \mathbf{S}^0_{\omega,+}$, and Theorem A(iii) reduces to the standard Parseval equation.

Let $\gamma(x) = x + ia(x)$, $-\infty < x < \infty$, be a bounded Lipschitz curve, and define (see also [CM1]) $\mathcal{A}(\gamma)$ to be the space of functions f which are, for some $\eta > 0$ depending on f, holomorphic in the strip $\min a(x) - \eta < \operatorname{Im} z < \max a(x) + \eta$, and $\sup_y \int_{-\infty}^{\infty} |f(x+iy)|^2\,dx \le C_\eta < \infty$.

For $f \in \mathcal{A}(\gamma)$ one can define the Fourier transform of f:

$$\widehat{f}_\gamma(\xi) = \int\limits_\gamma \exp(-i\xi z)f(z)\,dz, \qquad \xi \in \mathbb{R}.$$

It is easy to verify, by using the Cauchy theorem, that the Parseval formula holds:

$$\int\limits_\gamma f(z)g(z)\,dz = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \widehat{f}(\xi)\widehat{g}(-\xi)\,d\xi, \qquad f,g \in \mathcal{A}(\gamma).$$

Let $\omega > \arctan\|a'\|_\infty$. For a function $b \in H^\infty(\mathbf{S}^0_\omega)$ one can formally define a multiplier operator on $\mathcal{A}(\gamma)$:

$$m_b(f)(z) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \exp(i\xi z)b(\xi)\widehat{f}_\gamma(\xi)\,d\xi.$$

For a pair of functions (ϕ,ϕ_1) as specified in Theorems A and B, with $\omega > \arctan\|a'\|_\infty$, one can define the following integral operator on $\mathcal{A}(\gamma)$:

$$T_{(\phi,\phi_1)}(f)(z) = \lim_{\varepsilon\to 0}\Big\{\int\limits_{|\zeta-z|>\varepsilon,\,\zeta\in\gamma} \phi(z-\zeta)f(\zeta)\,d\zeta + \phi_1(\varepsilon t(z))f(z)\Big\}, \qquad z \in \gamma,$$

where $t(z)$ is the normalized tangent vector to γ at the point $z \in \gamma$. It lies inside $\mathbf{S}^0_{\omega,+}$.

Now we are ready to state

THEOREM C. *Let $\omega \in (\arctan\|a'\|_\infty, \pi/2]$ and (ϕ,ϕ_1) be associated with $b \in H^\infty(\mathbf{S}^0_\omega)$ as in Theorem A. Then $m_b = T_{(\phi,\phi_1)}$, and m_b extends to a bounded operator on $L^2(\gamma,|d\gamma|)$.*

The converse to Theorem C is proved in [McQ2]. The following results were proved in [Q].

THEOREM D. *Let $\omega \in (0,\pi/2]$ and (Φ,Φ_1) be a pair of holomorphic functions defined on $\exp(i\mathbf{S}^0_\omega(\pi))$ and $\exp(i\mathbf{S}^0_{\omega,+}(\pi))$, respectively, that satisfy*

(i) *there is a constant c_0 such that*

$$|\Phi(z)| \le c_0/|1-z|, \qquad z \in \exp(i\mathbf{S}^0_\omega(\pi)); \tag{14}$$

(ii) *there is a constant c_1 such that $\|\Phi_1\|_{H^\infty(\exp(i\mathbf{S}^0_{\omega,+}(\pi)))} \le c_1$, and*

$$\Phi_1'(z) = \frac{1}{iz}(\Phi(z)+\Phi(z^{-1})), \qquad z \in \exp(i\mathbf{S}^0_{\omega,+}(\pi)).$$

