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SOME REMARKS ON INACCESSIBLE ALEPHS

BY

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If s is a sequence composed of zeros and ones and of the ordinal type α, say $s = \{\delta_\beta\}_{\beta<\alpha}$, where $\delta_\beta = 0$ or 1, then let us define

$$\varepsilon|s|\eta = \{\delta_\beta\}_{\varepsilon\leqslant\beta<\eta} \quad \text{for} \quad \varepsilon < \eta \leqslant \alpha.$$

Thus $0|s|\eta$ (or simpler $s|\eta$) is a segment of type η of s. For a set S of sequences of type α we define $\varepsilon|S|\eta = \{\varepsilon|s|\eta : s \in S\}$ and $S|\eta = 0|S|\eta$.

Let α be a limit number. A sequence of type α of subsets $A_0, A_1, \ldots, A_\beta, \ldots$ of a set M is called *convergent to a set* $A \subset M$ if for each $x \in M$ there exists a $\gamma < \alpha$ such that $x \notin A \dot{-} A_\beta$ for $\gamma < \beta < \alpha$ ($A \dot{-} A_\beta$ denotes the set of those $y \in M$ which belong to strictly one of the sets A, A_β).

A function m defined on all subsets of a set M and taking the values 0, 1 will be called *additive with each power less than* $\overline{\overline{M}} = \aleph_\mu$ if $m \bigcup_{\beta<\alpha} A_\beta = \sum_{\beta<\alpha} mA_\beta$ for pairwise disjoint sets $A_\beta \subset M$ and $\alpha < \omega_\mu$.

We call an infinite aleph $\aleph_\mu$ *inaccessible* if

(a) $\sum_{\beta<\alpha} \mathfrak{m}_\beta < \aleph_\mu$ whenever $\alpha < \omega_\mu$ and $\mathfrak{m}_\beta < \aleph_\mu$ for $\beta < \alpha$;

(b) $\mathfrak{n}^{\mathfrak{m}} < \aleph_\mu$ for $\mathfrak{n}, \mathfrak{m} < \aleph_\mu$.

The definition of inaccessible cardinals given in [3] coincides with the above definition when applied to alephs.

Consider the following statements:

(H$_1$) If to each ordinal $\alpha < \omega_\mu$ corresponds a non-empty family S_α of sequences which are composed of zeros and ones and are all of the type α and $S_\alpha|\eta = S_\eta$ whenever $\eta < \alpha$, then there exists a sequence s of type ω_μ, composed of zeros and ones, such that $s|\alpha \in S_\alpha$ for each $\alpha < \omega_\mu$.

(H$_2$) If $\overline{\overline{M}} = \aleph_\mu$, then every sequence of type ω_μ of subsets of M contains a convergent subsequence of type ω_μ.

(H_3) There exists a zero-one valued measure m defined on all subsets of M which is additive with each power less than $\overline{\overline{M}}$, and satisfies $mM = 1$.

In this paper we shall prove the following theorem:

THEOREM. *If $\overline{\overline{M}} = \aleph_\mu$ is inaccessible, then* (H_1) *and* (H_2) *are equivalent statements and they follow from* (H_3).

Under the assumption that $\overline{\overline{M}} = \aleph_\mu$ is inaccessible, (H_1), (H_2) and (H_3) are unsolved hypotheses (cf. [1], [4], [5]). (H_1) and (H_2) were put forward by R. Sikorski; (H_1) is the yet unsolved part of P 19 (see [2], [4]) and (H_2) is equivalent with Problem 23 in the "New Scottish Book" (inscribed at 13. XII. 1946).

Proof of the theorem. We shall prove the implications $(H_3) \to (H_1) \to (H_2) \to (H_1)$.

(H_1) implies (H_2). Denote by $A_0, A_1, \ldots, A_\beta, \ldots$ a sequence of subsets of M which is of the ordinal type ω_μ. We may assume that M is the set of all ordinals which are smaller than ω_μ. Thus the characteristic function of each set A_β is a sequence composed of zeros and ones and of type ω_μ. We denote this sequence by $s^{(\beta)}$.

LEMMA. *If $B \subset M$ is of power $\aleph_\mu$ and $\varepsilon < \eta < \omega_\mu$, then there is an $\alpha \in B$ such that*

$$(*) \qquad \varepsilon|s^{(\alpha)}|\eta = \varepsilon|s^{(\beta)}|\eta$$

holds for $\aleph_\mu$ ordinals $\beta \in B$.

