To prove property (H) for all compacts $C \subset \hat{G}$ it suffices to prove (H) for all compacts C of the form

$$C = E \times I \times T^{m},$$

where E is a finite subset of $\hat{B}$ and I is an interval in R^n of the form

$$I = \bigcap_{j=1}^{n} \{\langle t_1, t_2, \ldots, t_n\rangle : |t_j| \leqslant a\}.$$

It is clear that the characters

$$\exp\left(i\frac{\pi}{a}t_j k\right) \qquad (j = 1, 2, \ldots, n;\ k = 0, \pm 1, \pm 2, \ldots)$$

of R^n separate points of I. Further, there is a finite system of characters of $\hat{\hat{B}}$, i. e. elements of B which separates points of E. Finally, the character group of T^m, i. e. Δ^m, is finitely generated. Hence it follows that there is a finitely generated algebraic subgroup D, of G which separates points of the set C. From the Stone-Weierstrass Theorem ([2], p. 9) it follows that every continuous function on C can be uniformly approximated on C by finite linear combinations of elements belonging to D, considered as functions on C. Let $\tilde{\mathcal{F}}(C)$ be the image of $\mathcal{F}(C)$ under the Fourier-Plancherel transformation. Every function belonging to $\tilde{\mathcal{F}}(C)$ can be approximated in $\mathcal{L}^2(\hat{G})$-norm by continuous functions vanishing off C (see [1], § 55). Consequently, every function g belonging to $\tilde{\mathcal{F}}(C)$ is uniquely determined by inner products

$$\int_C g(\chi)\chi(y)\mu(d\chi) \qquad (y \in D). \tag{2}$$

By the continuity of $f \in \mathcal{F}(C)$ we have the equality

$$f(y) = \int_C \tilde{f}(\chi)\chi(y)\mu(d\chi).$$

Therefore by (2) it follows that the Fourier-Plancherel transform of f and, consequently, the function f itself is uniquely determined by the values $f(y)$ $(y \in D)$. The theorem is thus proved.

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A LIMIT THEOREM FOR RANDOM VARIABLES IN COMPACT TOPOLOGICAL GROUPS

BY

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I. Let G be a compact (not necessarily Abelian) topological group. A regular completely additive measure μ defined on the class of all Borel subsets of G, with $\mu(G) = 1$, will be called a *probability distribution*. A sequence of probability distributions $\mu_1, \mu_2, \ldots$ is said to be *weakly convergent* to a probability distribution μ if

$$\lim_{n\to\infty} \int_G f(g)\mu_n(dg) = \int_G f(g)\mu(dg)$$

for any complex-valued continuous function f defined on G.

A G-valued random variable is called *symmetric* if its probability distribution μ is invariant under the transformation $g \to g^{-1}$, i. e. if $\mu(E) = \mu(E^{-1})$ for each Borel subset $E \subset G$, where $E^{-1} = \{g^{-1} : g \in E\}$.

Let $X_1, X_2, \ldots$ be a sequence of independent G-valued random variables with probability distributions $\mu_1, \mu_2, \ldots$ Put

$$Y_n = X_1 \cdot X_2 \cdot \ldots \cdot X_n \qquad (n = 1, 2, \ldots),$$

where the product is taken in the sense of group multiplication in G. It is well known that the probability distribution ν_n of the random variable Y_n is given by the formula

$$\nu_n = \mu_1 * \mu_2 * \ldots * \mu_n \qquad (n = 1, 2, \ldots),$$

where the convolution $*$ is defined by

$$\nu * \lambda(E) = \int_G \nu(Eg^{-1})\lambda(dg)\ (^1).$$

The limiting distribution of the sequence $Y_1, Y_2, \ldots$ is the weak limit of the probability distributions $\nu_1, \nu_2, \ldots$.

(1) $Eg^{-1} = \{hg^{-1} : h \in E\}$.

A probability distribution λ is called *positive* if $\lambda(V) > 0$ for every open non-empty subset $V \subset G$. For example the uniform distribution on G, i. e. the normed Haar measure on G, is positive.