Then there exists a unique function b in $H^\infty(\mathbf{S}^0_\mu)$, $\mu \in (0,\omega)$, such that

$$\|b\|_{H^\infty(\mathbf{S}^0_\mu)} \le C_{\omega,\mu}(c_0+c_1)$$

and

$$\sum_{\substack{n=-\infty\\ n\ne 0}}^{\infty} b(n)\widehat{F}_{\mathbb{T}}(-n) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\Big\{\int_{\pi\ge|\log z|>\varepsilon,\, z\in\mathbb{T}} \Phi(z)F(z)\frac{dz}{z} + \Phi_1(\exp(i\varepsilon))F(1)\Big\} \tag{15}$$

for all smooth functions F on $\mathbb{T}$ such that $\widehat{F}_{\mathbb{T}}(0)=0$, where $\widehat{F}_{\mathbb{T}}(n)$ is the n-th standard Fourier coefficient of F. Moreover,

$$b(\eta) = \lim_{\varepsilon\to 0}\Big\{\int_{\pi\ge|\log z|>\varepsilon,\, z\in\mathbb{T}} z^{-\eta}\Phi(z)\frac{dz}{z} + \Phi_1(\exp(i\varepsilon))\Big\}$$

for any $\eta \in \mathbf{S}^0_\omega$.

THEOREM E. *Let $\omega\in(0,\pi/2]$ and $b\in H^\infty(\mathbf{S}^0_\omega)$. Then the functions*

$$\Phi^{\pm}(z) = \pm\sum_{n=\pm1}^{\pm\infty} b(n)z^n, \qquad |z^{\pm1}|<1, \tag{16}$$

can be holomorphically extended to $\exp(i\mathbf{C}^0_{\omega\pm}(\pi))$. If $\Phi=\Phi^++\Phi^-$, then the inequality (14) *is satisfied for $0<\mu<\omega$, and some constants $C_{\omega,\mu}$. Let $\Phi_1^{\pm}$ be defined on $\exp(i\mathbf{S}^0_\omega(\pi))$ by*

$$\Phi_1^{\pm}(z) = \int_{\delta^{\pm}(z)} \Phi^{\pm}(\eta)\frac{d\eta}{\eta}, \qquad z\in\exp(iS^0_{\omega,+}(\pi)), \tag{17}$$

where $\delta^{\pm}(z)$ is a path in $\exp(i\mathbf{C}^0_{\omega,\pm}(\pi))$ from z^{-1} to z. Let $\Phi_1=\Phi_1^++\Phi_1^-$. Then the Parseval formula (15) *holds with respect to the function b and the* *pair* (Φ,Φ_1). *Moreover*, *the* H^∞ *function determined by* (Φ,Φ_1) *according to Theorem D is identical to* b.

Statement of results. For a function $b\in H^\infty(\mathbf{S}^0_\omega)$ we define the operator $M_b:\mathcal{A}(\Gamma)\to\mathcal{A}(\Gamma)$ by

$$M_bF(z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} b(n)\widehat{F}_\Gamma(n)z^n.$$

Let (Φ,Φ_1) be determined by b as in Theorem E. There is a corresponding operator associated with the function pair (Φ,Φ_1). Its action on smooth functions on Γ is given by

$$\begin{aligned}(18)\quad & T_{(\Phi,\Phi_1)}F(z)\\ &=\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\Big\{\int\limits_{|\log(\eta z^{-1})|>\varepsilon,\,\eta\in\Gamma}\Phi(z\eta^{-1})F(\eta)\frac{d\eta}{\eta}+\Phi_1(t(z)^\varepsilon)F(z)\Big\},\end{aligned}$$

where $t(z)$ is the normalized tangent vector to Γ at z.

With this background, we can now state the Main Theorem.

THEOREM 2.1 (Main Theorem). *Let* $\omega\in(\arctan N,\pi/2]$, $b\in H^\infty(\mathbf{S}^0_\omega)$ *and* (Φ,Φ_1) *be associated with* b *as in Theorem E. Then*

(i) $T_{(\Phi,\Phi_1)}=M_b$ *on* $\mathcal{A}(\Gamma)$;
(ii) M_b *extends to a bounded operator on* $L^2(\Gamma)$.

The assertion (i) of the Main Theorem was proved in [Q]. We shall devote ourselves to proofs of the assertion (ii) in the rest of the paper.

Remark. Theorem 2.1 has important consequences. For instance, let Φ be as above, and let T be the operator

$$TF(z)=\int\limits_\Gamma \Phi(z\zeta^{-1})F(\zeta)\frac{d\zeta}{\zeta},$$

defined, *a priori*, for $z\notin\Gamma$ and F, say, continuous on Γ. Then the non-tangential boundary limit $TF(z_0)$ of $TF(z)$ exists for a.e. $z\in\Gamma$, and defines an operator which is L^2-bounded from $C(\Gamma)$ to $L^2(\Gamma)$.

