COLLOQUIUM MATHEMATICUM

VOL. VII 1960 FASC, 2

A CHARACTERIZATION OF STEP FUNCTIONS

BY

H. FAST AND K. URBANIK (WROCŁAW)

To every Lebesgue measurable function f on an interval I assign a functional $\alpha_I(f)$, to be called its *index of variability*, defined by the formula

$$\alpha_I(f) = \lim_{\varepsilon\to 0+} \varliminf_{h\to 0+} \frac{1}{h} \left|\{x: |f(x+h)-f(x)| > \varepsilon h,\ x\in I,\ x+h\in I\}\right|,$$

where $|A|$ denotes the Lebesgue measure of a set A. It is easy to see that $\alpha_I(f_1) = \alpha_I(f_2)$ if f_1 is equivalent to f_2 in I, i. e. if f_1 is equal to f_2 almost everywhere in I. Moreover, we have the inequality $\alpha_{I_1}(f) \leqslant \alpha_{I_2}(f)$ when $I_1 \subset I_2$.

A function f defined in an interval I will be called a *step function with n jumps* if it has n points of discontinuity only, all of them belonging to the interior of I, if it is continuous on the right and if it is identically constant in every interval of continuity.

The aim of the present note is to give a characterisation of step functions by the index of variability. Namely, we shall prove the following

THEOREM. *The values of the index of variability are either non-negative integers or the infinity. Moreover, $\alpha_I(f) = n$ $(n = 0, 1, \ldots)$ if and only if f is equivalent to a step function with n jumps.*

Before proving the Theorem we shall prove three Lemmas.

LEMMA 1. *If f is continuous in I and $\alpha_I(f) < \infty$, then it is a constant function.*

Proof. Let I_0 be a closed interval contained in the interior of I. We choose a positive number h_0 so that $x+h_0\in I$, when $x\in I_0$. Setting

(1) $$A(\varepsilon, h) = \{x: |f(x+h)-f(x)| > \varepsilon h,\ x\in I,\ x+h\in I\},$$

(2) $$B(\varepsilon, h) = \{x: |f(x+h)-f(x)| \leqslant \varepsilon h,\ x\in I,\ x+h\in I\},$$

we have for every $\varepsilon > 0$ and $0 < h \leqslant h_0$ the following inequality:

$$\int_{I_0} \frac{|f(x+h)-f(x)|}{h}\,dx \leqslant \int_{A(\varepsilon,h)} \frac{|f(x+h)-f(x)|}{h}\,dx + \int_{B(\varepsilon,h)} \frac{|f(x+h)-f(x)|}{h}\,dx$$
$$\leqslant \frac{1}{h}\,|A(\varepsilon,h)|\cdot\max_{x\epsilon I_0}|f(x+h)-f(x)| + \varepsilon|B(\varepsilon,h)|.$$

Taking into account the inequality $a_I(f) < \infty$, the continuity of f and the arbitrariness of ε, we infer that

$$\lim_{\overline{h\to 0+}} \int_{I_0} \frac{|f(x+h)-f(x)|}{h}\,dx = 0. \tag{3}$$

Further, for every pair $y_1 < y_2$ $(y_1, y_2 \epsilon I_0)$ we have the equality

$$f(y_2)-f(y_1) =$$
$$\lim_{h\to 0+}\frac{1}{h}\int_{y_2}^{y_2+h} f(x)\,dx - \lim_{h\to 0+}\int_{y_1}^{y_1+h} f(x)\,dx = \lim_{h\to 0+}\int_{y_1}^{y_2} \frac{f(x+h)-f(x)}{h}\,dx.$$

Hence and from (3) it follows that f is constant in I_0 and, consequently, in I.