In the present paper we shall prove the following theorem, which is a solution of a problem raised when constructing generators of stochastic processes:

THEOREM. *Let λ be a positive probability distribution on G and let $X_1, X_2, \ldots$ be a sequence of symmetric independent G-valued random variables with probability distributions $\mu_1, \mu_2, \ldots$ If for every Borel subset $E \subset G$ we have the inequality*

(1) $$\mu_n(E) \geqslant a_n \lambda(E) \qquad (0 \leqslant a_n \leqslant 1;\ n = 1, 2, \ldots),$$

where

(2) $$\sum_{n=1}^{\infty} a_n = \infty,$$

then the limiting distribution of products $X_1 \cdot X_2 \cdot \ldots \cdot X_n$ is uniform on G

Moreover, condition (2) is essential, i. e. for every sequence $a_1, a_2, \ldots$. $(0 \leqslant a_n \leqslant 1)$ for which

$$\sum_{n=1}^{\infty} a_n < \infty,$$

there is a sequence $X_1, X_2, \ldots$ of symmetric independent G-valued random variables with probability distributions satisfying (1) such that the limiting distribution of $X_1 \cdot X_2 \cdot \ldots \cdot X_n$ $(n = 1, 2, \ldots)$ is not uniform.

II. Before proving the Theorem we shall give some elementary properties of characteristic functions.

$\mathfrak{U}(G)$ will denote the class of all continuous finitely dimensional irreducible unitary representations of the group G (2). The matrix-valued function

$$\varphi_\mu(\mathcal{U}) = \int_G \mathcal{U}(g)\mu(dg) \qquad (\mathcal{U} \in \mathfrak{U}(G))$$

is called the *characteristic function* of the probability distribution μ. It is well known that the probability distribution is uniquely determined by its characteristic function. Moreover, the weak convergence of probability distributions is equivalent to the convergence of their characteristic functions. Further, it is easy to prove that

$$\varphi_{\mu * \nu}(\mathcal{U}) = \varphi_\mu(\mathcal{U})\varphi_\nu(\mathcal{U}).$$

(2) See A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris 1940, Chapitre IV.

Z^n will denote the n-dimensional complex Euclidean space with inner product

$$(x, y) = \sum_{j=1}^{n} x_j \bar{y}_j$$

and norm

$$\|x\| = \sqrt{(x, x)},$$

where $x = \langle x_1, x_2, \ldots, x_n \rangle$, $y = \langle y_1, y_2, \ldots, y_n \rangle$. Let us consider a linear transformation of Z^n determined by a matrix $A: x \to Ax$ $(x \in Z^n)$. The norm of A is defined as follows:

$$\|A\| = \sup_{\|x\|=1} \|Ax\|.$$

If A is any matrix, then there exist two uniquely determined Hermitian matrices $\mathcal{R}A$ and $\mathcal{I}A$ such that

$$A = \mathcal{R}A + i\mathcal{I}A.$$

Moreover, the inequalities

$$\|\mathcal{R}A\| \leqslant \|A\|, \qquad \|\mathcal{I}A\| \leqslant \|A\|$$

hold.

LEMMA 1. *If A is a matrix with a norm not greater than 1, $|h| = 1$ and*

(3) $$(Ax_0, x_0) = h(x_0, x_0) \qquad (x_0 \neq 0),$$

then x_0 is a proper vector of A for the proper value h.

Proof. Setting

(4) $$Ax_0 = px_0 + y,$$

where y is orthogonal to x_0 and p is a complex constant, we have the equality

$$(Ax_0, x_0) = p(x_0, x_0).$$

Hence, in view of (3), we get the equality $p(x_0, x_0) = h(x_0, x_0)$ and, consequently,

(5) $$p = h.$$

Further, taking into account equality (4), we obtain, by the Pythagorean theorem,

$$\|Ax_0\|^2 = |p|^2\|x_0\|^2 + \|y\|^2 = \|x_0\|^2 + \|y\|^2.$$

Since $\|A\| \leqslant 1$, the last equality implies

$$\|x_0\|^2 + \|y\|^2 \leqslant \|x_0\|^2.$$

Thus $y = 0$. Consequently, according to (4) and (5), we have the equality $Ax_0 = hx_0$, which was to be proved.