3. A proof using Littlewood–Paley theory. We begin by introducing the space $H^2(\Gamma)$ by using non-tangential maximal functions and non-tangential limits of holomorphic functions. We then use a characterisation of $H^2(\Gamma)$ in terms of an appropriate g-function. This is analogous to the approach of Jerison and Kenig [K] for (non-periodic) unbounded Lipschitz curves.

DEFINITION 1. Let Δ and Δ^{c} be the bounded and the unbounded connected components of $\mathbb{C}\setminus\Gamma$, and $\alpha>0$. The *non-tangential approach* *regions* $\Lambda_\alpha(z)$ and $\Lambda^{\mathrm{c}}_\alpha(z)$ to a point $z \in \Gamma$ are defined by

$$\Lambda_\alpha(z) = \Lambda_\alpha(z, \Delta) = \{\zeta \in \Delta : |z - \zeta| < (1+\alpha)\operatorname{dist}(\zeta, \Gamma)\}.$$

The definition of $\Lambda^{\mathrm{c}}_\alpha(z)$ uses Δ^{c} in place of Δ.

It is known ([K, [JK]) that there exists a positive constant α_0, depending only on the Lipschitz constant N, such that $\Lambda_\alpha(z) \subset \Delta$ and $\Lambda^{\mathrm{c}}_\alpha(z) \subset \Delta^{\mathrm{c}}$ for $0 < \alpha < \alpha_0$ and all $z \in \Gamma$. The arguments given below are valid for each fixed α in the range $(0, \alpha_0)$, and give rise to the same spaces irrespective of the choice of α. We choose and fix α.

Definition 2. The *interior non-tangential maximal function* $N_\alpha(F)$ of a function F on Δ is

$$N_\alpha(F)(z) = \sup\{|F(\zeta)| : \zeta \in \Lambda_\alpha(z)\}, \qquad z \in \Gamma.$$

A similar definition applies to the *exterior maximal function* $N^{\mathrm{c}}_\alpha(F)$, for functions defined on Δ^{c}.

Definition 3. Given $0 < p < \infty$, the Hardy space $H^p(\Delta)$ is defined by

$$H^p(\Delta) = \{F : F \text{ is holomorphic in } \Delta, \text{ and } N_\alpha(F) \in L^p(\Gamma)\}.$$

The norm $\|F\|_{H^p(\Delta)}$ is defined by

$$\|F\|_{H^p(\Delta)} = \|N_\alpha(F)\|_{L^p(\Gamma)} = \Big(\int\limits_\Gamma N_\alpha(F)^p(z)\,|dz|\Big)^{1/p}.$$

The space $H^p(\Delta^{\mathrm{c}})$ is defined similarly, except that the functions in $H^p(\Delta^{\mathrm{c}})$ are assumed to vanish at infinity.

Proposition 3.1. *If $F \in H^p(\Delta)$, $p > 1$, then the non-tangential limit of F,*

$$\lim_{\zeta\to z,\,\zeta\in\Lambda_\alpha} F(\zeta),$$

exists almost everywhere with respect to arc-length measure on Γ. If the limit is denoted by $F_\Gamma(z)$, $z \in \Gamma$, then $F_\Gamma \in L^p(\Gamma, |d\Gamma|)$, and

$$C_N\|F\|_{H^p(\Delta)} \le \|F_\Gamma\|_{L^p(\Gamma,|dz|)} \le C'_N\|F\|_{H^p(\Delta)}, \qquad F \in H^p(\Delta),$$

where C_N, C'_N depend on p and the Lipschitz constant N. A similar statement holds for the exterior component Δ^{c}.

Proposition 3.2. *Suppose that $F \in H^2(\Delta)$. Then the norm $\|F\|_{H^2(\Delta)}$ is equivalent to the norm*

$$(19)\quad \Big(\int\limits_{-\pi}^{\pi}\int\limits_0^\infty (1-\exp(-\eta))^{2j-1}|D^jF(\exp i(x+iA(x)+i\eta))|^2\,d\eta\,dx\Big)^{1/2},$$

where $D = iz\frac{d}{dz}$. A similar statement applies to $F \in H^p(\Delta^{\mathrm{c}})$.

