sing an irreducible generating system and having a non-normal subgroup C with a finite generating system (e. g. the symmetric group S_n of degree $n \geqslant 3$). It is evident that the subgroup $H_0 = H \times C$ is not normal in the group $G_0 = H \times A \times B$ and according to Theorem 2 has no irreducible generating system; the group G_0 has, of course, property P.

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EMBEDDINGS IN GROUPS OF COUNTABLE PERMUTATIONS

BY

M. KNESER (MÜNCHEN) AND S. ŚWIERCZKOWSKI (WROCŁAW)

The aim of this note is to answer a question put forward by J. Mycielski. The question is whether, given an arbitrary group G,

(*) *G is isomorphic to a group of permutations of a set X such that every permutation displaces not more than countably many elements of X.*

We shall prove

THEOREM 1 (1). *(*) is true for every abelian group G.*

THEOREM 2. *If F is a non-abelian free group with more than $2^{\aleph_0}$ free generators, F' is the commutator subgroup of F and F'' is the commutator subgroup of F', then the group $G = F/F''$ does not satisfy (*).*

If G is an abelian group of order $2^{\aleph_0}$, then Theorem 1 follows from a result of N. G. De Bruijn [1]: Every abelian group of order $2^{\mathfrak{n}}$, where $\mathfrak{n}$ is an arbitrary infinite cardinal, is isomorphic to a group of permutations of a set of $\mathfrak{n}$ elements.

Our proof of Theorem 2 can easily be generalized to a proof of the following result: *If $\mathfrak{n}$ is an arbitrary infinite cardinal and F is a non-abelian free group with more than $2^{\mathfrak{n}}$ free generators, then $G = F/F''$ is not isomorphic to a group of permutations of a set X such that every permutation displaces at most $\mathfrak{n}$ elements of X.*

PROOF OF THEOREM 1. We start with three lemmas:

(i) *If G is countable, then (*) is true.*

To see this it is enough to regard each $g \in G$ as the permutation $x \to gx$ on the set $X = G$.

(ii) *If $\{G_\tau : \tau \in T\}$ is a collection of groups and each G_τ satisfies (*), then the direct sum $G = \sum_{\tau \in T} G_\tau$ also satisfies (*).*

To prove this let us denote by X_τ disjoint sets such that (*) holds with G_τ, X_τ instead of G, X. Each $g \in G_\tau$ then acts as a permutation on

(1) We have been informed that A. Hulanicki found independently a proof of this theorem.

the set $X = \bigcup_{\tau\epsilon T} X_\tau$ such that only elements in X_τ are displaced. Hence G acts as a group of permutations on X and (*) is easily verified.

(iii) *If a group G satisfies* (*), *then so does every subgroup.*

Let us assume now that G is an abelian group. Consequently G is a subgroup of an abelian divisible group H (see [2], p. 167). Since H is a direct sum of countable groups (see ibid., p. 165), we infer (*) by (i), (ii) and (iii).

PROOF OF THEOREM 2. Let F be a non-abelian free group with more than $2^{\aleph_0}$ free generators f_i, and let g_i be the image of f_i under the canonical homomorphism of F onto $G = F/F''$. Then g_i and $g_j g_k^{-1}$ are not commutative if i, j, k are different. For, let h be the endomorphism of F, defined by $h(f_k) = 1$, $h(f_l) = f_l$ for $l \neq k$, and $\bar{h}$ the induced endomorphism of G. Then $\bar{h}(g_i) = g_i$ and $\bar{h}(g_j g_k^{-1}) = g_j$ are not commutative, and thus g_i and $g_j g_k^{-1}$ cannot commute.

Now, suppose that G operates as a permutation group on a set X. Let us denote the image of $x \epsilon X$ under g by gx. Suppose further that every $g \epsilon G$ displaces at most countably many elements $x \epsilon X$. We shall then obtain three different indices i, j, k such that the permutations induced on X by g_i and $g_j g_k^{-1}$ commute, thus proving that the representation of G as a permutation group is certainly not an isomorphism.

As a first step, we prove that for each $x \epsilon X$ the domain of transitivity $Gx = \{gx: g \epsilon G\}$ is at most countable. This will be a consequence of the following lemma, applied first to the case $H = G' \cong F'/F''$, and then to $H = G/G' = F/F'$.

(j) *If H is an abelian group which acts as a group of permutations on a set S so that each permutation displaces not more than countably many elements of S, then each domain of transitivity Hs is at most countable.*

Proof. Let $s \epsilon S$. Suppose first that $hs \neq s$ for some $h \epsilon H$. Then, for each $g \epsilon H$, we have $ghs \neq gs$ and this implies $hgs \neq gs$. Hence every element in Hs is displaced by h and thus Hs is at most countable. If $hs = s$ holds for every $h \epsilon H$, then $Hs = \{s\}$.

Returning to our group G, we know by (*) and (j), that each domain of transitivity $G'x$ under the commutator subgroup G' of G is at most countable. Now, denote by S the set of all domains of transitivity $G'x$ of G' in X, and let an element $G'g$ of the factor group $H = G/G'$ operate on S as a permutation defined by $G'gG'x = G'gx$. Since g displaces at most countable many elements $x \epsilon X$, $G'g$ displaces at most countably many elements $G'x \epsilon S$. By lemma (j) it follows that any domain of transitivity Gx of G contains at most countably many sets $G'y$, and since each of these is at most countable, Gx must be at most countable.

Now let g_i be one of the generators of G and denote by Y the union of all domains of transitivity Gx for which $g_i x \neq x$. The set Y is at most countable and it is transformed onto itself by all permutations induced on X by G. Since there are not more than $2^{\aleph_0}$ permutations of Y, two different generators g_j, g_k^{-1} $(j, k \neq i)$ must induce the same permutation on Y. Consequently all elements of Y are fixed under $g_j g_k^{-1}$. Since all elements in the complement of Y are fixed under g_i, g_i and $g_j g_k^{-1}$ induce commutative permutations on X. As stated above, this concludes our proof of Theorem 2.

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MATHEMATICAL INSTITUTE OF THE MUNICH UNIVERSITY
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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