LEMMA 2. *If U is a unitary matrix and if x_0 is a proper vector of $\mathcal{R}U$ for the proper value h, where $|h| = 1$, then x_0 is also a proper vector of U for the same proper value.*

Proof. It is well known that for every unitary matrix U the matrix $\mathcal{R}U$ commutes with the matrix $\mathcal{I}U$. Consequently,

$$(\mathcal{R}U)^2+(\mathcal{I}U)^2 = UU^* = \mathcal{E},$$

where U^* denotes the adjoint of U and $\mathcal{E}$ denotes the unit matrix. Hence we get the equality

$$\begin{aligned}\|x_0\|^2 = (UU^*x_0, x_0) &= \big((\mathcal{R}U)^2x_0, x_0\big)+\big((\mathcal{I}U)^2x_0, x_0\big)\\ &= (\mathcal{R}Ux_0, \mathcal{R}Ux_0)+(\mathcal{I}Ux_0, \mathcal{I}Ux_0)\\ &= (hx_0, hx_0)+\|\mathcal{I}Ux_0\|^2 = \|x_0\|^2+\|\mathcal{I}Ux_0\|^2,\end{aligned}$$

which implies $\|\mathcal{I}Ux_0\|^2 = 0$ and, consequently, $\mathcal{I}Ux_0 = 0$. From the last equality it follows that $Ux_0 = \mathcal{R}Ux_0+i\mathcal{I}Ux_0 = hx_0$. The lemma is thus proved.

It is well known that for every Hermitian matrix A the inner product (Ax, x) is real for every vector x. In particular, $(\mathcal{R}Ax, x)$ is real for every matrix A and for every vector x.

LEMMA 3. *Let λ be a positive probability distribution on G. For every representation $\mathcal{U} \in \mathfrak{U}(G)$ and for $h = 1$ or -1 there exists a positive c such that*

$$\inf_{\|x\|=1} \lambda\{g: 1-h\big(\mathcal{R}\mathcal{U}(g)x, x\big) > c\} > 0.$$

Proof. Contrary to the statement of our lemma let us suppose that for every integer r there exists a sequence of vectors $x_1^{(r)}, x_2^{(r)}, \ldots,$ with $x_j^{(r)} = 1$ $(j = 1, 2, \ldots)$, for which

$$\lim_{j\to\infty} \lambda\{g: 1-h\big(\mathcal{R}\mathcal{U}(g)x_j^{(r)}, x_j^{(r)}\big) > r^{-1}\} = 0. \tag{6}$$

Since the sphere $\|x\| = 1$ is compact, we suppose, without loss of generality of our consideration, that the sequence $x_1^{(r)}, x_2^{(r)}, \ldots$ converges to a vector $x^{(r)}$. Obviously, the following inclusion holds:

$$\{g: 1-h\big(\mathcal{R}\mathcal{U}(g)x^{(r)}, x^{(r)}\big) > r^{-1}\} \subset \liminf_{j\to\infty} \{g: 1-h\big(\mathcal{R}\mathcal{U}(g)x_j^{(r)}, x_j^{(r)}\big) > r^{-1}\}.$$

Hence, according to (6),

$$\lambda\{g: 1-h\big(\mathcal{R}\mathcal{U}(g)x^{(r)}, x^{(r)}\big) > r^{-1}\} = 0.$$

Further, without loss of generality, we may suppose that the sequence $x^{(1)}, x^{(2)}, \ldots$ converges to a vector x_0, with $\|x_0\| = 1$. It is easy to prove that for every a the inclusion

$$\{g: 1-h\big(\mathcal{R}\mathcal{U}(g)x_0, x_0\big) > a\} \subset \liminf_{r\to\infty} \{g: 1-h\big(\mathcal{R}\mathcal{U}(g)x^{(r)}, x^{(r)}\big) > a\}$$

is true. Consequently, in virtue of (7), we get the equality

$$\lambda\{g: 1-h\big(\mathcal{R}\mathcal{U}(g)x_0, x_0\big) > 0\} = 0. \tag{8}$$

