But the set $\{x \epsilon X: f(x, y) \geqslant a - \varepsilon$ for some $y$ in $Y\}$ is the projection upon the $X$-axis of the closed set $\{\langle x, y\rangle \epsilon X \times Y: f(x, y) \geqslant a - \varepsilon\}$, whence, according to the lemma, it is closed. Consequently, $L^a$ is also closed. Finally, $g(x)$ is continuous on $X$, and thus it is bounded. It implies that $f(x, y)$ is also bounded and (ii) is proved.

Theorem (iii) may easily be proved using the following result of Smirnov [5]:

A subset $P$ of a topological space $R$ is said to be *normally disposed in* $R$ if for each closed set $F$ lying in $R \setminus P$ there exists a $G_\delta$-set containing $F$ and disjoint from $P$. Then:

*if $X$ is a Lindelöf space, then $X$ is normally disposed in any of its compactifications*;

*if $X$ is normally disposed in some of its compactifications, then $X$ is a Lindelöf space.*

By a compactification we understand here any compact space which contains the given space as a dense subset.

Now (iii) can be proved in a few words. Assume that $X^*$ is a compactification of $X$. Then $X^* \setminus Y$ is a compactification of $X \times Y$. Let $F$ be any closed set lying in $X^* \times Y \setminus X \times Y$ and $F_1$ — the projection of $F$ upon the $X$-axis. Of course, $F_1$ is disjoint from $X$, and, by the lemma, it is closed. Thus there exists a $G_\delta$-set $G$ which contains $F_1$ and does not meet $X$. Of course, the counter-image of $G$ under the projection is a $G_\delta$-set which contains $F$ and is disjoint from $X \times Y$. Thus $X \times Y$ is normally disposed in $X^* \times Y$ and it follows that $X \times Y$ is a Lindelöf space.

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# *ON THE POTENCY OF SUBSETS OF $\beta N$*

BY

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Let $N$ be the space of positive integers and $\beta N$ — the maximal Stone-Čech compactification of $N$. In [5] B. Pospišil has shown the following:

(i) *The potency of $\beta N$ is equal to* $2^{\mathfrak{c}}$.

In [4] J. Novàk has given another proof of (i) and deduced from (i) the following:

(ii) *Each closed infinite subset of $\beta N$ is of the power* $2^{\mathfrak{c}}$.

Now we shall give a very simple proof of (i). Let us consider the Cartesian product $I^{\mathfrak{c}}$ of continuously many unit intervals $I = [0, 1]$. Of course, $I^{\mathfrak{c}}$ is a compact space of the power $2^{\mathfrak{c}}$. On the other hand, $I^{\mathfrak{c}}$ may be considered as the set of all functions from $I$ to $I$ and it is clear that the set $M \subset I^{\mathfrak{c}}$ consisting of all polynomials with rational coefficients is dense in $I^{\mathfrak{c}}$. Let $\varphi$ be any mapping from $N$ onto $M$. Then $\varphi$ is a continuous mapping (because $N$ has the discrete topology), whence $\varphi$ can be continuously extended over the whole $\beta N$; let $\varphi^*$ denote this extension. Of course, the image $\varphi^*(\beta N)$ is a closed subset of $I^{\mathfrak{c}}$ and since it contains $M$, it coincides with $I^{\mathfrak{c}}$. Thus $\overline{\overline{\beta N}} \geqslant 2^{\mathfrak{c}}$. On the other hand, it is plain that $\beta N$, having an enumerable dense subset, is of the power $\leqslant 2^{\mathfrak{c}}$. Thus (i) is proved.

Now, following Novák, we can easily show (ii).

Let $F$ be any infinite closed subset of $\beta N$. Note that $F$ contains an enumerable subset $E$ which is homeomorphic to $N$. Indeed, this results for instance from the following lemma (see [3], Lemma 1):

*If $X$ is a compact space and $F$ is a closed infinite subset of $X$, then there exists a sequence $G_1, G_2, \ldots$ of mutually disjoint open subsets of $X$ such that $F \cap G_n \neq 0$ $(n = 1, 2, \ldots)$.*

Of course, if $p_n \epsilon F \cap G_n$ $(n = 1, 2, \ldots)$, then the set $E = \{p_1, p_2, \ldots\}$ is homeomorphic to $N$.