Indeed, let us write $\alpha \sim \beta$ for any ordinals $\alpha, \beta \in B$ such that $(*)$ holds. It is clear that $\sim$ is a congruence relation on B. The number of classes of congruent ordinals into which B is decomposed by $\sim$ is evidently at most $2^{\bar{\bar{\eta}}}$. Since $\aleph_\mu$ is inaccessible, we have $2^{\bar{\bar{\eta}}} < \aleph_\mu$. This implies, again by the inaccessibility of $\aleph_\mu$, that at least one of these congruence classes is of power $\aleph_\mu$. That means that there exists an $\alpha \in B$ such that $(*)$ holds for $\aleph_\mu$ ordinals $\beta \in B$.

It follows from our lemma (in the case $\varepsilon = 0$, $B = M$) that for each $\eta < \omega_\mu$ there exists a sequence $s^{(\alpha)}$ such that $s^{(\alpha)}|\eta = s^{(\beta)}|\eta$ for $\aleph_\mu$ ordinals β. We shall denote the family of all such sequences $s^{(\alpha)}|\eta$ by S_η.

Let us verify that $S_\eta|\varepsilon = S_\varepsilon$ whenever $\varepsilon < \eta$. Evidently $S_\eta|\varepsilon \subset S_\varepsilon$. To prove the inverse inclusion suppose that $s^{(\delta)}|\varepsilon \in S_\varepsilon$. Let B be the set of all ordinals β for which $s^{(\beta)}|\varepsilon = s^{(\delta)}|\varepsilon$. It follows that $\overline{\overline{B}} = \aleph_\mu$.

By our lemma (here we apply it with $\varepsilon \neq 0$) there exists such an $\alpha \in B$ that $(*)$ holds for $\aleph_\mu$ ordinals β. Thus by $\alpha, \beta \in B$ we have $s^{(\alpha)}|\eta = s^{(\beta)}|\eta$ for $\aleph_\mu$ ordinals β. This implies $s^{(\alpha)}|\eta \in S_\eta$. Since $s^{(\alpha)}|\varepsilon = s^{(\delta)}|\varepsilon$ we obtain $s^{(\delta)}|\varepsilon \in S\ |\varepsilon$.

It follows from (H_1) that there exists a 0-1 sequence s satisfying $s|\eta \in S_\eta$ for each η. Let A be the set with the characteristic function s. For each ordinal $\eta < \omega_\mu$ let us consider the ordinals α which satisfy $s|\eta = s^{(\alpha)}|\eta \in S_\eta$. Since there are $\aleph_\mu$ such ordinals α, there is among them an ordinal $\alpha_\eta > \eta$. Thus we can define a sequence $\{\alpha_\eta\}_{\eta<\omega_\mu}$ such that $s|\eta = s^{(\alpha_\eta)}|\eta$ for each η. These equalities imply

$$\eta \notin A \dot{-} A_{\alpha_\beta} \quad \text{for} \quad \beta > \eta$$

what proves that the sequence $\{A_{\alpha_\eta}\}_{\eta<\omega_\mu}$ converges to A. It follows from $\alpha_\eta > \eta$ that $\{\alpha_\eta\}_{\eta<\omega_\mu}$ is confinal with ω_μ.

(H_2) implies (H_1). Let M be the set of all ordinals smaller than ω_μ. Suppose that to each $\alpha < \omega_\mu$ corresponds a family S_α of 0-1 sequences of type α and that $S_\alpha|\beta = S_\beta$ for $\beta < \alpha$. Denote by $A_\alpha \subset M$ any set which characteristic function is a 0-1 sequence $s^{(\alpha)}$ of type ω_μ satisfying $s^{(\alpha)}|\alpha \in S_\alpha$. By ($H_2$) there exists a subsequence $\{A_{\alpha_\eta}\}_{\eta<\omega_\mu}$ convergent to some set $A \subset M$. Let s be the characteristic function of A. Let us prove that s satisfies the conclusion of (H_1), i. e. that $s|\gamma \in S_\gamma$ for each γ. Establish some $\gamma < \omega_\mu$. For each $\beta < \gamma$ there exists an ordinal $\delta_\beta < \omega_\mu$ such that

$$\beta \notin A \dot{-} A_{\alpha_\eta} \quad \text{for} \quad \eta > \delta_\beta.$$

Since $\aleph_\mu$ is inaccessible, it follows that the sequence $\{\delta_\beta\}_{\beta<\gamma}$ cannot be confinal with ω_μ and thus there exists a number $\delta < \omega_\mu$ which is greater than each δ_β with $\beta < \gamma$. Consequently

$$\beta \notin A \dot{-} A_{\alpha_\eta} \quad \text{for every } \beta < \gamma \text{ and } \eta > \delta.$$

This proves that $s|\gamma \in S_{\alpha_\eta}|\gamma$ for $\eta > \delta$ and thus $s|\gamma \in S_\gamma$ by $S_\gamma = S_{\alpha_\eta}|\gamma$ for $\alpha_\eta > \gamma$.