From the inequality $|\big(\mathcal{R}\mathcal{U}(g)x_0, x_0\big)| \leqslant \|\mathcal{R}\mathcal{U}(g)\| \leqslant \|\mathcal{U}(g)\| = 1$ it follows that $1-h\big(\mathcal{R}\mathcal{U}(g)x_0, x_0\big) \geqslant 0$ for each $g \in G$. Consequently, formula (8) implies the equality

$$1-h\big(\mathcal{R}\mathcal{U}(g)x_0, x_0\big) = 0 \tag{9}$$

for λ-almost all $g \in G$. By the continuity of the representation U and the posititivity of λ, equality (9) holds for every $g \in G$. Since $\|x_0\| = 1$, we infer, in view of (9), that

$$\big(\mathcal{R}\mathcal{U}(g)x_0, x_0\big) = h(x_0, x_0)$$

for all $g \in G$. Thus, in virtue of lemma 1, x_0 is a proper vector of $\mathcal{R}\mathcal{U}(g)$ $(g \in G)$ for the proper value h and, moreover, in virtue of Lemma 2, x_0 is a proper vector of $\mathcal{U}(g)$ $(g \in G)$ for the proper value h, which contradicts the irreducibility of the representation $\mathcal{U}$. The Lemma is thus proved.

LEMMA 4. *If μ is a symmetric probability distribution, then, for every $\mathcal{U} \in \mathfrak{U}(G)$, $\varphi_\mu(\mathcal{U})$ is a Hermitian matrix and*

$$\varphi_\mu(\mathcal{U}) = \int_G \mathcal{R}\mathcal{U}(g)\mu(dg).$$

Proof. By the symmetry of μ we obtain the equality

$$\begin{aligned}\varphi_\mu(\mathcal{U}) = \int_G \mathcal{U}(g)\mu(dg) &= \int_G \mathcal{U}(g^{-1})\mu(dg) = \int_G \mathcal{U}^{-1}(g)\mu(dg)\\ &= \int_G \mathcal{U}^*(g)\mu(dg) = \varphi_\mu^*(\mathcal{U}).\end{aligned}$$

Consequently, $\varphi_\mu(\mathcal{U})$ is a Hermitian matrix. Since $\mathcal{R}\mathcal{U}(g)$ $(g \in G)$ are Hermitian matrices, $\int_G \mathcal{R}\mathcal{U}(g)\mu(dg)$ is also a Hermitian matrix. Thus the matrix

$$i\int_G \mathcal{I}\mathcal{U}(g)\mu(dg) = \varphi_\mu(\mathcal{U})-\int_G \mathcal{R}\mathcal{U}(g)\mu(dg)$$

is Hermitian. Consequently,

$$\varphi_\mu(\mathcal{U})-\int_G \mathcal{R}\mathcal{U}(g)\mu(dg) = \tfrac{1}{2}\Big\{i\int_G \mathcal{I}\mathcal{U}(g)\mu(dg)+\Big(i\int_G \mathcal{I}\mathcal{U}(g)\mu(dg)\Big)^*\Big\}$$

$$= \tfrac{1}{2}\Big(i\int_G \mathcal{I}\mathcal{U}(g)\mu(dg)-i\int_G \mathcal{I}\mathcal{U}(g)\mu(dg)\Big) = 0.$$

The lemma is thus proved.

LEMMA 5. *Let λ be a positive probability distribution. For every $\mathcal{U}\in\mathfrak{U}(G)$ there exists a positive constant b such that for every symmetric probability distribution μ satisfying the condition*

(10) $$\mu(E)\geqslant a\lambda(E) \qquad (E\subset G)$$

the following inequality holds:

$$\|\varphi_\mu(\mathcal{U})\|\leqslant 1-ab.$$

Proof. Let $\mathcal{U}\in\mathfrak{U}(G)$. By Lemma 3 there exists a positive number c such that the greatest lower bounds of the λ-measures of sets

$$E_x = \{g: 1-h(\mathcal{R}\mathcal{U}(g)x, x) > c\},$$

where $\|x\|=1$ and $h=1$ or -1, are positive:

(11) $$d = \inf_{\substack{\|x\|=1\\ h=\pm1}} \lambda(E_x) > 0.$$

Since $1-h(\mathcal{R}\mathcal{U}(g)x, x)\geqslant 0$ for $\|x\|=1$, $h=1$ or -1, we have, applying lemma 4, the inequality

$$1-h(\varphi_\mu(\mathcal{U})x, x) = \int_G \big(1-h(\mathcal{R}\mathcal{U}(g)x, x)\big)\mu(dg)$$

$$\geqslant \int_{E_x}\big(1-h(\mathcal{R}\mathcal{U}(g)x, x)\big)\mu(dg)\geqslant c\mu(E_x)$$

for every vector x with $\|x\|=1$. If μ satisfies condition (10), then the last inequality and (11) imply

$$1-h(\varphi_\mu(\mathcal{U})x, x)\geqslant ca\lambda(E_x)\geqslant cad$$

for every vector x with $\|x\|=1$ and for $h=1$ or -1. Hence, putting $b=cd>0$, we get the inequality