Proofs of Propositions 3.1 and 3.2 can be found in [JK]. The following is a consequence of [CMcM].

PROPOSITION 3.3. *Suppose that* $F_\Gamma \in L^2(\Gamma)$. *Then there exist* $F^+ \in H^2(\Delta)$, $F^- \in H^2(\Delta^c)$ *such that their boundary values* $F^\pm_\Gamma$ *lie in* $L^2(\Gamma)$, *and* $F_\Gamma = F^+_\Gamma + F^-_\Gamma$. *The mappings* $F_\Gamma \mapsto F^\pm_\Gamma$ *are continuous on* $L^2(\Gamma)$.

In the following lemma, we use the notation Γ_η, $\eta > 0$, for the curve $\exp i(x + iA(x) + i\eta)$.

LEMMA 3.4. *Suppose* $z_0 = \exp(i(x_0 + iA(x_0))) \in \Gamma$. *Let* $\eta > 0$, *and* $z = \exp(i(x_0 + iA(x_0) + i\eta))$, $\eta > 0$, *be the corresponding point on* Γ_η. *Then there is a constant* C_N, *depending on the Lipschitz constant of* Γ, *such that*

$$|1 - z\zeta^{-1}| \geq C_N\{(1 - \exp(-\eta/2))^2 + |x - x_0|^2\}^{1/2}, \tag{20}$$

for all points $\zeta = \exp(i(x + iA(x) + i\eta/2))$, $-\pi \leq x \leq \pi$, *in* $\Gamma_{\eta/2}$.

Proof. In fact,

$$\begin{aligned}|1 - z\zeta^{-1}|^2 &= |1 - \beta\exp(i(x - x_0))|^2 \\ &= (1-\beta)^2 + 4\beta\sin^2\left(\frac{x - x_0}{2}\right) \\ &\geq (1-\beta)^2 + \frac{4\beta}{\pi^2}(x - x_0)^2,\end{aligned}$$

where $\beta = \exp(A(x) - A(x_0) - \eta/2)$.

We have two cases to consider.

Case 1: $\eta > 2(1 + N\pi)$. In this case $1 - \beta$ has a positive lower bound. Since the right-hand side of (20) is bounded above by a constant, it is less than a constant multiple of the right side of (21).

Case 2: $\eta \leq 2(1 + N\pi)$. In this case $\beta > \exp(-(1 + 2N\pi))$, and so

$$|1 - z\zeta^{-1}|^2 \geq (1-\beta)^2 + \frac{4}{\pi^2}\exp(-(1 + 2N\pi))|x - x_0|^2. \tag{22}$$

There are three subcases to Case 2.

(i) $A(x) - A(x_0) \leq \eta/4$. In this case $1 - \beta > 1 - \exp(-\eta/4) > C(1 - \exp(-\eta/2))$. Sustituting in (22), we get the desired inequality.

(ii) $\eta/4 < A(x) - A(x_0) \leq \eta$. In this case

$$\eta/4 < A(x) - A(x_0) \leq N|x - x_0|,$$

and so

$$|x - x_0| > \frac{\eta}{4N} \geq \frac{1}{4N}\left(1 - \exp\left(-\frac{\eta}{2}\right)\right).$$

Therefore,

$$|x - x_0| > \frac{1}{2}|x - x_0| + \frac{1}{8N}\left(1 - \exp\left(-\frac{\eta}{2}\right)\right).$$

Substituting in (22) and ignoring the entry related to $1-\beta$, we obtain the desired inequality.

(iii) $A(x)-A(x_0)>\eta$. In this case $\beta>\exp(\eta/2)>1$, and so

$$(1-\beta)^2>(1-\exp(\eta/2))^2\geq(1-\exp(-\eta/2))^2.$$

Substituting in (7), we obtain the desired inequality. ∎

Proof of Theorem 2.1(ii). Let $F\in\mathcal{A}(\Gamma)$, and suppose Φ is a kernel satisfying the conditions of Theorem 2.1. Define, for $z\notin\Gamma$,

$$T_\Phi F(z)=\frac{1}{2\pi i}\int_\Gamma \Phi(z\zeta^{-1})F(\zeta)\frac{d\zeta}{\zeta}. \tag{23}$$

It can be shown, using the techniques of [Q, Theorem 1], that the non-tangential limit of (23) as $z\to z_0$, $z_0\in\Gamma$, is equal to the right-hand side of (18).

