On account of (11) the ratio $\nu_{p(c)}(Z'_{p(c)})/\nu_{p(c)}(Z_{p(c)}) = k = \text{const}$, whence $\lambda(Z') = k\int\limits_B \chi(c)\,d\lambda(c) = k\lambda(Z)$, q. e. d.

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Reçu par la Rédaction le 5. 4. 1958

COLLOQUIUM MATHEMATICUM

VOL. VII 1959 FASC. 1

COMPACTNESS AND PRODUCT SPACES

BY

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In this paper we are concerned with the preserving of different sorts of compactness under the Cartesian multiplication. We shall use the following terminology:

countably compact = each countable open covering contains a finite subcovering;

compact = each open covering contains a finite subcovering;

Lindelöf space = each open covering contains a countable subcovering;

pseudo-compact = each real-valued continuous function is bounded (see [2]).

I. M. Katětov has proved the following theorem (see [3]):

The Cartesian product of two countably compact spaces, one of which is compact, is also countably compact.

In [6] C. Ryll-Nardzewski has proved a similar theorem:

The Cartesian product of two countably compact spaces, one of which satisfies the first axiom of countability, is also countably compact.

Using the theory of Moore-Smits nets (for the definition, properties, notation and terminology see [4], p. 65) we may obtain, by a uniform method, the following theorem:

(i) *The Cartesian product of two countably compact spaces, X and Y, one of which is either compact or sequentially compact, is also countably compact.*

We recall that a space is said to be *sequentially compact* if each sequence of elements of the space contains a convergent subsequence. Of course, each countably compact space satisfying the first axiom of countability is sequentially compact, but not conversely.

Proof of (i). In order to prove that $X\times Y$ is countably compact it suffices to show that each sequence $\{p_n\}$, $p_n = \langle x_n, y_n\rangle$ has a cluster point. Assume that Y is compact. Since X is countably compact, the sequence $\{x_n\}$ contains a subnet $\{x_{n_s}: s\in S\}$ which converges to some point x_0. Since Y is compact, the net $\{y_{n_s}: s\in S\}$ has a cluster point y_0. Of course, the point $p_0 = \langle x_0, y_0\rangle$ is a cluster point of the sequence $\{p_n\}$. Now assume X is sequentially compact. Then the sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ which converges to some point x_0. Since Y is countably compact, the sequence $\{y_{n_k}\}$ has a cluster point y_0 and it is plain that the point $p_0 = \langle x_0, y_0\rangle$ is a cluster point of the sequence $\{p_n\}$. Thus (i) is proved.

In connection with (i) it seems to be of interest to find a necessary and sufficient condition for a space X under which the product $X\times Y$ would be countably compact for any countably compact space Y.

II. In [3] J. Novák has shown that the Cartesian product of two countably compact spaces is not necessarily countably compact. Namely, Novák has shown that there exist two countably compact subsets M and P of βN (N denotes here the space of non-negative integers and βN the maximal Stone-Čech compactification of N) such that $M\cap P = N$. It follows that the diagonal D of the product $M\times P$ (i. e. the set of all $\langle x, y\rangle\in M\times P$ for which $x = y$) consists of all points $p = \langle x, y\rangle$ for which $x\in N$. Thus D is an infinite closed subset of $M\times P$ having no accumulation point.

A similar situation may be observed in connection with other sorts of compactness. Using the above example of Novák, we can show that the product of two pseudo-compact spaces is not necessarily pseudo-compact. Indeed, each point of N is isolated in βN and it follows that each point of the diagonal D is isolated in $M\times P$. Hence, setting $f(p_n) = n$, where p_n is the n-th point of D and $f(p) = 0$ for $p\in M\times P\setminus D$, we obtain a continuous unbounded function on $M\times P$.

