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ON THE HARDY-LANDAU THEOREM

BY

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A. Zygmund [3] gave a simple proof of the following Hardy-Landau Theorem:

If $na_n \leqslant M$ *(or* $na_n \geqslant M$*) and* $\sum a_n$ *is* $(C, 1)$*-summable to* s*, then* $\sum a_n$ *converges to* s*.*

In this note I will show that the ideas of A. Zygmund are applicable to the proof of the generalised Hardy-Landau Theorem:

If $na_n \leqslant M$ *(or* $na_n \geqslant M$*) and* $\sum a_n$ *is* (C, a)*-summable to* s *for any* $a \geqslant 1$*, then* $\sum a_n$ *converges to* s*.*

The proof of this theorem given in Hardy's paper [1] (p. 122-123) is not a simple one.

1. We shall consider, instead of the summability of the series, the limitability of the sequence $f = \{f_n\}$, which of course may always be treated as the sequence of the partial sums of the series $\sum \Delta_n f$, where $\Delta_0 f = f_0$ and $\Delta_n f = f_n - f_{n-1}$ for $n \geqslant 1$.

The sequence $\{na_n\}$ is called the Kronecker sequence of the series $\sum a_n$. In the following the sequence $\{n\Delta_n f\}$ is called the *Kronecker sequence of the sequence* $f = \{f_n\}$ and denoted by $\mathcal{K}f = \{\mathcal{K}_n f\}$.

By $\mathcal{H}$ we denote the Hölder operator, which transforms any sequence $f = \{f_n\}$ into the sequence of its arithmetical means,

$$\mathcal{H}f = \left\{\frac{1}{n+1}\sum_{\nu=0}^{n} f_\nu\right\} = \{\mathcal{H}_n f\},$$

and by $\mathcal{C}^a$ the Cesàro operator, which transforms any sequence $f = \{f_n\}$ into the sequence

$$\mathcal{C}^a f = \left\{\frac{1}{A_n^a}\sum_{\nu=0}^{n} A_{n-\nu}^{a-1} f_\nu\right\} = \{\mathcal{C}_n^a f\},$$

where A_n^{α} are the Cesàro numbers:

$$A_0^{\alpha} = 1, \quad A_n^{\alpha} = \frac{(\alpha+1)\dots(\alpha+n)}{n!} \quad \text{for} \quad n \geqslant 1.$$

Of course $\mathcal{H} = \mathcal{C}^1$.

Operators $\mathcal{C}^{\alpha}$ are regular for $\alpha \geqslant 0$: if $\lim f = s$, then $\lim \mathcal{C}^{\alpha} f = s$ ([1], p. 101). Operators $\mathcal{C}^{\alpha+1}$ and $\mathcal{H}\mathcal{C}^{\alpha}$ for $\alpha \geqslant 0$ are equivalent: if $\lim \mathcal{C}^{\alpha+1} f = s$, then $\lim \mathcal{H}\mathcal{C}^{\alpha} f = s$ and vice versa ([1], p. 102). If $f \leqslant M$ (or $f \geqslant M$) for some real M, then, for $\alpha \geqslant 0$, $\mathcal{C}^{\alpha} f \leqslant M$ (or $\mathcal{C}^{\alpha} f \geqslant M$) (here $f \leqslant M$ denotes that $f_n \leqslant M$ holds for any $n = 0, 1, 2, \dots$).

2. Let us denote by $\mathcal{S}^{\alpha}$ the operator of summation of order α:

$${}^{\alpha}f = \Big\{\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} f_{\nu}\Big\} = \{\mathcal{S}_n^{\alpha} f\}.$$

In particular $\mathcal{S}^{-1} = \Delta$. It is well known that for any α and β

$$\mathcal{S}^{\alpha+\beta} = \mathcal{S}^{\alpha} \cdot \mathcal{S}^{\beta}.$$

Now we shall prove a formula interesting in itself.

LEMMA 1. *For any α and any sequences $f = \{f_n\}$ and $g = \{g_n\}$ there holds the formula*

$$\mathcal{S}_n^{\alpha} fg = \sum_{\nu=0}^{n} (-1)^{\nu} A_{\nu}^{\alpha-1} \mathcal{S}_n^{-\nu} f \mathcal{S}_{n-\nu}^{\alpha+\nu} g.$$