Using the Laurent expansion (3), write $F=F^++F^-$, where F^+ is holomorphic in Δ and F^- is holomorphic in Δ^{c}. Using Theorem E, in particular (16), write $\Phi=\Phi^++\Phi^-$. Note that Φ^+ and Φ^- satisfy the same kernel conditions as Φ. It follows from (16) that

$$T_\Phi(F)=T_{\Phi^+}(F^+)+T_{\Phi^-}(F^-).$$

In order to prove Theorem 2.1(ii), it suffices, by Propositions 3.1 and 3.3, to show that the operators T_{Φ^+} and T_{Φ^-} are bounded on $H^2(\Delta)$ and $H^2(\Delta^{\mathrm{c}})$ respectively. We shall establish the boundedness of T_{Φ^+} by using Proposition 3.2. The proof of the boundedness of the complementary operator is similar.

We suppress the superscript "+" , and consider the operator T_Φ on $H^2(\Delta)$. It is easy to verify that

$$DT_\Phi F(z)=\int_\Gamma \Phi(z\zeta^{-1})DF(\zeta)\frac{d\zeta}{\zeta}.$$

Repeating the procedure and changing the integral contour, we have

$$D^2TF(z)=\int_{\Gamma_{\eta/2}} D\Phi(z\zeta^{-1})DF(\zeta)\frac{d\zeta}{\zeta}.$$

Now

$$|D\Phi(z)|\leq C\frac{1}{|1-z|^2},\qquad z\in\exp(i\mathbf{S}^0_\mu(\pi)).$$

So, by Lemma 3.4, we have

$$
\begin{aligned}
&|D^2TF(z)|\\
&\quad\le C\Big(\int\limits_{\Gamma_{\eta/2}} |D\Phi(z\zeta^{-1})|\frac{|d\zeta|}{|\zeta|}\Big)^{1/2}\Big(\int\limits_{\Gamma_{\eta/2}} |D\Phi(z\zeta^{-1})||DF(\zeta)|^2\frac{|d\zeta|}{|\zeta|}\Big)^{1/2}\\
&\quad\le C\Big(\int\limits_{\Gamma_{\eta/2}} \frac{1}{|1-z\zeta^{-1}|^2}\frac{|d\zeta|}{|\zeta|}\Big)^{1/2}\Big(\int\limits_{\Gamma_{\eta/2}} \frac{1}{|1-z\zeta^{-1}|^2}|DF(\zeta)|^2\frac{|d\zeta|}{|\zeta|}\Big)^{1/2}\\
&\quad\le C\Big(\int\limits_{-\pi}^{\pi} \frac{1}{(x-x_0)^2+(1-e^{-\eta/2})^2}\,dx\Big)^{1/2}\\
&\qquad\times\Big(\int\limits_{\Gamma_{\eta/2}} \frac{1}{(x-x_0)^2+(1-e^{-\eta/2})^2}\\
&\qquad\times|DF(\exp(i(x+iA(x)+i\eta/2)))|^2\,dx\Big)^{1/2}\\
&\quad\le C(1-e^{-\eta/2})^{-1/2}\Big(\int\limits_{\Gamma_{\eta/2}} \frac{|DF(\exp(i(x+iA(x)+i\eta/2)))|^2}{(x-x_0)^2+(1-e^{-\eta/2})^2}\,dx\Big)^{1/2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int\limits_{-\pi}^{\pi}\int\limits_{0}^{\infty}(1-e^{-\eta})^3|D^2F(\exp(i(x_0+iA(x_0)+i\eta)))|^2\,d\eta\,dx_0\\
&\qquad\le C\int\limits_{-\pi}^{\pi}\int\limits_{0}^{\infty}(1-e^{-\eta/2})^2\\
&\qquad\quad\times\Big(\int\limits_{-\pi}^{\pi}\frac{|DF(\exp(i(x+iA(x)+i\eta/2)))|^2}{(x-x_0)^2+(1-e^{-\eta/2})^2}dx\Big)d\eta\,dx_0\\
&\qquad\le C\int\limits_{-\pi}^{\pi}\int\limits_{0}^{\infty}|DF(\exp(i(x+iA(x)+i\eta/2)))|^2\\
&\qquad\quad\times\Big(\int\limits_{-\pi}^{\pi}\frac{(1-e^{-\eta/2})^2}{(x-x_0)^2+(1-e^{-\eta/2})^2}dx_0\Big)d\eta\,dx\\
&\qquad\le C\int\limits_{-\pi}^{\pi}\int\limits_{0}^{\infty}(1-e^{-\eta/2})|DF(\exp(i(x+iA(x)+i\eta/2)))|^2\,d\eta\,dx.
\end{aligned}
$$