On the other hand, it is known that the product of two Lindelöf spaces is not necessarily a Lindelöf space (see e. g. [4], p. 59, example L). Nevertheless, we can show:

(ii) *If X is a pseudo-compact space and Y is compact, then the product $X\times Y$ is pseudo-compact.*

(iii) *If X is a Lindelöf space and Y is compact, then the product $X\times Y$ is a Lindelöf space.*

We start with the following lemma:

LEMMA. *If Y is a compact space, then the projection of any closed subset of $X\times Y$ upon the X-axis is closed in X.*

Proof. Suppose that S is a closed subset of $X\times Y$ and let S_1 be the projection of S upon the X-axis. Let x_0 be any point of S_1. Then there exists a net $\{x_r; r\in R\}$ of points of S_1 which converges to x_0. Now, for each $r\in R$ we can find $y_r\in Y$ such that $\langle x_r, y_r\rangle\in S$. Since Y is compact, the net $\{y_r; r\in R\}$ contains a convergent subnet $\{y_{r_q}; q\in Q\}$. Then the net $\{\langle x_{r_q}, y_{r_q}\rangle; q\in Q\}$ is also convergent; let p_0 denote its limit. Of course, $p_0\in S$ and x_0 is the projection of p_0. Thus $x_0\in S_1$.

Remark. This lemma may also be formulated in the following way:

If Y is a compact space, then for each space X the projection of the product $X\times Y$ upon the X-axis is a closed mapping (i. e. it carries closed subsets of $X\times Y$ into closed subsets of X).

In this form the lemma admits a converse. Indeed, assume that Y is not compact. Then there exists a subset Z of Y of regular potency which does not possess a point of full accumulation (see [1], Théorème 3'). Let ξ be the initial ordinal of the power $\overline{\overline{Z}}$ and let $\{y_0, \ldots, y_\xi, \ldots\}_{\xi<\omega_\lambda}$ be a transfinite sequence (without repetitions) of points of Z. Let X be the space of all ordinals $\xi\leqslant\omega_\lambda$ with the order topology and let S be the set of all points of the product $X\times Y$ of the form $\langle\xi, y_\xi\rangle$ $(\xi<\omega_\lambda)$. We assert that $\overline{S}$ does not contain any point of the form $\langle\omega_\lambda, y\rangle$. Indeed, since y is not a point of full accumulation of Z, there exists a neighbourhood U of y such that $\overline{\overline{U\cap Z}} < \overline{\overline{Z}}$, whence, ω_λ being a regular ordinal, an ordinal $\xi_0<\omega_\lambda$ may be found so that $y_\xi\notin U$ for each $\xi>\xi_0$. Hence $\{\xi: \xi_0<\xi<\omega_\lambda\}\times U$ is a neighbourhood of $\langle\omega_\lambda, y\rangle$ which is free from points of S. Thus $\langle\omega_\lambda, y\rangle$ does not belong to $\overline{S}$. Consequently, the projection of $\overline{S}$ upon the X-axis is nothing else then the set of all ordinals $\xi<\omega_\lambda$, whence it is not closed in X.

Now we pass to the proof of (ii). Let $f(x, y)$ be any continuous real-valued function on $X\times Y$. We can assume, without loss of generality, that $f(x, y)$ is a non-negative function. Since Y is compact, the number $\sup_{y\in Y} f(x, y)$ is finite for each x in X, whence we can define the function

$$g(x) = \sup_{y\in Y} f(x, y).$$

We shall show that $g(x)$ is continuous on X. To prove that it suffices to show that for each real number a the sets $L^a = \{x\in X: g(x)\geqslant a\}$ and $L_a = \{x\in X: g(x)\leqslant a\}$ are closed. But

$$L_a = \bigcap_{y\in Y}\{x\in X: f(x, y)\leqslant a\},$$

whence L_a is closed as the intersection of closed sets. On the other hand,

$$L^a = \bigcap_{\varepsilon>0}\{x\in X: f(x, y)\geqslant a-\varepsilon \text{ for some } y \text{ in } Y\}.$$

But the set $\{x \in X: f(x, y) \geqslant a-\varepsilon$ for some y in $Y\}$ is the projection upon the X-axis of the closed set $\{\langle x, y\rangle \in X\times Y: f(x, y) \geqslant a-\varepsilon\}$, whence, according to the lemma, it is closed. Consequently, L^a is also closed. Finally, $g(x)$ is continuous on X, and thus it is bounded. It implies that $f(x, y)$ is also bounded and (ii) is proved.