(12) $$\sup_{\|x\|=1}|(\varphi_\mu(\mathcal{U})x, x)|\leqslant 1-ab.$$

It is well known that

(13) $$\|A\| = \sup_{\|x\|=1}|(Ax, x)|$$

for every Hermitian matrix A. Since, by Lemma 4, $\varphi_\mu(\mathcal{U})$ is a Hermitian matrix for symmetric probability distributions, we have, in view of (12) and (13), the assertion of our Lemma.

Proof of the theorem. To prove the first part of the theorem it suffices to show that for symmetric probability distributions $\mu_1, \mu_2, \ldots$ satisfying conditions (1) and (2) the weak limit of $\mu_1*\mu_2*\ldots*\mu_n$ $(n=1,2,\ldots)$ is the Haar measure on G.

Let $\mathcal{U}\in\mathfrak{U}(G)$. By Lemma 5 and condition (1) there exists a positive constant b such that

$$\|\varphi_{\mu_n}(\mathcal{U})\|\leqslant 1-a_n b \qquad (n=1,2,\ldots).$$

Hence we get the inequality

$$\|\varphi_{\mu_1*\mu_2*\ldots*\mu_n}(\mathcal{U})\| = \|\varphi_{\mu_1}(\mathcal{U})\varphi_{\mu_2}(\mathcal{U})\ldots\varphi_{\mu_n}(\mathcal{U})\|$$

$$\leqslant \prod_{j=1}^{n}\|\varphi_{\mu_j}(\mathcal{U})\|\leqslant\prod_{j=1}^{n}(1-a_n b).$$

Therefore, in view of (2),

$$\lim_{n\to\infty}\|\varphi_{\mu_1*\mu_2*\ldots*\mu_n}(\mathcal{U})\| = 0$$

for every $\mathcal{U}\in\mathfrak{U}(G)$. In other words,

(14) $$\lim_{n\to\infty}\varphi_{\mu_1*\mu_2*\ldots*\mu_n}(\mathcal{U}) = 0$$

for every $\mathcal{U}\in\mathfrak{U}(G)$.

Since the characteristic function of the Haar measure m on G is equal to 0: $\varphi_m(\mathcal{U}) = 0$ $(\mathcal{U}\in\mathfrak{U}(G))$ (see op. cit. [2], Chapitre V), we have, according to (14),

$$\lim_{n\to\infty}\varphi_{\mu_1*\mu_2*\ldots*\mu_n}(\mathcal{U}) = \varphi_m(\mathcal{U})$$

for every $\mathcal{U}\in\mathfrak{U}(G)$, which implies the weak convergence of $\mu_1*\mu_2*\ldots*\mu_n$ to m. The first part of the theorem is thus proved.

Now let us suppose that a sequence $a_1, a_2, \ldots$ $(0\leqslant a_n\leqslant 1)$ satisfies the condition

(15) $$\sum_{n=1}^{\infty} a_n < \infty.$$

For given $\mathcal{U}_0\in\mathfrak{U}(G)$ we set

(16) $$V_n = \{g: \|\mathcal{U}_0(g)-\mathcal{E}\| < n^{-2}\} \qquad (n=1,2,\ldots).$$

Obviously, $V_n = V_n^{-1}$ and $\lambda(V_n) > 0$ $(n = 1, 2, \ldots)$. Further, we define the sequence of symmetric probability distributions $\mu_1, \mu_2, \ldots$ by the following formula:

$$\mu_n(E) = \frac{1 - a_n + a_n\lambda(V_n)}{\lambda(V_n)}\lambda(E \cap V_n) + a_n\lambda(E \setminus V_n) \qquad (n = 1, 2, \ldots).$$