So, by Proposition 3.3, the boundedness of T_Φ on $H^2(\Delta)$ has been established. ∎

4. Restriction of multipliers. Assume that $b \in H^\infty(\mathbf{S}^0_\omega)$ with $\omega \in (\arctan \|A'\|_\infty, \pi/2]$. Suppose $f \in \mathcal{A}(\gamma_p)$ and $g \in \mathcal{A}(\gamma)$. Define

$$m_b f(z) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{itz} b(t) \widehat{f}_{\gamma_p}(t)\, dt, \tag{24}$$

$$M_b g(z) = \sum_{n\neq 0} b(n) \widehat{g}_\gamma(n) e^{inz}, \tag{25}$$

where, if f is integrable on γ_p, then

$$\widehat{f}_{\gamma_p}(t) = \int\limits_{\gamma_p} \exp(-itz) f(z)\, dz,$$

$$\widehat{g}_\gamma(n) = \frac{1}{2\pi} \int\limits_{\gamma} \exp(-inz) g(z)\, dz.$$

We will show that the boundedness of m_b, which was proved in [McQ2], implies the boundedness of M_b. In fact, we shall prove more: we shall show how to "restrict" multipliers of $L^p(\gamma_p)$ to multipliers of $L^p(\gamma)$, thus extending well-known techniques of de Leeuw [deL]. Our arguments are modifications of the corresponding arguments in [SW]. See also [EG].

Denote by w_t, $t > 0$, the Gauss–Weierstrass kernels $w_t(z) = e^{-\pi t z^2}$.

LEMMA 4.1. (i) *If $1 \le p < \infty$, then the set $\{e^{imz}\}_{m=-\infty}^{\infty}$ spans a dense subspace of $L^p(\gamma, |d\gamma|)$.*

(ii) *If $f \in L^1(\gamma)$, and f_p is its periodic extension to γ_p, then*

$$\lim_{\varepsilon\to 0^+} \sqrt{\varepsilon} \int\limits_{\gamma_p} |f_p(z)|\, |e^{-\varepsilon\pi z^2}|\, |dz| = \int\limits_{\gamma} |f(z)|\, |dz|. \tag{26}$$

(iii) *If f and g are in the span of the sets $\{e^{imz}\}_{m\neq 0}$ and $\{e^{imz}\}_{m=-\infty}^{\infty}$ respectively, and f_p and g_p are their periodic extensions, then*

$$\lim_{\varepsilon\to 0} \sqrt{\varepsilon} \int\limits_{\gamma_p} m_b(f_p w_{\varepsilon\alpha}) g_p w_{\varepsilon\beta}\, dz = \int\limits_{\gamma} (M_b f) g\, dz \tag{27}$$

if $\alpha > 0$, $\beta > 0$, and $\alpha + \beta = 1$.

Proof. The mappings γ and γ_p have Jacobian determinant equal to $\sqrt{1 + A'(t)^2}$, which is uniformly bounded, and uniformly bounded away from 0. By change of variables, γ induces identifications, with norm equivalence, between the L^p-spaces on γ, with respect to arc length, with those on $[-\pi, \pi]$. In a similar way, the change of variables γ_p identifies the (nonperiodic) L^p spaces on γ_p with those on $(-\infty, \infty)$. The statement (i) therefore follows immediately from the density of the space of trigonometric polynomials in $L^p[-\pi, \pi]$.