Proof. Starting from the second part of the formula we obtain

$$\begin{aligned}
\sum_{\nu=0}^{n} & (-1)^{\nu} A_{\nu}^{\alpha-1} \mathcal{S}_n^{-\nu} f \mathcal{S}_{n-\nu}^{\alpha+\nu} g \\
&= \sum_{\nu=0}^{n} (-1)^{\nu} A_{\nu}^{\alpha-1} \sum_{\mu=n-\nu}^{n} A_{n-\mu}^{-\nu-1} f_{\mu} \sum_{\lambda=0}^{n-\nu} A_{n-\nu-\lambda}^{\alpha+\nu-1} g_{\lambda} \\
&= \sum_{\mu=0}^{n} f_{\mu} \sum_{\nu=n-\mu}^{n} (-1)^{\nu} A_{\nu}^{\alpha-1} A_{n-\mu}^{-\nu-1} \sum_{\lambda=0}^{n-\nu} A_{n-\nu-\lambda}^{\alpha+\nu-1} g_{\lambda} \\
&= \sum_{\mu=0}^{n} f_{\mu} \sum_{\lambda=0}^{n} g_{\lambda} \sum_{\nu=n-\mu}^{n-\lambda} (-1)^{\nu} A_{\nu}^{\alpha-1} A_{n-\mu}^{-\nu-1} A_{n-\nu-\lambda}^{\alpha+\nu-1} \\
&= \sum_{\mu=0}^{n} f_{\mu} \sum_{\lambda=0}^{\mu} g_{\lambda} A_{n-\lambda}^{\alpha-1} \binom{n-\lambda}{n-\mu} \sum_{\nu=n-\mu}^{n-\lambda} (-1)^{\nu-n+\mu} \binom{\mu-\lambda}{\nu-n+\mu} \\
&= \sum_{\mu=0}^{n} f_{\mu} \sum_{\lambda=0}^{n} g_{\lambda} A_{n-\lambda}^{\alpha-1} \binom{n-\lambda}{n-\mu} \sum_{\nu=0}^{\mu-\lambda} (-1)^{\nu} \binom{\mu-\lambda}{\nu} \\
&= \sum_{\mu=0}^{n} A_{n-\mu}^{\alpha-1} f_{\mu} g_{\mu} = \mathcal{S}_n^{\alpha} fg,
\end{aligned}$$

since

$$\sum_{\nu=0}^{\mu-\lambda} (-1)^{\nu} \binom{\mu-\lambda}{\nu} = \begin{cases} 0 & \text{for} \quad \mu > \lambda \\ 1 & \text{for} \quad \mu = \lambda. \end{cases}$$

If $\alpha = -k$, where k is a positive integer, we obtain the well-known formula for $\Delta^k fg$.

It is well known (see Kogbetliantz [2], p. 23) that the operators $\mathcal{C}^{\alpha}$ and $\mathcal{K}$ (where $\mathcal{K}$ denotes the Kronecker operator) are commutative. For the sake of completeness I give a proof of this fact.

LEMMA 2. *For any $\alpha \neq -1, -2, \dots$ and for any sequence $f = \{f_n\}$*

$$\mathcal{C}^{\alpha} \mathcal{K} f = \mathcal{K} \mathcal{C}^{\alpha} f.$$

Proof. We have $\mathcal{K} f = \Delta A^1 f - f$, whence

$$\mathcal{S}_n^{\alpha} \mathcal{K} f = \mathcal{S}_n^{\alpha-1} A^1 f - \mathcal{S}_n^{\alpha} f.$$

By lemma 1

$$\mathcal{S}_n^{\alpha-1} A^1 f = A_n^1 \mathcal{S}_n^{\alpha-1} f - (\alpha-1) \mathcal{S}_{n-1}^{\alpha} f,$$

whence

$$\begin{aligned}
\mathcal{S}_n^{\alpha} \mathcal{K} f &= A_n^1 \mathcal{S}_n^{\alpha-1} f - (\alpha-1) \mathcal{S}_{n-1}^{\alpha} f - \mathcal{S}_n^{\alpha} f \\
&= A_n^1 (\mathcal{S}_n^{\alpha} f - \mathcal{S}_{n-1}^{\alpha} f) - (\alpha-1) \mathcal{S}_{n-1}^{\alpha} - \mathcal{S}_n^{\alpha} f \\
&= n \mathcal{S}_n^{\alpha} f - (n+\alpha) \mathcal{S}_{n-1}^{\alpha} f,
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathcal{C}_n^{\alpha} \mathcal{K} f &= n \mathcal{C}_n^{\alpha} f - \frac{(n+\alpha) A_{n-1}^{\alpha}}{A_n^{\alpha}} \mathcal{C}_{n-1}^{\alpha} f \\
&= n \mathcal{C}_n^{\alpha} f - n \mathcal{C}_{n-1}^{\alpha} f = n \Delta_n \mathcal{C}^{\alpha} f = \mathcal{K}_n \mathcal{C}^{\alpha} f.
\end{aligned}$$

3. THEOREM 1. *If $\mathcal{K} f \leqslant M$ for some real M, and* $\lim \mathcal{C}^{\alpha+1} f = s$ *for $\alpha \geqslant 0$, then* $\lim \mathcal{C}^{\alpha} f = s$.

