On account of (11) the ratio \( r_{\psi_0}(Z_{\psi_0})/r_{\psi_0}(Z_{\psi_0}) = k = \text{const.} \),
whereas \( \lambda(Z) = k \int_{Z} \omega(c) \omega(c) = k \delta(Z), \) q. e. d.

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COMPACTNESS AND PRODUCT SPACES

BY

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In this paper we are concerned with the preserving of different sorts of compactness under the Cartesian multiplication. We shall use the following terminology:

- **countably compact** — each countable open covering contains a finite subcovering;
- **compact** — each open covering contains a finite subcovering;
- **Lindelöf** — each open covering contains a countable subcovering;
- **pseudo-compact** — each real-valued continuous function is bounded (see [2]).

I. M. Katětov has proved the following theorem (see [3]):

*The Cartesian product of two countably compact spaces, one of which is compact, is also countably compact.*

In [6] C. Ryll-Nardzewski has proved a similar theorem:

*The Cartesian product of two countably compact spaces, one of which satisfies the first axiom of countability, is also countably compact.*

Using the theory of Moore-Smith nets (for the definition, properties, notation and terminology see [4], p. 65) we may obtain, by a uniform method, the following theorem:

(i) The Cartesian product of two countably compact spaces, \( X \) and \( Y \), one of which is either compact or sequentially compact, is also countably compact.

We recall that a space is said to be sequentially compact if each sequence of elements of the space contains a convergent subsequence. Of course, each countably compact space satisfying the first axiom of countability is sequentially compact, but not conversely.
Proof of (i). In order to prove that \( X \times Y \) is countably compact, it suffices to show that each sequence \((p_n)\), \( p_n = \langle x_n, y_n \rangle \) has a cluster point. Assume that \( Y \) is compact. Since \( X \) is countably compact, the sequence \((x_n)\) contains a subnet \((x_{n_k}; k \in \mathbb{N})\) which converges to some point \( x_0 \). Since \( Y \) is compact, the net \( \langle y_{n_k}; k \in \mathbb{N} \rangle \) has a cluster point \( y_0 \). Of course, the point \( p_0 = \langle x_0, y_0 \rangle \) is a cluster point of the sequence \((p_n)\).

Now assume \( X \) is sequentially compact. Then the sequence \((x_n)\) contains a subsequence \((x_{n_k})\) which converges to some point \( x_0 \). Since \( Y \) is countably compact, the sequence \((y_{n_k})\) has a cluster point \( y_0 \), and it is plain that the point \( p_0 = \langle x_0, y_0 \rangle \) is a cluster point of the sequence \((p_n)\). Thus (i) is proved.

In connection with (i) it seems of interest to find a necessary and sufficient condition for a space \( X \) under which the product \( X \times Y \) would be countably compact for any countably compact space \( Y \).

II. In [3] J. Novák has shown that the Cartesian product of two countably compact spaces is not necessarily countably compact. Namely, Novák has shown that there exist two countably compact subsets \( M \) and \( P \) of \( \mathbb{N}^* \) (\( \mathbb{N}^* \) denotes here the space of non-negative integers and \( \mathbb{N}^* \) the maximal Stone-Čech compactification of \( \mathbb{N}^* \)) such that \( M \times P = \mathbb{N}^* \). It follows that the diagonal \( D \) of the product \( M \times P \) (i.e., the set of all \( \langle x, y \rangle \in M \times P \) for which \( x = y \)) consists of all points \( p = \langle x, y \rangle \) for which \( x, y \in \mathbb{N}^* \). Thus \( D \) is an infinite closed subset of \( M \times P \) having no accumulation point.

A similar situation may be observed in connection with other sorts of compactness. Using the above example of Novák, we can show that the product of two pseudo-compact spaces is not necessarily pseudo-compact. Indeed, each point of \( N \) is isolated in \( \mathbb{N}^* \) and it follows that each point of the diagonal \( D \) is isolated in \( M \times P \). Hence, setting \( f(p_n) = n \), where \( p_n \) is the \( n \)-th point of \( D \), and \( f(p) = 0 \) for \( p \in M \times P \setminus D \), we obtain a continuous unbounded function on \( M \times P \).

On the other hand, it is known that the product of two Lindelöf spaces is not necessarily a Lindelöf space (see e.g. [4], p. 59, example 1).