(ii) It follows from the periodicity of f_p and γ_p that

$$(28)\quad \begin{aligned}&\sqrt{\varepsilon}\int_{\gamma_p} |f_p(z)|\,|e^{-\varepsilon\pi z^2}|\,|dz|\\ &= \int_{-\infty}^{\infty} |f_p(t+iA(t))|e^{-\varepsilon\pi(t^2-A(t)^2)}\sqrt{1+A'(t)^2}\,dt\\ &= \sqrt{\varepsilon}\int_{-\pi}^{\pi} |f(t+iA(t))|e^{\varepsilon\pi A(t)^2}\sqrt{1+A'(t)^2}\sum_{k=-\infty}^{\infty} e^{-\varepsilon\pi(t+2k\pi)^2}\,dt.\end{aligned}$$

The limit, as $\varepsilon\to 0$, of the expression on the right side of (28) is

$$\int_{-\pi}^{\pi} |f(t+iA(t))|\sqrt{1+A'(t)^2}\,dt,$$

by a familiar argument [SW, p. 261].

(iii) Let $f(z)=e^{imz}$, $g(z)=e^{-ikz}$ with $m,k\in\mathbb{Z}$ and $m\neq 0$. Then by (24), and Cauchy's theorem,

$$\begin{aligned}&\sqrt{\varepsilon}\int_{\gamma_p} m_b(f_p w_{\varepsilon\alpha})g_p w_{\varepsilon\beta}\,dz\\ &= \frac{\sqrt{\varepsilon}}{2\pi}\int_{-\infty}^{\infty} b(x)\int_{\gamma_p} e^{-ix\zeta}e^{im\zeta}e^{-\pi\varepsilon\alpha\zeta^2}\,d\zeta\int_{\gamma_p} e^{ixz}e^{-ikz}e^{-\pi\varepsilon\beta z^2}\,dz\,dx\\ &= \frac{\sqrt{\varepsilon}}{2\pi}\int_{-\infty}^{\infty} b(x)\int_{-\infty}^{\infty} e^{-i(x-m)y}e^{-\pi\varepsilon\alpha y^2}\,dy\int_{-\infty}^{\infty} e^{i(x-k)u}e^{-\pi\varepsilon\beta u^2}\,du\,dx\\ &= \frac{1}{2\pi\sqrt{\varepsilon\alpha\beta}}\int_{-\infty}^{\infty} b(x)\,e^{-(x-m)^2/(4\varepsilon\alpha\pi)}e^{-(x-k)^2/(4\varepsilon\beta\pi)}\,dx.\end{aligned}$$

According to the argument in [SW, pp. 261–262], the limit, as $\varepsilon\to 0$, is 0 if $m\neq k$, and $b(m)$ if $m=k\neq 0$, since b is bounded and continuous away from 0. This equals the right-hand side of (27). ∎

THEOREM 4.2. *Suppose that* $1\le r<\infty$, *and that* m_b *defined by* (24) *is a bounded operator on* $L^r(\gamma_p)$. *Then the operator* M_b *defined by* (25) *is bounded on* $L^r(\gamma)$ *and*

$$\|M_b\|_r \le C\|m_b\|_r,$$

where C *is a constant dependent only on the Lipschitz constant of the curve* γ.

Proof. Let f, g, f_p and g_p be as in Lemma 4.1(iii). Since m_b is a bounded operator on $L^r(\gamma_p)$, we have, for any $\varepsilon > 0$,

$$\Big| \int\limits_{\gamma_p} m_b(f_p w_{\varepsilon\alpha}) g_p w_{\varepsilon\beta}\, dz \Big| \le \|m_b\|\, \|f_p w_{\varepsilon\alpha}\|_{L^r(\gamma_p)} \|g_p w_{\varepsilon\beta}\|_{L^{r'}(\gamma_p)}.$$

Multiplying both sides of the above inequality by $\sqrt{\varepsilon}$ and taking the limit as $\varepsilon \to 0+$, the left-hand side becomes

$$\Big| \int\limits_{\gamma} (M_b f) g\, dz \Big|$$

by Lemma 4.1(iii), while the right-hand side becomes

$$\begin{aligned}\lim_{\varepsilon\to 0+} \|m_b\| \Big(\sqrt{\varepsilon} \int\limits_{\gamma_p} |f_p(z)\, e^{-\varepsilon\alpha z^2}|^r\, |dz|\Big)^{1/r} \Big(\sqrt{\varepsilon} \int\limits_{\gamma_p} |g_p(z)\, e^{-\varepsilon\beta z^2}|^{r'}\, |dz|\Big)^{1/r'} \\ = \|m_b\| \Big(\int\limits_{\gamma} |f(z)|^r\, |dz|\Big)^{1/r} \Big(\int\limits_{\gamma} |g(z)|^{r'}\, |dz|\Big)^{1/r'}\end{aligned}$$

if we choose $\alpha = 1/r$, $\beta = 1/r'$ and use Lemma 4.1(ii). ∎

5. A proof using the $T(b)$ theorem. It is possible to prove the boundedness of the operator T_Φ by using an appropriate version of the $T(b)$ theorem. We sketch the initial steps only. The details may be filled in by adapting the ideas of [GLQ] and [T].