LEMMA 2. *If f is measurable and bounded on I and if $a_I(f) < \infty$, then it is equivalent in I to a function of bounded variation.*

Proof. We may suppose, without loss of generality, that f is periodic on the line of period $|I|$. Setting

$$M = \sup_{x\epsilon I}|f(x)|, \qquad C(h) = \{x: x\epsilon I,\ x+h \notin I\}$$

and using notations (1) and (2) we get the inequality

$$\int_I \frac{|f(x+h)-f(x)|}{h}\,dx \leqslant \int_{A(\varepsilon,h)} \frac{|f(x+h)-f(x)|}{h}\,dx + \int_{B(\varepsilon,h)} \frac{|f(x+h)-f(x)|}{h}\,dx +$$
$$+\int_{C(h)} \frac{|f(x+h)-f(x)|}{h}\,dx \leqslant 2M\frac{1}{h}\,|A(\varepsilon,h)| + \varepsilon|B(\varepsilon,h)| + 2M\frac{1}{4}\,|C(h)|,$$

whence for $\varepsilon\to 0+$, $h\to 0+$ we obtain the inequality

$$\lim_{\overline{h\to 0+}} \int_I \frac{|f(x+h)-f(x)|}{h}\,dx \leqslant 2M\big(1+a_I(f)\big) < \infty. \tag{4}$$

It follows from this inequality, in virtue of a modified theorem of Hardy and Littlewood (see [1]), that f is equivalent in I to a function of bounded variation. The proof of this modified Theorem is exactly the same as that of the original one, given in [2] (p. 106). Namely, from (4) we get

$$\int_I |\sigma_n'(x)|\,dx = \lim_{h\to 0+}\int_I \frac{|\sigma_n(x+h)-\sigma_n(x)|}{h}\,dx$$
$$\leqslant \lim_{\overline{h\to 0+}}\int_I \frac{|f(x+h)-f(x)|}{h}\,dx \qquad (n = 1, 2, \ldots)$$

where $\sigma_n(x)$ is the n-th Fejér mean of the Fourier series of f. Whence, in view of Theorem 4.325 in [2] (p. 82), follows the Theorem.

LEMMA 3. *If a measurable function f has at least n jumps in the interior of I, then $a_I(f) \geqslant n$.*

Proof. Let us consider a system of n points of jumps $a_1 < a_2 < \ldots \ldots < a_n$ belonging to the interior of I. For every $\varepsilon > 0$ we choose a positive number $h(\varepsilon)$ so that

$$|f(a_{j+})-f(x_1)| < \varepsilon \quad \text{for} \quad a_j < x_j < a_j+h(\varepsilon),$$
$$|f(a_{j-})-f(x_2)| < \varepsilon \quad \text{for} \quad a_j-h(\varepsilon) < x_2 < a_j \quad (j = 1, 2, \ldots, n).$$

Consequently for $a_j-h < x < a_j$ $(j = 1, 2, \ldots, n;\ 0 < h \leqslant h(\varepsilon))$ we have the inequality

$$|f(a_{j+})-f(a_{j-})| \leqslant |f(a_{j+})-f(x+h)| + |f(x+h)-f(x)| +$$
$$+|f(a_{j-})-f(x)| \leqslant 2\varepsilon + |f(x+h)-f(x)|.$$

Hence, when $0 < \varepsilon < \frac{1}{3}\min\limits_{1\leqslant j\leqslant n}|f(a_{j+})-f(a_{j-})|$ and $0 < h < \min(1, h(\varepsilon))$, we get the inequality

$$|f(x+h)-f(x)| > \varepsilon h$$

for $a_j-h < x < a_j$ $(j = 1, 2, \ldots, n)$. In other words, all the intervals $a_j-h < x < a_j$ $(j = 1, 2, \ldots, n)$ are contained in the set $A(\varepsilon, h)$ defined by formula (1). For sufficiently small h the intervals $a_j-h < x < a_j$ $(j = 1, 2, \ldots, n)$ are disjoint, which implies the inequality $|A(\varepsilon,h)| \geqslant nh$. Consequently

$$a_I(f) = \lim_{\varepsilon\to 0+}\lim_{\overline{h\to 0+}}\frac{1}{h}|A(\varepsilon,h)| \geqslant n.$$

The Lemma is thus proved.