Nevertheless, we can show:

(ii) If \( X \) is a pseudo-compact space and \( Y \) is compact, then the product \( X \times Y \) is pseudo-compact.

(iii) If \( X \) is a Lindelöf space and \( Y \) is compact, then the product \( X \times Y \) is a Lindelöf space.

We start with the following lemma:

**Lemma.** If \( Y \) is a compact space, then the projection of any closed subset of \( X \times Y \) upon the \( X \)-axis is closed in \( X \).

Proof. Suppose that \( S \) is a closed subset of \( X \times Y \) and let \( S_t \) be the projection of \( S \) upon the \( X \)-axis. Let \( x_0 \) be any point of \( S_t \). Then there exists a net \( (x_{\xi}, \xi \in \mathbb{R}) \) of points of \( S \) which converges to \( x_0 \). Now, for each \( r > 0 \), we can find \( y_{r, \xi} \in Y \) such that \( \langle x_{\xi}, y_{r, \xi} \rangle \in S_t \). Since \( Y \) is compact, the net \( Y_{\xi} = \{ y_{r, \xi}; r > 0 \} \) contains a convergent subnet \( Y_{\xi} = \{ y_{r, \xi}; r > 0 \} \). Then the net \( (x_{\xi}, y_{r, \xi}) \rightarrow (x_0, y_0) \) is also convergent; let \( y_{r, \xi} \) denote its limit. Of course, \( y_{r, \xi} \in S_t \) and \( x_0 \) is the projection of \( y_{r, \xi} \). Thus \( x_0 \in S_t \).

Remark. This lemma may also be formulated in the following way:

If \( Y \) is a compact space, then for each space \( X \) the projection of the product \( X \times Y \) upon the \( X \)-axis is a closed mapping (i.e., it carries closed subsets of \( X \times Y \) into closed subsets of \( X \)).

In this form the lemma admits a converse. Indeed, assume that \( Y \) is not compact. Then there exists a subset \( Z \) of \( Y \) of regular potency which does not possess a point of full accumulation (see [1], Théorème 3).

Let \( \mathfrak{f} \) be the initial ordinal of the power \( Z \) and let \( y_1, y_2, \ldots, y_{\mathfrak{f}} \) be a transfinite sequence (without repetitions) of points of \( Z \). Let \( X \) be the space of all ordinals \( \xi \leq \mathfrak{f} \) with the order topology and let \( S \) be the set of all points of the product \( X \times Y \) of the form \( \langle \xi, y \rangle \) \( (\xi < \mathfrak{f}) \). We assert that \( S \) does not contain any point of the form \( \langle \mathfrak{f}, y \rangle \). Indeed, since \( y \) is not a point of full accumulation of \( Z \), there exists a neighbourhood \( U \) of \( y \) such that \( U \cap Z \neq \emptyset \), whence, \( a_0 \) being a regular ordinal, an ordinal \( \xi_0 < a_0 \) may be found so that \( y_0 \notin U \) for all \( \xi > \xi_0 \). Hence \( \xi_0 < \xi < a_0 \) \( \preceq \xi \) is a point of \( S \) which is free from points of \( S \). Thus \( \langle a_0, y \rangle \) does not belong to \( S \). Consequently, the projection of \( S \) upon the \( X \)-axis is nothing else than the set of all ordinals \( \xi < a_0 \) whence it is not closed in \( X \).

Now we pass to the proof of (ii). Let \( f(x, y) \) be any continuous real-valued function on \( X \times Y \). We can assume, without loss of generality, that \( f(x, y) \) is a non-negative function. Since \( Y \) is compact, the number \( \sup x \in X \) is finite for each \( x \in X \), whence we can define the function \( x \in X \times Y \)

\[
g(x) = \sup_{y \in Y \times Y} f(x, y).
\]

We shall show that \( g(x) \) is continuous on \( X \). To prove that it suffices to show that for each real number \( a \) the sets \( L_* = \{ x \in X \times Y \mid g(x) < a \} \) and \( L_0 = \{ x \in X \times Y \mid g(x) < a \} \) are closed. But

\[
L_0 = \bigcap_{y \in Y \times Y} \{ x \in X \mid f(x, y) < a \},
\]

whence \( L_0 \) is closed as the intersection of closed sets. On the other hand, \( L_* = \bigcap_{y \in Y \times Y} \{ x \in X \mid f(x, y) < a - \varepsilon \} \text{ for some } y \in Y \).
But the set \( \{ x \in X : f(x, y) \geq a - \varepsilon \text{ for some } y \in Y \} \) is the projection upon the \( X \)-axis of the closed set \( \{ (x, y) \in X \times Y : f(x, y) \geq a - \varepsilon \} \), whence, according to the lemma, it is closed. Consequently, \( D' \) is also closed. Finally, \( g(\varepsilon) \) is continuous on \( X \), and thus it is bounded. It implies that \( f(x, y) \) is also bounded and (ii) is proved.