Let Φ, with associated primitive function Φ_1, be as in Section 2. Let h be the function defined on $\mathbf{S}^0_\omega(\pi)$ (see (5) and (8)) by setting

$$h(z) = \Phi(\exp iz). \tag{29}$$

Let h_1 be the associated function $h_1(z) = \Phi_1(\exp iz)$, defined on $\mathbf{S}^0_{\omega,+}(\pi)$. It may be regarded as a function on the periodic set $\mathbf{S}^0_{\omega,p}(\pi)$ (see (9)).

If we use the parameterization $\gamma(x) = x + iA(x)$, then, modulo a correction term dominated by the Hardy–Littlewood maximal function, the singular integral operator (18) is equal to the principal value operator

$$\begin{aligned} Sf(x) &= \lim_{\varepsilon\to 0} \Big(\int\limits_{y\in[-\pi,\pi],\, |y-x|>\varepsilon} h(x + iA(x) - y - iA(y)) f(y)(1 + iA'(y))\, dy \\ &\quad + h_1(\varepsilon t(x + iA(x))) f(x)\Big), \end{aligned} \tag{30}$$

where $t(x+iA(x)) = (1+iA'(x))/\sqrt{1 + A'(x)^2}$ is the normalized derivative of γ at $x + iA(x)$. The correction term arises from the fact that the region of integration in (30) does not correspond exactly to that in (18) under the change of variables $z \mapsto (1/i)\log z$. The operator S in (30) acts on functions on $Q = \mathbb{R}/2\pi\mathbb{Z}$. Regard Q as an additive group, with the natural distance function $|\cdot|$.

Replace (30) by the operator

$$\begin{aligned}(31)\qquad Tf(x) = \lim_{\varepsilon\to 0}\Big(&\int\limits_{y\in Q,\,|y-x|>\varepsilon} h(x+iA(x)-y-iA(y))f(y)\,dy \\ &+ h_1(\varepsilon t(x+iA(x)))f(x)\Big).\end{aligned}$$

Since $1 \le |b(x)| \le 1+\|A\|_\infty$, the boundedness of the operator S is reduced to that of T. Denote by b the function

$$(32)\qquad b(x) = 1 + iA'(x).$$

Then b is *pseudo-accretive*: there exists a constant C such that

$$(33)\qquad \frac{1}{C} \le \Big|\frac{1}{|I|}\int\limits_I b(x)\,dx\Big| \le C$$

for all non-degenerate subintervals I of Q.

Consider the family of finite σ-algebras $\mathcal{F}_k$ $(k\ge 0)$, $\mathcal{F}_k$ being generated by the set of dyadic intervals of Q of order k, also regarded as periodic subsets of $\mathbb{R}$. The conditional expectation operators E_k associated with the weight b are given by $\{E_k\}_{k=0}^\infty$:

$$E_k(f)(x) = \Big(\int\limits_I f(y)b(y)\,dy\Big)\Big/\Big(\int\limits_I b(y)\,dy\Big) \qquad (x\in I)$$

for each dyadic subinterval of order k. The corresponding martingale difference sequence

$$(34)\qquad \Delta_k f = E_k f - E_{k+1} f$$

has an associated Littlewood–Paley theorem (cf. [CGQ], [GLQ]).

The kernel

$$K(x,y) = h(x+iA(x)-y-iA(y))$$

associated with the operator T in (31) is of Calderón–Zygmund type. The operator T can be shown to satisfy the conditions of the $T(b)$ theorem by using Cauchy's theorem together with the decay properties of the kernel and the fact that the curve satisfies a Lipschitz condition. The details may be filled in by adapting the approach in [GLQ] and [T].

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