Theorem (iii) may easily be proved using the following result of Smirnov [5]:

A subset P of a topological space R is said to be *normally disposed in R* if for each closed set F lying in $R\setminus P$ there exists a G_δ-set containing F and disjoint from P. Then:

if X is a Lindelöf space, then X is normally disposed in any of its compactifications;

if X is normally disposed in some of its compactifications, then X is a Lindelöf space.

By a compactification we understand here any compact space which contains the given space as a dense subset.

Now (iii) can be proved in a few words. Assume that X^* is a compactification of X. Then $X^*\setminus Y$ is a compactification of $X\times Y$. Let F be any closed set lying in $X^*\times Y\setminus X\times Y$ and F_1 — the projection of F upon the X-axis. Of course, F_1 is disjoint from X, and, by the lemma, it is closed. Thus there exists a G_δ-set G which contains F_1 and does not meet X. Of course, the counter-image of G under the projection is a G_δ-set which contains F and is disjoint from $X\times Y$. Thus $X\times Y$ is normally disposed in $X^*\times Y$ and it follows that $X\times Y$ is a Lindelöf space.

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Reçu par la Rédaction le 10. 10. 1958

COLLOQUIUM MATHEMATICUM

VOL. VII 1959 FASC. 1

ON THE POTENCY OF SUBSETS OF βN

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Let N be the space of positive integers and βN — the maximal Stone-Čech compactification of N. In [5] B. Pospíšil has shown the following:

(i) *The potency of βN is equal to $2^{\mathfrak{c}}$.*

In [4] J. Novàk has given another proof of (i) and deduced from (i) the following:

(ii) *Each closed infinite subset of βN is of the power $2^{\mathfrak{c}}$.*

Now we shall give a very simple proof of (i). Let us consider the Cartesian product $I^{\mathfrak{c}}$ of continuously many unit intervals $I = [0, 1]$. Of course, $I^{\mathfrak{c}}$ is a compact space of the power $2^{\mathfrak{c}}$. On the other hand, $I^{\mathfrak{c}}$ may be considered as the set of all functions from I to I and it is clear that the set $M \subset I^{\mathfrak{c}}$ consisting of all polynomials with rational coefficients is dense in $I^{\mathfrak{c}}$. Let φ be any mapping from N onto M. Then φ is a continuous mapping (because N has the discrete topology), whence φ can be continuously extended over the whole βN; let φ^* denote this extension. Of course, the image $\varphi^*(\beta N)$ is a closed subset of $I^{\mathfrak{c}}$ and since it contains M, it coincides with $I^{\mathfrak{c}}$. Thus $\overline{\overline{\beta N}} \geqslant 2^{\mathfrak{c}}$. On the other hand, it is plain that βN, having an enumerable dense subset, is of the power $\leqslant 2^{\mathfrak{c}}$. Thus (i) is proved.

Now, following Novák, we can easily show (ii).

Let F be any infinite closed subset of βN. Note that F contains an enumerable subset E which is homeomorphic to N. Indeed, this results for instance from the following lemma (see [3], Lemma 1):

If X is a compact space and F is a closed infinite subset of X, then there exists a sequence $G_1, G_2, \ldots$ of mutually disjoint open subsets of X such that $F \cap G_n \neq 0$ $(n = 1, 2, \ldots)$.

Of course, if $p_n \in F \cap G_n$ $(n = 1, 2, \ldots)$, then the set $E = \{p_1, p_2, \ldots\}$ is homeomorphic to N.