Proof of the Theorem. Let us assume that $a_I(f) < \infty$. For every positive integer N we set

$$f_N(x) = \begin{cases} f(x) & \text{if } |f(x)| \leqslant N, \\ N & \text{if } f(x) > N, \\ -N & \text{if } f(x) < -N. \end{cases}$$

Obviously, $a_I(f_N) \leqslant a_I(f)$. Further, by Lemma 2, f_N is equivalent in I to a function f_N^* of bounded variation. Moreover, we may suppose that f_N^* is continuous on the right. By Lemma 3, f_N^* has at most $[a_I(f)]$ points of jumps, in the interior of I. By Lemma 1, f_N^* is identically constant in every interval of continuity. Hence it follows that f_N $(N = 1, 2, \ldots)$ are equivalent in I to step functions with at most $[a_I(f)]$ jumps. Thus f is equivalent in I to a step function. It is easy to verify that for a step function its index of variability is equal to the number of jumps. Hence directly follows the assertion of our Theorem.

REFENCES

[1] G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals,* Mathematische Zeitschrift 27 (1928), p. 565-606.

[2] A. Zygmund, *Trigonometrical Series,* Warszawa-Lwów 1935.

Reçu par la Rédaction le 16. 3. 1959

COLLOQUIUM MATHEMATICUM

VOL. VII 1960 FASC. 2

SUR LES THÉORÈMES DE J. MYCIELSKI ET W. GUSTIN CONCERNANT LES DÉCOMPOSITIONS DE L'INTERVALLE

PAR

Á. CSÁSZÁR (BUDAPEST) ET S. MARCUS (BUCAREST)

Dans un travail récent J. Mycielski a démontré le théorème suivant [2]:

Pour tout nombre transfini $\mathfrak{m}$, inférieur ou égal à la puissance du continu, chacun des intervalles $(0,1)$, $\langle 0,1)$, $\langle 0,1\rangle$ est une réunion de $\mathfrak{m}$ ensembles disjoints, superposables deux à deux par translation.

En ce qui concerne l'intervalle $\langle 0,1)$, il est évident que le théorème de J. Mycielski reste valable aussi dans le cas où $\mathfrak{m}$ est un nombre entier positif. Ce n'est pas le cas pour les intervalles $(0,1)$ et $\langle 0,1\rangle$, comme il résulte du théorème suivant, établi par W. Gustin [1]:

Pour tout nombre entier $n > 1$, aucun des intervalles $(0,1)$ et $\langle 0,1\rangle$ ne peut être décomposé en n ensembles disjoints superposables par translation ou par rotation.

La démonstration de ce théorème donnée par W. Gustin est assez compliquée et occupe dix pages (d'ailleurs on y démontre un résultat plus général). Il ne sera peut être pas dépourvu d'intérêt de montrer que le théorème de W. Gustin admet une démonstration simple et directe, dès qu'on supprime dans l'énoncé les derniers trois mots: „ou par rotation". On obtiendra ainsi une démonstration rapide du fait que dans le théorème de J. Mycielski on ne peut pas remplacer $\mathfrak{m}$ par un entier supérieur à 1.

THÉORÈME. *Il n'existe aucune décomposition de l'intervalle* $(0,1)$ *ou* $\langle 0,1\rangle$ *en* $n > 1$ *ensembles disjoints, superposables, deux à deux, par translation.*

Démonstration. Soit $I = (0,1)$ ou $\langle 0,1\rangle$, $I = \bigcup_{1}^{n} E_i$, $E_i \cap E_j = 0$ $(i \neq j)$, $E_i + a_{ij} = E_j$.

Le nombre des composantes des E_i est fini. En effet, si ce nombre était infini, il existerait dans $\langle 0,1\rangle$ un point dont chaque voisinage rencontre une infinité de ces composantes. L'ensemble des points de ce