Now let us notice that each bounded real-valued function $f$ defined on $E$ can be continuously extended to a bounded function defined over $E \cup N$ (the space $E \cup N$, as an enumerable completely regular space, is normal; on the other hand, $N$, as a locally compact space which is dense in $E \cup N$, is open in $E \cup N$ and it follows that $E$ is closed in $E \cap N$, whence the Tietze theorem can be applied); thus $f$ can be continuously extended over $N$ and, in particular, $f$ can be extended over $\bar{E}$ (the bar indicates closure with respect to $\beta N$). It follows that $\bar{E}$ coincides with $\beta E$ and, since $E$ is homeomorphic to $N$, $\bar{E}$ is homeomorphic to $\beta N$. Since $\bar{E} \subset F$, $F$ is of the power $2^{\mathfrak{c}}$.

Of course, there are finite and countable open subsets of $\beta N$; for instance, each subset of $N$ is open. Nevertheless, from (ii) we immediately obtain:

(iii) *Each uncountable open subset $G$ of $\beta N$ is of the power $2^{\mathfrak{c}}$.*

Indeed, $G$ contains a point $p_0$ from $\beta N \setminus N$ which is not isolated. Let $U$ be any neighbourhood of $p_0$ such that $\bar{U} \subset G$. Since $p_0$ is non-isolated, $U$ is infinite, whence $\bar{U}$ is of the power $2^{\mathfrak{c}}$ and so is $G$.

Similarly to (i) we can prove:

(iv) *Each pseudo-compact subset $P$ of $\beta N$ which contains $N$ is of the power $\geqslant \mathfrak{c}$.*

We recall that a space is said to be *pseudo-compact* if each continuous real-valued function on the space is bounded (see [1]).

In order to prove (iv) suppose that $\varphi$ is any mapping of $N$ onto the set $W$ of all rational numbers of the interval $[0, 1]$. Then $\varphi$ can be continuously extended to a mapping $\varphi^*$ of $\beta N$ into the interval $[0, 1]$. Since $N \subset P$, $W \subset \varphi^*(P)$. But $\varphi^*(P)$, as a continuous image of a pseudo-compact space, is again a pseudo-compact space and it follows that $\varphi^*(P)$ is closed in the interval $[0, 1]$ (a pseudo-compact space is closed in each super-space which satisfies the first axiom of countability, see [2]), whence it coincides with the interval $[0, 1]$. Thus $\bar{\bar{P}} \geqslant \mathfrak{c}$.

Now, in the same way as we have deduced we can show:

(v) *Each infinite pseudo-compact subset of $\beta N$ is of the power $\geqslant \mathfrak{c}$.*

In particular: *each infinite countably compact subset of $\beta N$ is of the power $\geqslant \mathfrak{c}$.*

The above estimation of the powers of countably compact subsets of $\beta N$ is exact; indeed, there exists a countably compact subset of $N$ which is of the potency $\mathfrak{c}$. In fact, let us define by means of transfinite induction the sequence $\{N_0, N_1, \ldots, N_\xi, \ldots\}_{\xi<\Omega}$ of subsets of $\beta N$:



1° $N_0 = N$;

2° Assume that for some $\xi_0 < \Omega$ the sets $N_\xi$ are defined for each $\xi < \xi_0$, are of the power $\leqslant \mathfrak{c}$ and form an increasing sequence. Let $\mathfrak{R}$ be the family of all countably infinite subsets of $\bigcup\{N_\xi : \xi < \xi_0\}$. Let us assign to each $C \in \mathfrak{R}$ a point $P_C$ which is an accumulation point of $C$ and let

$$N_{\xi_0} = U\{N_\xi : \xi < \xi_0\} \cup \{p_C : C \in \mathfrak{R}\}.$$

Of course, $N_{\xi_0}$ is also of the power $\leqslant \mathfrak{c}$ and contains each $N_\xi$ with $\xi < \xi_0$. Thus, by transfinite induction, the sets $N_\xi$ are defined for each $\xi < \Omega$. Let

$$D = U\{N_\xi : \xi < \Omega\}.$$

Of course, $\bar{\bar{D}} \leqslant \mathfrak{c}$. On the other hand, if $C$ is any countably infinite subset of $D$ then there exists $\xi_0 < \Omega$ such that $C$ lies in $U\{N_\xi : \xi < \xi_0\}$, and this $C$ possesses an accumulation point lying in $N_{\xi_0} \subset D$. This shows that $D$ is countably compact.

According to (ii), $D$ is a countably compact infinite space which contains no compact subsets except finite subsets. It is interesting to show, without the use of the hypothesis of the continuum, the existence of such a space of the power $\aleph_1$.

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