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REMARKS ON COMPACTLY GENERATED ABELIAN TOPOLOGICAL GROUPS

BY

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Let G be a locally compact Abelian topological group and let $\hat{G}$ be its character group. By $\mathcal{L}^2(G)$ we shall denote the space of all measurable and square integrable with respect to the Haar measure of G complex-valued functions. We set for every integrable function f belonging to $\mathcal{L}^2(G)$

$$\tilde{f}(\chi) = \int_G f(x)\overline{\chi(x)}m(dx) \qquad (\chi \in \hat{G}),$$

where m denotes the Haar measure of G. For a suitably normalized Haar measure m the mapping $f \to \tilde{f}$ has a unique extension to a unitary mapping of the whole of $\mathcal{L}^2(G)$ onto the whole of $\mathcal{L}^2(\hat{G})$, which will be called the *Fourier-Plancherel transformation* (see [2], p. 145). In the sequel $\tilde{f}$ will denote the Fourier-Plancherel transform of a function f belonging to $\mathcal{L}^2(G)$.

Let C be a compact subset of $\hat{G}$. By $\mathcal{F}(C)$ we denote the space of all functions belonging to $\mathcal{L}^2(G)$ whose Fourier-Plancherel transform vanishes off C almost everywhere with respect to the Haar measure of $\hat{G}$. It is evident that $\mathcal{F}(C)$ is a closed subspace of $\mathcal{L}^2(G)$. Moreover, every function $f \in \mathcal{F}(C)$ is equivalent to a continuous function. This follows from the continuity of the inverse transform $\int_{\hat{G}} \tilde{f}(\chi)\chi(x)\mu(dx)$ where μ is the suitably normalized Haar measure of $\hat{G}$. Therefore $\mathcal{F}(C)$ will be regarded as a space of continuous functions.

Let us introduce the following property:

(H) *For every compact $C \subset \hat{G}$ there exists a finitely generated algebraic subgroup D of G such that every function belonging to $\mathcal{F}(C)$ is uniquely determined by its values on D.*

The additive group R of all real numbers with its natural topology has property (H). In fact, every function f whose Fourier-Plancherel transform vanishes off an interval $-a \leqslant x \leqslant a$ can be represented by an orthogonal expansion

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{a}\right)\cdot\frac{\sin(ax-n\pi)}{ax-n\pi},$$

whence it follows that f is determined by its values on the cyclic group $n\pi/a$ $(n = 0, \pm 1, \pm 2, \ldots)$.

In the present note we shall prove the following theorem, which is an answer to a problem raised by S. Hartman:

THEOREM. *A locally compact Abelian group is compactly generated if and only if it has property* (H).

Before proving the theorem we shall prove a lemma.

LEMMA. *Let G_1 and G_2 be two locally compact Abelian groups. If the direct sum $G_1 \times G_2$ has property* (H), *then the summands G_1 and G_2 have the same one.*

Proof. Obviously, to prove our lemma is sufficient to show that G_1 has property (H). Let us assume that C_1 and C_2 are compact subsets of $\hat{G}_1$ and $\hat{G}_2$ respectively and, moreover, $\operatorname{Int} C_2 \neq 0$. From the compactness of $C_1 \times C_2$ and property (H) of $G_1 \times G_2$ follows the existence of a finitely generated algebraic subgroup D of $G_1 \times G_2$ such that every function belonging to $\mathcal{F}(C_1 \times C_2)$ is uniquely determined by its values on D. Further, from the relation $\operatorname{Int} C_2 \neq 0$ it follows that there is a function g belonging to $\mathcal{F}(C_2)$ and being not identically zero. Let p be the projection of $G_1 \times G_2$ onto G_1 and let f_1, f_2 be functions belonging to $\mathcal{F}(C_1)$, satisfying the equality $f_1(x) = f_2(x)$ for any $x \in p(D)$. Setting $h_1(\langle x_1, x_2\rangle) = f_1(x_1)g(x_2)$, $h_2(\langle x_1, x_2\rangle) = f_2(x_1)g(x_2)$ $(\langle x_1, x_2\rangle \in G_1 \times G_2)$, we obtain functions belonging to $\mathcal{F}(C_1 \times C_2)$ which are identical on D. Thus, by property (H), h_1 and h_2 are identical on $G_1 \times G_2$. Since g is not identically zero, we have the equality $f_1(x) = f_2(x)$ for any $x \in G_1$. Consequently, every function belonging to $\mathcal{F}(C_1)$ is uniquely determined by its values on $p(D)$. Property (H) for the group G_1 is thus proved.