Theorem (iii) may easily be proved using the following result of Smirnov [5]:

A subset \( P \) of a topological space \( R \) is said to be normally disposed in \( R \) if for each closed set \( F \) lying in \( R \setminus P \) there exists a \( G_P \)-set containing \( F \) and disjoint from \( P \). Then:

(i) if \( X \) is a Lindelöf space, then \( X \) is normally disposed in any of its compactifications;

(ii) if \( X \) is normally disposed in some of its compactifications, then \( X \) is a Lindelöf space.

By a compactification we understand here any compact space which contains the given space as a dense subset.

Now (iii) can be proved in a few words. Assume that \( X' \) is a compactification of \( X \). Then \( X' \setminus X \) is a compactification of \( X \times X \). Let \( F \) be any closed set lying in \( X' \times X' \setminus X \times X \) and \( F_1 \) the projection of \( F \) upon the \( X \)-axis. Of course, \( F_1 \) is disjoint from \( X \), and, by the lemma, it is closed. Thus there exists a \( G_{F_1} \)-set \( G \) which contains \( F_1 \) and does not meet \( X \). Of course, the counter-image of \( G \) under the projection is a \( G_P \)-set which contains \( F \) and is disjoint from \( X \times X \). Thus \( X' \times X' \) is normally disposed in \( X' \times X' \) and it follows that \( X \times X \) is a Lindelöf space.

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ON THE POTENCY OF SUBSETS OF \( \beta N \)

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Let \( N \) be the space of positive integers and \( \beta N \) — the maximal Stone-Cech compactification of \( N \). In [5] B. Pospíšil has shown the following:

(i) The potency of \( \beta N \) is equal to \( 2^\alpha \).

In [4] J. Novák has given another proof of (i) and deduced from (i) the following:

(ii) Each closed infinite subset of \( \beta N \) is of the power \( 2^\alpha \).

Now we shall give a very simple proof of (i). Let us consider the Cartesian product \( \Gamma \) of continuously many unit intervals \( I = [0, 1] \). Of course, \( \Gamma \) is a compact space of the power \( 2^\alpha \). On the other hand, \( \Gamma \) may be considered as the set of all functions from \( I \) to \( I \) and it is clear that the set \( M \subset \Gamma \) consisting of all polynomials with rational coefficients is dense in \( \Gamma \). Let \( \varphi \) be any mapping from \( N \) onto \( M \). Then \( \varphi \) is a continuous mapping (because \( N \) has the discrete topology), whence \( \varphi \) can be continuously extended over the whole \( \beta N \); let \( \varphi' \) denote this extension. Of course, the image \( \varphi'(\beta N) \) is a closed subset of \( \Gamma \) and since it contains \( M \), it coincides with \( \Gamma \). Thus \( \beta N \cong \Gamma \). On the other hand, it is plain that \( \beta N \), having an enumerable dense subset, is of the power \( \leq 2^\alpha \). Thus (i) is proved.

Now, following Novák, we can easily show (ii).

Let \( F \) be an infinite closed subset of \( \beta N \). Note that \( F \) contains an enumerable subset \( E \) which is homeomorphic to \( N \). Indeed, this results for instance from the following lemma (see [3], Lemma 1):

If \( X \) is a compact space and \( F \) is a closed infinite subset of \( X \), then there exists a sequence \( G_1, G_2, \ldots \) of mutually disjoint open subsets of \( X \) such that \( F \cap G_n \neq 0 \ (n = 1, 2, \ldots) \).

Of course, if \( p_n \in F \cap G_n \ (n = 1, 2, \ldots) \), then the set \( E = \{ p_1, p_2, \ldots \} \) is homeomorphic to \( N \).