ON THE COMPLEXITY OF $H$ SETS OF THE UNIT CIRCLE

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Let $K(T)$ be the space of all compact subsets of the circle group $T$ equipped with its natural (metric, compact) topology. Recently T. Linton ([8]) showed that an important class of thin sets from harmonic analysis, the $H$ sets, form a true $\Sigma^0_3\; (G_{\sigma\delta})$ subset of $K(T)$, that is, a $\Sigma^0_3$ set which is not $\Pi^0_3\; (F_{\sigma\delta})$. In this note, we generalize his result by showing that if $E \in K(T)$ is an $M$ set (see the definition below), then the $H$ sets contained in $E$ also form a true $\Sigma^0_3$ subset of $K(T)$. In fact, the result is somewhat more general and shows that several related classes of thin sets are true $\Sigma^0_3$ within any $M$ set.

Before stating precisely the result, we have to introduce some definitions. We denote by $A$ the Banach algebra of all continuous complex-valued functions on $T$ with absolutely convergent Fourier series. The norm of $f \in A$ is given by $\|f\|_A = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|$. Thus the Fourier transform identifies $A$ with $l^1(\mathbb{Z})$. The dual space of $A$ is the space $PM(\sim l^\infty)$ of all distributions on $T$ with bounded Fourier coefficients, and $A$ itself is the dual of the space $PF(\sim c_0)$ of pseudofunctions (distributions with Fourier coefficients tending to 0).

**Definition 1.** A closed set $E \subseteq T$ is said to be a set of uniqueness, or a $U$ set (resp. a $U_0$ set) if it supports no non-zero pseudofunction (resp. no probability measure in $PF$). $E$ is an $M$ set ($M_0$ set) if it is not in $U$ ($U_0$).

**Definition 2.** A closed set $E \subseteq T$ is said to be an $H$ set if there exists a non-empty open set $V \subseteq T$ and an infinite sequence $(m_k)$ of positive integers such that for all $k$, $m_k E \cap V = \emptyset$ (where $mE = \{mx : x \in E\}$).

Evidently $U \subseteq U_0$ but the converse is not true. It is well known that every $H$ set is a set of uniqueness; in fact, $H$ is only a very small part of $U$. All these sets have a long history and we refer to [1], [4] or [7] for much more information.

It is not difficult to check that $H$ is $\Sigma^0_3$ in $K(T)$ (a proof is given in [8]). Now the easiest way to show that it is not $\Pi^0_3$ is to produce a continuous

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map \( \varphi : X \to \mathcal{K}(T) \) from some Polish space \( X \), such that \( \varphi^{-1}(H) \) is not \( \Pi^0_3 \) in \( X \). The Polish space we use is the space \( \omega^\omega \) of all infinite sequences of non-negative integers, with the product topology. Let \( W \) be the following subset of \( \omega^\omega \):

\[
W = \{ \alpha \in \omega^\omega : \alpha(p) \to \infty \text{ as } p \to \infty \}.
\]

It is well known that \( W \) is a true \( \Pi^0_3 \) subset of \( \omega^\omega \) (see [9], pp. 92–96, for a proof). In [8] T. Linton constructs a continuous \( \varphi : \omega^\omega \to \mathcal{K}(T) \) such that \( \varphi^{-1}(H) = \omega^\omega \setminus W \). We now state a similar but more general result. Below, if \( E \in \mathcal{K}(T) \) we let \( \mathcal{K}(E) = \{ K \in \mathcal{K}(T) : K \subseteq E \} \).

**Theorem.** (1) Let \( E \subseteq T \) be an \( M \) set. Then there is a continuous map \( \varphi : \omega^\omega \to \mathcal{K}(E) \) such that

- if \( \alpha \in W \) then \( \varphi(\alpha) \) is an \( M \) set,
- if \( \alpha \notin W \) then \( \varphi(\alpha) \) is an \( H \) set.