It is easy to verify that the inequality

$$\mu_n(E) \geqslant a_n\lambda(E) \qquad (n = 1, 2, \ldots)$$

is true for every Borel subset $E \subset G$. Moreover, in virtue of (16),

$$\|\varphi_{\mu_n}(\mathcal{U}_0) - \mathcal{E}\| \leqslant \int_G \|\mathcal{U}_0(g) - \mathcal{E}\|\mu_n(dg)$$

$$= \frac{1 - a_n + a_n\lambda(V_n)}{\lambda(V_n)} \int_{V_n} \|\mathcal{U}_0(g) - \mathcal{E}\|\lambda(dg) + a_n \int_{G\setminus V_n} \|\mathcal{U}_0(g) - \mathcal{E}\|\lambda(dg) \leqslant n^{-2} + 2a_n$$
$$(n = 1, 2, \ldots).$$

Hence, according to (15), we get the inequality

$$\sum_{n=1}^{\infty} \|\varphi_{\mu_n}(\mathcal{U}_0) - \mathcal{E}\| < \infty,$$

which implies the convergence

$$\lim_{n\to\infty} \varphi_{\mu_1 * \mu_2 * \ldots * \mu_n}(\mathcal{U}_0) = \lim_{n\to\infty} \varphi_{\mu_1}(\mathcal{U}_0)\varphi_{\mu_2}(\mathcal{U}_0) \ldots \varphi_{\mu_n}(\mathcal{U}_0) \neq 0.$$

Consequently, the Haar measure is not the weak limit of the probability distributions $\mu_1 * \mu_2 * \ldots * \mu_n$ $(n = 1, 2, \ldots)$. The Theorem is thus proved.

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ON THE POWER OF COMPACT SPACES

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Let X be a compact (= bicompact) infinite space. The character $\Theta(p)$ of a point $p \in X$ is the least cardinal of a family of open sets containing the point p and having that point as its intersection.

Recently S. Mrówka [4] has shown the following

THEOREM. *If* $\mathfrak{m} \leqslant \Theta(p)$ *for all* $p \in X$, *then* $\overline{\overline{X}} \geqslant 2^{\mathfrak{m}}$.

In this note we present a somewhat simpler proof of this theorem. The main idea of the proof is similar to that of my papers [1] and [2] and has some connections with the idea used by F. B. Jones in [3].

Proof. Denote by I^α the set of all 0-1 sequences of the ordinal type α. Let $\xi \in I^\alpha$ and $\beta < \alpha$. Denote by ξ_β the segment of the type β of the sequence ξ. For each α we are going to define a family $\{V(\xi): \xi \in I^\alpha\}$ of open sets of the space X. We use the transfinite induction: for $\alpha = 1$ we have two one-element sequences 0 and 1. Let $V(0)$ and $V(1)$ be two arbitrary disjoint open subsets of X. Suppose we have defined $V(\xi)$ for all $\xi \in I^\beta$ and $\beta < \alpha$. In order to define $V(\xi)$ for $\xi \in I^\alpha$ consider two possibilities: (a) α has a precedent and (b) α is a limit-number.

(a) Put $\alpha = \alpha' + 1$. Then either for some $\xi \in I^{\alpha'}$ the set $\bigcap_{\beta \leqslant \alpha'} V(\xi_\beta)$ contains at most one point or for every $\xi \in I^{\alpha'}$ it consists of two points at least. In the first case, the sets $V(\xi)$ for $\xi \in I^\alpha$ will not be defined. In the second case, given a $\xi \in I^{\alpha'}$, let x_0 and x_1 be two points of $\bigcap_{\beta \leqslant \alpha'} V(\xi_\beta)$.

There are exactly two different sequences η^0 and η^1 belonging to I^α such that $\eta^0_{\alpha'} = \eta^1_{\alpha'} = \xi$. Let $V(\eta^0)$ and $V(\eta^1)$ be two disjoint open sets such that $x_0 \in V(\eta^0)$, $x_1 \in V(\eta^1)$ and $\overline{V(\eta^0)} \subset V(\xi)$, $\overline{V(\eta^1)} \subset V(\xi)$.

(b) If α is a limit-number, then for $\xi \in I^\alpha$ put $V(\xi) = X$.

Note that:

(i) For every α the set $\bigcap_{\beta \leqslant \alpha} V(\xi_\beta)$ is non-void.