(H_3) implies (H_1). Suppose the families S_α do satisfy the assumptions of (H_1). Consider the set $S = \bigcup_{\alpha<\omega_\mu} S_\alpha$. Clearly $\overline{\overline{S_\alpha}} \leqslant 2^{\bar{\bar{\alpha}}} < \aleph_\mu$ for each α and thus $\overline{\overline{S}} = \aleph_\mu$ (since S_α are pairwise disjoint). Denote for $\beta < \omega_\mu$ and $i = 0$ or 1 by $T_i^{(\beta)}$ the set of all sequences in S which are of ordinal type greater than β and the β-th term of which is i. Thus we have the decompositions $S = T^{(\beta)} \cup T_0^{(\beta)} \cup T_1^{(\beta)}$ where $T^{(\beta)}$ denotes the set of all sequences belonging to S and of ordinal types not greater than β.

Let m be a 0-1 measure defined on all subsets of S which exists by (H_3). It is $mT^{(\beta)} = 0$ since $\overline{\overline{T^{(\beta)}}} < \aleph_\mu$ by the inaccessibility of $\aleph_\mu$. Thus $mS = 1$ implies that for $i_\beta = 0$ or 1 we have $mT_{i_\beta}^{(\beta)} = 1$ and $mT_{1-i_\beta}^{(\beta)} = 0$. We define $s = \{i_\beta\}_{\beta<\omega_\mu}$. ($H_1$) follows if we prove that $s|\alpha \in S_\alpha$ for each

$\alpha < \omega_\mu$. Suppose some $t \in \bigcap_{\beta<\alpha} T_{i_\beta}^{(\beta)}$. Then $t|\alpha \in S_\alpha$ and $t|\alpha = s|\alpha$ what implies $s|\alpha \in S_\alpha$. Now it follows from

$$m \bigcap_{\beta<\alpha} T_{i_\beta}^{(\beta)} \geqslant 1 - \sum_{\beta<\alpha} m(T^{(\beta)} \cup T_{1-i_\beta}^{(\beta)}) = 1$$

that $\bigcap_{\beta<\alpha} T_{i_\beta}^{(\beta)}$ is not empty for $\alpha < \omega_\mu$.

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ON THE REPRESENTATION OF FIELDS AS FINITE UNIONS OF SUBFIELDS

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The purpose of this paper is to prove the following theorem:

THEOREM. *An algebraic field cannot be represented as a union (in the sense of the theory of sets) of a finite number of proper subfields.*

LEMMA 1. *If $G, G_1, G_2, \ldots, G_n$ are groups, $G = \bigcup_{i=1}^{n} G_i$,*

(1) $$G \neq \bigcup_{i=1}^{j-1} G_i \cup \bigcup_{i=j+1}^{n} G_i \quad \textit{for} \quad 1 \leqslant j \leqslant n$$

and G is an infinite set, then $\bigcap_{i=1}^{n} G_i$ is infinite.

Proof. We shall prove by induction that for each $k \leqslant n$ there exists such a sequence $i_1, i_2, \ldots, i_k$ of different natural numbers $\leqslant n$ that

(2) $$\bigcap_{j=1}^{k} G_{i_j} \text{ is infinite.}$$

For $k = 1$, (2) follows from the fact that $\bigcup_{i=1}^{n} G_i$ is infinite. Suppose that (2) holds for $k < n$ and let $\{a_n\}$ be an infinite sequence of different elements of the group $\bigcap_{j=1}^{k} G_{i_j}$. By (1) $G \neq \bigcup_{j=1}^{k} G_{i_j}$ and so there exists a $b \in G - \bigcup_{j=1}^{k} G_{i_j}$.

Consequently $a_n b \notin \bigcup_{j=1}^{k} G_{i_j}$ and $a_n b \in \bigcup_{i \neq i_1, i_2, \ldots, i_k} G_i$. Hence there exists a number $i_{k+1} \neq i_1, i_2, \ldots, i_k$ such that infinitely many elements of the sequence $\{a_n b\}$ belong to $G_{i_{k+1}}$. Let $a_{m_n} b \in G_{i_{k+1}}$ $(n=1,2,\ldots)$. Then $a_{m_n} a_{m_1}^{-1} = (a_{m_n} b)(a_{m_1} b)^{-1} \in G_{i_{k+1}}$ and by the definition of $\{a_n\}$: $a_{m_n} a_{m_1}^{-1} \in \bigcap_{j=1}^{k} G_{i_j}$. Then $a_{m_n} a_{m_1}^{-1} \in \bigcap_{j=1}^{k+1} G_{i_j}$, and $\bigcap_{j=1}^{k+1} G_{i_j}$ is an infinite set, which completes our inductive proof.