In particular, there is no \( \Pi^0_3 \) set \( B \subseteq \mathcal{K}(T) \) such that \( H \cap \mathcal{K}(E) \subseteq B \subseteq U_0 \).

(2) If \( E \) is an \( M_0 \) set, the same conclusion holds with \( M \) replaced by \( M_0 \).

**Remark.** It follows from a result of N. Bary ([2], Théorème V) that the proof in [8] gives the above conclusion for \( E = T \).

For the proof of our theorem we will make use of two standard lemmas.

**Lemma 1 (see [7], p. 234).** Let \( h \) be a function in \( A \) and \( S \in \mathcal{P}F \). For \( m \in \mathbb{Z} \) define \( h^m \in A \) by \( h^m(x) = h(mx) \) and \( S^m = h^m \cdot S \in \mathcal{P}F \). Then

(i) \( S^m \to \hat{h}(0) \cdot S \) weakly in \( \mathcal{P}F \) as \( |m| \to \infty \);
(ii) \( \|S^m\|_{PM} \to \|\hat{h}\|_{PM} \|S\|_{PM} \).

**Lemma 2.** Let \( \varepsilon \) be any positive number. Then one can find \( h \in A \), \( h \geq 0 \), such that

- \( \hat{h}(0) = 1 \),
- \( |\hat{h}(k)| < \varepsilon \) if \( k \neq 0 \),
- \( h \equiv 0 \) in a neighbourhood of \( 0 \).

One can take \( 1 - \tau_\eta \) suitably normalized, where \( \tau_\eta \) is the usual trapezoidal function and \( \eta \) is small enough.

Before turning to the proof of the theorem, let us fix some notations. The set of non-negative integers is denoted by \( \omega \), and \( \omega^{<\omega} \) is the set of all finite sequences of (non-negative) integers. If \( s \in \omega^{<\omega} \), \(|s|\) is the length of \( s \). If \( s = (n_0, \ldots, n_k) \in \omega^{<\omega} \) and \( n \in \omega \), \( s^{-n} \) is the sequence \((n_0, \ldots, n_k, n) \). If \( s \in \omega^{<\omega} \) and \( \alpha \in \omega^\omega \), \( s \leq \alpha \) means that \( \alpha(i) = s(i) \) for all \( i < |s| \). Finally, if \( \alpha \in \omega^\omega \) and \( N \) is a positive integer, we denote by \( \alpha|_N \) the sequence \((\alpha(0), \ldots, \alpha(N-1)) \).
We can at last begin the proof of the theorem. The two parts will be treated together.

First, according to Lemma 2, we choose for each $n \in \omega$ a non-negative function $h_n \in A$ and an open set $U_n \subseteq \mathbb{T}$ such that $h_n \equiv 0$ on $U_n$, $\hat{h}_n(0) = 1$ and $|\hat{h}(k)| < 1/(2(n + 1))$ if $k \neq 0$.

Let now $E \in \mathcal{K}(\mathbb{T})$ be an $M$ set and $T$ be a non-zero pseudofunction with $\text{supp}(T) \subseteq E$, $\|T\|_{PM} = 1 = \hat{T}(0)$.

We construct for each $s \in \omega^\omega$ a closed set $E_s \subseteq \mathbb{T}$, a pseudofunction $T_s$ and positive integers $N_s$, $m_s$ satisfying the following conditions:

(0) $T_\emptyset = T$, $E_\emptyset = \text{supp}(T)$;
(1) $N_s \sim n > N_s$, $m_s \sim n > m_s$, $E_s \sim n \subseteq E_s$ for all $n \in \omega$;
(2) $E_s$ is a perfect set and $\text{supp}(T_s) \subseteq E_s$;
(3) $\delta(E_s \sim n, E_s) < 2^{-|s|}$ for all $n$, where $\delta$ is the Hausdorff metric on $\mathcal{K}(\mathbb{T})$;
(4) $\sup\{|\hat{T}_s(k)| : |k| > N_s\} < 2^{-|s|}$;
(5) $\|T_s\|_{PM} < 2$,
   - $|\hat{T}_s(k) - \hat{T}_s(k)| < 2^{-|s|} - 1$ for all $n \in \omega$ and $k$, $|k| \leq N_s$;
(6) $\|T_s - T_{s \sim n}\|_{PM} < 2^{-|s|} + 1/(n + 1)$ for all $n$;
(7) $m_s \sim n \cdot E_s \sim n \cap U_n = \emptyset$ for all $n$.

By condition (0) we must let $T_\emptyset = T$, $E_\emptyset = \text{supp}(T_\emptyset)$. Then $E_\emptyset$ is perfect because $T \in PF$, so that (2) is true. We can also choose $N_0$ big enough to ensure (4).

Assume $E_s$, $T_s$, $N_s$, $m_s$ have been constructed and fix $n \in \omega$. Let $h = h_n$ and, as in Lemma 1, $S^m = h(mx) \cdot T_s$ for $m \in \mathbb{Z}$.

If we apply Lemma 1 to $h = 1$ and $T_s$, then by the definition of $h$ and condition (5) (i.e. $\|T_s\|_{PM} < 2$) we obtain

$$\lim_{|m| \to \infty} \|S^m - T_s\|_{PM} < \frac{1}{n + 1}.$$  

Lemma 1 also gives that $S^m \to T_s$ weakly and $\|S^m\|_{PM} \to \|T_s\|_{PM}$. Thus we can find a positive integer $M > m_s$ such that for every $m \geq M$,

- $|\hat{S}_m(k) - \hat{T}_s(k)| < 2^{-|s|} - 1$ if $|k| \leq N_s$,
- $\|S^m\|_{PM} < 2$,
- $\|S^m - T_s\|_{PM} < 1/(n + 1)$.

Then we almost get what we want, except that perhaps there will be no $m \geq M$ such that $\delta(E_s, \text{supp}(S^m)) < 2^{-|s|}$. To overcome this difficulty, we introduce another definition: a set $K \in \mathcal{K}(\mathbb{T})$ is said to be a Kronecker set if the exponentials $e^{int}$ are uniformly dense in $S(K) = \{f \in C(K) : |f(t)| = 1$ for all $t \in K\}$. We shall use two results about Kronecker sets. The first one is almost obvious: if $K$ is a Kronecker set, then for any non-empty open set
V ⊆ T and any integer L one can find l ≥ L such that lK ⊆ V. The second result is essentially due to R. Kaufman (see [5], or [7], pp. 337–338): for any perfect set F ⊆ T, the perfect Kronecker sets contained in F are dense in \( K(F) \).

After this detour we complete the inductive step as follows. Since \( E_s \) is perfect (by (2)), we choose a Kronecker set \( K \subseteq E_s \) with \( \delta(K, E_s) < 2^{-|s|} \). Then we pick \( m \geq M \) such that \( mK \cap U_n = \emptyset \), and let \( T_{s_n} = S^m \), \( E_{s_n} = K \cup \text{supp}(T_{s_n}) \), \( m_{s_n} = m \). Finally, we take \( N_{s_n} > N_s \) large enough to ensure (4). Then conditions (1), . . . , (7) are clearly satisfied and this concludes the inductive step.

Now if \( \alpha \in \mathbb{ω}^\mathbb{ω} \) it follows from (1) and (5) that the sequence \( (T_{\alpha_n})_{N \geq 1} \) converges \( \text{w}^* \) to a pseudomeasure \( T_\alpha \). By (5), \( \hat{T}_\alpha(0) \geq \hat{T}(0) - 1/2 = 1/2 \), hence \( T_\alpha \neq 0 \). If we set \( E_\alpha = \bigcap_{N \geq 1} E_{\alpha_n} \), then by (1) and (2), \( \text{supp}(T_\alpha) \subseteq E_\alpha \subseteq E \). Moreover, condition (3) implies that the map \( \alpha \mapsto E_\alpha \) is continuous.