Proof of the theorem. *The sufficiency*. Let G be a locally compact Abelian group having property (H). According to a well-known theorem the group G decomposes into the direct sum $G = R^n \times G_0$ of an n-dimensional vector group R^n and a group G_0 having a compact subgroup G_1 such that the quotient group G_0/G_1 is discrete (see [3], § 29). Obviously, to prove that the group G is compactly generated, it suffices to show that the group G_0 is of the same kind.

By the lemma, the group G_0 also has property (H). Since G_0 has no direct summands of the form R^m $(m \geqslant 1)$, the character group $\hat{G}_0$ does not contain direct summands of this form. There is then a compact subgroup C_0 of $\hat{G}_0$ such that the quotient group $\hat{G}_0/C_0$ is discrete. Let A be the annihilator for the group C_0 in the group G_0, i. e. the set of all elements $x \in G_0$ satisfying the equality $\chi(x) = 1$ for every $\chi \in C_0$. It is well known that C_0 and $\hat{G}_0/C_0$ are character groups of the groups G_0/A and A respectively. Since C_0 is compact, the quotient group G_0/A is discrete. Thus A is an open subgroup of G_0. Furthermore, since $\hat{G}_0/C_0$ is discrete, the group A is compact.

Let $\mathcal{S}$ be the class of all complex-valued functions defined on G_0 which are constant on cosets of the subgroup A and vanish off a finite number of cosets. It is evident that $\mathcal{S} \subset \mathcal{L}^2(G_0)$. Moreover, if a function f belonging to $\mathcal{S}$ assumes values $z_1, z_2, \ldots, z_k$ on disjoint cosets $x_1A, x_2A, \ldots, x_kA$ respectively and vanishes off $\bigcup_{j=1}^{k} x_jA$, then its Fourier-Plancherel transform is given by the formula

$$(1) \qquad \tilde{f}(\chi) = \sum_{j=1}^{k} \int_{x_jA} f(x)\overline{\chi(x)}\,m(dx) = \sum_{j=1}^{k} z_j\overline{\chi(x_j)}\int_{A}\overline{\chi(x)}\,m(dx),$$

where m is the suitably normalized Haar measure of G_0.

If $\chi \notin C_0$, then it is not identically 1 on A and, consequently, $\int_A \overline{\chi(x)}\,m(dx) = 0$ (see [3], § 20). Hence and from (1) it follows that the Fourier-Plancherel transforms of all functions belonging to $\mathcal{S}$ vanish off C_0. Since A is an open subgroup of G_0, all functions belonging to $\mathcal{S}$ are continuous. Thus $\mathcal{S} \subset \mathcal{F}(C_0)$. By the compactness of C_0 and property (H) of G_0 there is a finitely generated algebraic subgroup D of G_0 such that every function belonging to $\mathcal{S}$ is uniquely determined by its values on D. But this is possible if and only if the discrete group G_0/A is finitely generated. Hence, taking into account the compactness of A, we infer that the group G_0 is compactly generated. The sufficiency of our condition is thus proved.

The necessity. Every compactly generated Abelian group G decomposes into the direct sum

$$G = B \times R^n \times \Delta^m,$$

where B is a compact group, R^n is an n-dimensional vector group and Δ^m is the direct sum of m discrete additive groups of all integers (see [3], § 29). Hence we get the following decomposition of the character group:

$$\hat{G} = \hat{B} \times R^n \times T^m,$$

where T^m is an m-dimensional toroidal group.