Proof. Let us write, following the idea of A. Zygmund,

$$\sigma_{n,k} = \frac{1}{k} \sum_{\nu=n}^{n+k-1} \mathcal{C}_{\nu}^{\alpha} f.$$

A simple computation gives

$$\sigma_{n,k} = \mathcal{H}_{n+k-1} \mathcal{C}^{\alpha} f + \frac{n}{k} (\mathcal{H}_{n+k-1} \mathcal{C}^{\alpha} f - \mathcal{H}_{n-1} \mathcal{C}^{\alpha} f).$$

If $\lim \mathcal{C}^{\alpha+1} f = s$, then $\lim \mathcal{H}\mathcal{C}^{\alpha} f = s$, and, supposing $n/k \leqslant A$, we obtain $\lim_{n\to\infty} \sigma_{n,k} = s$.

On the other hand,

$$\sigma_{n,k}=\mathcal{C}_n^a f+\sum_{\nu=n+1}^{n+k-1}\left(1-\frac{\nu-n}{k}\right)\frac{1}{\nu}\mathcal{K}_\nu \mathcal{C}^a f,$$

or

$$\sigma_{n,k}=\mathcal{C}_{n+k-1}^a f-\sum_{\nu=n+1}^{n+k-1}\frac{\nu-n}{k\nu}\mathcal{K}_\nu \mathcal{C}^a f.$$

By lemma 2 and the hypothesis $\mathcal{K}f\leqslant M$

$$\mathcal{K}\mathcal{C}^a f=\mathcal{C}^a\mathcal{K}f\leqslant M,$$

and therefore

$$\sigma_{n,k}-\mathcal{C}_n^a f\leqslant M\sum_{\nu=n+1}^{n+k-1}\frac{1}{\nu}\leqslant M\frac{k-1}{n}$$

and

$$\mathcal{C}_{n+k-1}^a f-\sigma_{n,k}\leqslant M\sum_{\nu=n+1}^{n+k-1}\frac{1}{\nu}\leqslant M\frac{k-1}{n}.$$

Now, if $k=[n\varepsilon]+1$, where ε is an arbitrary positive real number, then

$$\frac{n}{k}=\frac{n}{[n\varepsilon]+1}<\frac{n}{n\varepsilon}=\frac{1}{\varepsilon},\qquad \frac{k-1}{n}=\frac{[n\varepsilon]}{n}<\frac{n\varepsilon+1}{n}=\varepsilon+\frac{1}{n},$$

whence

$$\lim_{n\to\infty}\sigma_{n,k}=s,\qquad s-\underline{\lim}\,\mathcal{C}^a f\leqslant M\varepsilon,\qquad \overline{\lim}\,\mathcal{C}^a f-s\leqslant M\varepsilon.$$

Thus

$$s-M\varepsilon\leqslant\underline{\lim}\,\mathcal{C}^a f\leqslant\overline{\lim}\,\mathcal{C}^a f\leqslant s+M\varepsilon$$

and the theorem is established.

COROLLARY. *If* $\mathcal{K}f\leqslant M$ *for some real* M *and* $\lim\mathcal{C}^{a+1}f=s$ *for* $a\geqslant 0$, *then* $\lim f=s$.

Proof. From $\lim\mathcal{C}^{a+1}f=s$ follows $\lim\mathcal{C}^{k+1}f=s$ for any integer $k\geqslant a$. Applying the theorem $k+1$ times, we obtain $\lim\mathcal{C}^0 f=\lim f=s$.

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ON MULTIPLICATIVE SEQUENCES

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1. Let $\{x(n)\}$, $n=1,2,\ldots$, denote a sequence satisfying the followong conditions:

1° $x(n)=1$ or -1 for $n=1,2,\ldots$,

2° $x(nm)=x(n)x(m)$ for $n,m=1,2,\ldots$

P. Erdős has asked (see [1], problem 9): if the limit

$$\lim_{n\to\infty}\frac{x(1)+\ldots+x(n)}{n}\tag{*}$$

exists for every sequence $\{x(n)\}$ satisfying conditions 1° and 2°.

The theorem proved in this section (*) concerns this problem but gives no final answer.

Now let $\{p_n\}$, $n=0,1,\ldots$, denote the sequence of all consecutive prime numbers. Let us write

$$n=p_0^{a_0^{(n)}}\ldots p_{k_n}^{a_{k_n}^{(n)}},$$

where $a_i^{(n)}$, $i=0,\ldots,k_n$, are non-negative integers and $a_{k_n}^{(n)}>0$. We denote by $\{r_n(t)\}$, $n=0,1,\ldots$, the set of all Rademacher functions (see e. g. [2], p. 42). Let us put

$$x(n,t)=[r_0(t)]^{a_0^{(n)}}\ldots[r_{k_n}(t)]^{a_{k_n}^{(n)}}.$$

Obviously, for almost all $t\in\langle 0,1\rangle$ conditions 1° and 2° for the sequences $\{x(n,t)\}$, $n=1,2,\ldots$, are satisfied.

THEOREM 1. *For almost all* $t\in\langle 0,1\rangle$

$$\lim_{n\to\infty}\frac{x(1,t)+\ldots+x(n,t)}{n}=0.$$

(*) (Added in proof.) After having written this note I learned that theorem 1 had been proved in another way by Wintner (see [4], p. 270, corollary).