We claim that if \( \alpha(p) \to \infty \) as \( p \to \infty \) then \( T_\alpha \in PF \), hence \( E_\alpha \) is an \( M \) set. Indeed, if \( k \) is any integer with \( |k| > N_0 \), then by (1) there is a unique \( (n, s) \in \omega \times \omega^{<\omega} \) such that \( s^\prec n \leq \alpha \) and \( N_s < |k| \leq N_{s^\prec n} \). Now by conditions (4), (5), (6) we get

\[
|\hat{T}_\alpha(k)| \leq |\hat{T}_\alpha(k) - \hat{T}_{s^\prec n}(k)| + |\hat{T}_{s^\prec n}(k) - \hat{T}_s(k)| + |\hat{T}_s(k)| < 3 \cdot 2^{-|s|} + \frac{1}{n+1}
\]

and the claim follows.

On the other hand, if \( \alpha(p) \not\to \infty \) as \( p \to \infty \) then conditions (1) and (7) readily imply that \( E_\alpha \) is an \( H \) set.

Thus we have proved the first part of the theorem.

Now if we assume that \( E \) is an \( M_0 \) set rather than an \( M \) set then the preceding construction begins with a positive measure in \( PF \) and since the functions \( h_n \) are non-negative we get a positive measure \( \mu_\alpha \) in the end. This completes the whole proof.

To conclude this note we point out very quickly some consequences of the above result (or of its proof). For all the notions involved below, we refer to [7] (and [10] for the definition of \( U'_2 \)).

1. Given \( T \in PF \) and \( \varepsilon > 0 \) there exists a pseudomeasure \( S \) whose support is an \( H \) set contained in \( \text{supp}(T) \) such that \( \|T - S\|_{PF} < \varepsilon \); if \( T \) is a probability measure, then \( S \) can be chosen to be a probability measure as well (see [3], or [7], pp. 217, 239, for comparison).

2. If \( E \) is an \( M \) set then the class \( U' \) and all the classes \( H^{(n)} \), \( n \geq 1 \), are true \( \Sigma^0_3 \) in \( K(E) \). If \( E \) is an \( M_0 \) set the same conclusion holds for the \( U'_0 \) and \( U'_2 \) sets contained in \( E \).
Let $H_\sigma$ be the sigma-ideal generated by the $H$ sets. Then if $E$ is an $M$ set there is no $\Sigma_1^1$ set $B \subseteq K(E)$ such that $H_\sigma \cap K(E) \subseteq B \subseteq U$; if $E$ is an $M_0$ set there is no $\Sigma_1^1$ set such that $H_\sigma \cap K(E) \subseteq B \subseteq U_0$.

The proof of (3) is as follows. Let $2^\omega$ be the space of all infinite sequences of 0’s and 1’s (with the product topology) and $D = \{ \alpha \in 2^\omega : \exists n \forall p > n \alpha(p) = 0 \}$. Then $D$ is $\Sigma_0^1$ in $2^\omega$, hence by the result just proved there is a continuous map $f : 2^\omega \to K(E)$ such that $f(\alpha) \in H$ if $\alpha \in D$ and $f(\alpha)$ is an $M$ (or $M_0$) set if $\alpha \notin D$. One can define a continuous map $F : K(2^\omega) \to K(E)$ by setting $F(K) = \bigcup \{ f(\alpha) : \alpha \in K \}$. Then $F(K)$ is an $H_\sigma$ set if $K \subseteq D$ and an $M$ (or $M_0$) set if $K \not\subseteq D$. This completes the proof since $K(D)$ is not a $\Sigma_1^1$ set (for a proof of this last result see e.g. [7], p. 119).

REFERENCES