To prove property (H) for all compacts $C\subset\hat{G}$ it suffices to prove (H) for all compacts C of the form

$$C = E\times I\times T^{m},$$

where E is a finite subset of $\hat{B}$ and I is an interval in R^n of the form

$$I = \bigcap_{j=1}^{n}\{\langle t_1, t_2, \ldots, t_n\rangle : |t_j| \leqslant a\}.$$

It is clear that the characters

$$\exp\left(i\frac{\pi}{a}t_j k\right) \qquad (j = 1, 2, \ldots, n;\ k = 0, \pm 1, \pm 2, \ldots)$$

of R^n separate points of I. Further, there is a finite system of characters of $\hat{B}$, i. e. elements of B which separates points of E. Finally, the character group of T^m, i. e. Δ^m, is finitely generated. Hence it follows that there is a finitely generated algebraic subgroup D, of G which separates points of the set C. From the Stone-Weierstrass Theorem ([2], p. 9) it follows that every continuous function on C can be uniformly approximated on C by finite linear combinations of elements belonging to D, considered as functions on C. Let $\tilde{\mathcal{F}}(C)$ be the image of $\mathcal{F}(C)$ under the Fourier-Plancherel transformation. Every function belonging to $\tilde{\mathcal{F}}(C)$ can be approximated in $\mathcal{L}^2(\hat{G})$-norm by continuous functions vanishing off C (see [1], § 55). Consequently, every function g belonging to $\tilde{\mathcal{F}}(C)$ is uniquely determined by inner products

$$\int_C g(\chi)\chi(y)\mu(d\chi) \qquad (y\in D). \tag{2}$$

By the continuity of $f\in\mathcal{F}(C)$ we have the equality

$$f(y) = \int_C \tilde{f}(\chi)\chi(y)\mu(d\chi).$$

Therefore by (2) it follows that the Fourier-Plancherel transform of f and, consequently, the function f itself is uniquely determined by the values $f(y)$ $(y\in D)$. The theorem is thus proved.

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A LIMIT THEOREM FOR RANDOM VARIABLES IN COMPACT TOPOLOGICAL GROUPS

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I. Let G be a compact (not necessarily Abelian) topological group. A regular completely additive measure μ defined on the class of all Borel subsets of G, with $\mu(G) = 1$, will be called a *probability distribution*. A sequence of probability distributions $\mu_1, \mu_2, \ldots$ is said to be *weakly convergent* to a probability distribution μ if

$$\lim_{n\to\infty}\int_G f(g)\mu_n(dg) = \int_G f(g)\mu(dg)$$

for any complex-valued continuous function f defined on G.

A G-valued random variable is called *symmetric* if its probability distribution μ is invariant under the transformation $g\to g^{-1}$, i. e. if $\mu(E) = \mu(E^{-1})$ for each Borel subset $E\subset G$, where $E^{-1} = \{g^{-1}: g\in E\}$.

Let $X_1, X_2, \ldots$ be a sequence of independent G-valued random variables with probability distributions $\mu_1, \mu_2, \ldots$ Put

$$Y_n = X_1\cdot X_2\cdot\ldots\cdot X_n \qquad (n = 1, 2, \ldots),$$

where the product is taken in the sense of group multiplication in G. It is well known that the probability distribution ν_n of the random variable Y_n is given by the formula

$$\nu_n = \mu_1 * \mu_2 * \ldots * \mu_n \qquad (n = 1, 2, \ldots),$$

where the convolution $*$ is defined by

$$\nu*\lambda(E) = \int_G \nu(Eg^{-1})\lambda(dg)\ (^1).$$

The limiting distribution of the sequence $Y_1, Y_2, \ldots$ is the weak limit of the probability distributions $\nu_1, \nu_2, \ldots$.

(1) $Eg^{-1} = \{hg^{-1}: h\in E\}$.