singing an irreducible generating system and having a non-normal subgroup $G$ with a finite generating system (e.g., the symmetric group $S_n$ of degree $n \geq 3$). It is evident that the subgroup $H_0 = H \times C$ is not normal in the group $G = H \times \Delta \times B$ and according to Theorem 2 has no irreducible generating system; the group $G_0$ has, of course, property $P$.

REFERENCES


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EMBEDDINGS IN GROUPS OF COUNTABLE PERMUTATIONS

BY

M. KNESEs (MÜNCHEN) AND S. ŚWIERCZKOWSKI (WROCLAW)

The aim of this note is to answer a question put forward by J. Mycielski. The question is whether, given an arbitrary group $G$,

$(*)$ if $G$ is isomorphic to a group of permutations of a set $X$ such that every permutation displaces not more than countably many elements of $X$.

We shall prove

THEOREM 1 $(*)$ is true for every abelian group $G$.

THEOREM 2. If $F$ is a non-abelian free group with more than $2^n$ free generators, $F'$ is the commutator subgroup of $F$ and $F''$ is the commutator subgroup of $F'$, then the group $G = F/F''$ does not satisfy $(*)$.

If $G$ is an abelian group of order $2^n$, then Theorem 1 follows from a result of N. G. De Bruijn [1]: Every abelian group of order $2^n$, where $n$ is an arbitrary infinite cardinal, is isomorphic to a group of permutations of a set of $n$ elements.

Our proof of Theorem 2 can easily be generalized to a proof of the following result: If $n$ is an arbitrary infinite cardinal and $F$ is a non-abelian free group with more than $2^n$ free generators, then $G = F/F''$ is not isomorphic to a group of permutations of a set $X$ such that every permutation displaces at most $n$ elements of $X$.

PROOF OF THEOREM 1. We start with three lemmas:

(i) If $G$ is countable, then $(*)$ is true.

To see this it is enough to regard each $g \in G$ as the permutation $x \rightarrow gx$ on the set $X = G$.

(ii) If $\{ G_r \mid r \in F \}$ is a collection of groups and each $G_r$ satisfies $(*)$, then the direct sum $G = \Sigma G_r$ also satisfies $(*)$.

To prove this let us denote by $X_r$ disjoint sets such that $(*)$ holds with $G_r, X_r$ instead of $G, X$. Each $g \in G$, then acts as a permutation on

$(i)$ We have been informed that A. Hulanicki found independently a proof of this theorem.

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the set $X = \bigcup_{a \in A} X_a$, such that only elements in $X_a$ are displaced. Hence $G$ acts as a group of permutations on $X$ and (2) is easily verified.

(iii) If a group $G$ satisfies (i), then so does every subgroup.

Let us assume now that $G$ is an abelian group. Consequently $G$ is a subgroup of an abelian divisible group $H$ (see [2], p. 167). Since $H$ is a direct sum of countable groups (see ibid., p. 165), we infer (2) by (i), (ii) and (iii).

Proof of Theorem 2. Let $F$ be a non-abelian free group with more than $2^n$ free generators $f_i$, and let $g_i$ be the image of $f_i$ under the canonical homomorphism of $F$ onto $G = F/F'$. Then $g_i$ and $g_j g^{-1}_j$ are not commutative if $i, j, k$ are different. For, let $h$ be the endomorphism of $F$, defined by $h(f_i) = f_i$, $h(f_j) = f_i$ for $i \neq k$, and $h$ the induced endomorphism of $G$. Then $h(g_i) = g_i$ and $h(g_j g^{-1}_j) = g_i$ are not commutative, and thus $g_i$ and $g_j g^{-1}_j$ cannot commute.

Now, suppose that $G$ operates as a permutation group on a set $X$. Let us denote the image of $x \in X$ under $g$ by $g \cdot x$. Suppose further that every $g \cdot x$ displaces at most countably many elements $x \in X$. We shall then obtain three different indices $i, j, k$ such that the permutations induced on $X$ by $g_i$ and $g_j g^{-1}_j$ commute, thus proving that the representation of $G$ as a permutation group is certainly not an isomorphism.

As a first step, we prove that for each $x \in X$ the domain of transitivity $Gx = \{g \cdot x : g \in G\}$ is at most countable. This will be a consequence of the following lemma, applied first to the case $H = G' = F/F'$, and then to $H = G/G' = F/F'$.

(i) If $H$ is an abelian group which acts as a group of permutations on a set $S$ so that each permutation does not displace more than countably many elements of $S$, then each domain of transitivity $Hs$ is at most countable.

Proof. Let $s \in S$. Suppose first that $hs = s$ for some $h \in H$. Then, for each $g \in H$, we have $gsh = gs$ and this implies $hgs = gs$. Hence every element in $Hs$ is displaced by $h$ and thus $Hs$ is at most countable. If $hs = s$ holds for every $h \in H$, then $Hs = \{s\}$.

Returning to our group $G$, we know by (2) and (i), that each domain of transitivity $G'x$ under the commutator subgroup $G'$ of $G$ is at most countable. Now, denote by $S$ the set of all domains of transitivity $G'x$ of $G'$ in $X$, and let an element $G'x$ of the factor group $H = G/G'$ operate on $S$ as a permutation defined by $G'x' x = G'x$. Since $g$ displaces at most countably many elements $x \in X$, $G'x$ displaces at most countably many elements $G'x \in S$. By lemma (i) it follows that any domain of transitivity $Gx$ of $G$ contains at most countably many sets $G'y$, and since each of these is at most countable, $Gx$ must be at most countable.