all t the convergence of the absolute moments of S_N/B_N to the absolute moment of the normalized Gaussian distribution. In other words, we have the relation

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\sqrt{2}}{\sqrt{N}}\sum_{k=1}^{N}\varphi_k(t)\cos kx\right|dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|e^{-x^2/2}dx=\sqrt{\frac{2}{\pi}}$$

for almost all t. Hence, using the well-known equality

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\int_0^{2\pi}\left|\sum_{k=1}^{N}\cos kx\right|dx=0,$$

we obtain the relation

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\int_0^{2\pi}\left|\sum_{k=1}^{N}(\varphi_k(t)+1)\cos kx\right|dx=2\sqrt{\pi} \tag{7}$$

for almost all t.

Let us fix an irrational number t_0 with this property. Let $n_1, n_2, \ldots$ denote the successive indices k for which $\varphi_k(t_0)=1$. Then

$$\sum_{k=1}^{n_N}(\varphi_k(t_0)+1)\cos kx=2\sum_{k=1}^{N}\cos n_k x$$

and, according to (7),

$$\int_0^{2\pi}\left|\sum_{k=1}^{N}\cos n_k x\right|dx=\sqrt{\pi}\sqrt{n_N}+o(\sqrt{n_N}),$$

which completes the proof of II.

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ON A PROBLEM OF MAZUR AND ULAM ABOUT IRREDUCIBLE GENERATING SYSTEMS IN GROUPS

BY

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1. INTRODUCTION

In 1935 S.Mazur and S. Ulam have stated the following problem [1] (for the terminology see the end of this section):

Let a group possess an irreducible generating system. Does each of its subgroups also have this property?

The problem mentioned has been solved negatively in paper [1]; e. g. the abelian group

$$G(p^\infty)+\sum_{i=1}^{\infty}G_i(p),$$

where $G(p^\infty)$ is the Prüfer group of the type p^∞ and $G_i(p)$ cyclic groups of the prime order p, possesses an irreducible generating system but none of its non-reduced subgroup with a finite reduced component has this property (it is easy to see that the subgroups in question are all those having no irreducible generating system; see also [2]). It is the purpose of this note to give some more general constructions of groups G with

PROPERTY P. *The group G possesses an irreducible generating system, but there exists a subgroup $H\subset G$ every generating system of which is reducible.*

Throughout this article we consider predominantly non-abelian groups written multiplicatively; $\times$ and (in the abelian case) $+$ denote the direct product. $\sum$ denotes the weak direct sum of abelian groups. G^n for a fixed natural n is the subset (of the group G) of all elements g^n with $g\in G$. The power of a set $\mathfrak{M}$ will be denoted by $m(\mathfrak{M})$ and the order of an element $g\in G$ by $O(g)$.

For any non-void subset $\mathfrak{M}$ of G, $\{\mathfrak{M}\}$ denotes the subgroup of G generated by the elements of $\mathfrak{M}$; thus $\{\mathfrak{G}\}=G$ means that $\mathfrak{G}$ is a *genera-*

[1] The Scottish Book, Problem 63, p. 27. I am indebted to Jan Mycielski for calling my attention to this problem.

ting system of the group G. A generating system $\mathfrak{G}$ of a group G is said to be *irreducible* if the set $\mathfrak{G}\setminus(g)$ is not a generating system of G for any element $g\in\mathfrak{G}$. In the contrary case it is called *reducible*. If $\mathfrak{G}\setminus(g)$ is a generating system of G for any element $g\in\mathfrak{G}$, then $\mathfrak{G}$ is called a *strongly reducible* generating system of G.

Let H be a normal subgroup of a group G and $\mathfrak{M}\subset G$. We denote by $\overline{\mathfrak{M}}$ the natural image of $\mathfrak{M}$ in G/H. Especially, if $\mathfrak{G}$ is a generating system of G, then $\overline{\mathfrak{G}}$ is obviously a generating system of G/H.

Let Π be a non-void set of primes; a group G is said to be a *Π-group* if the order of each element of the group G is finite and the primes of Π are its only prime divisors.

2. SOME TYPES OF GROUPS HAVING PROPERTY P

First of all we are going to prove the following simple lemmas:

LEMMA 1. *Let H be a normal subgroup of a group G and $\mathfrak{G}$ a generating system of G such that*

$$g_{\delta_1}g_{\delta_2}^{-1}\notin H \text{ for every pair } g_{\delta_1}, g_{\delta_2} \text{ of elements of } \mathfrak{G}. \tag{1}$$

If $\overline{\mathfrak{G}}$ is an irreducible generating system of the quotient group G/H, then $\mathfrak{G}$ is also irreducible.

Remark. Supposing only $g_{\delta_1}g_{\delta_2}^{-1}\notin H$ for every pair $g_{\delta_1}, g_{\delta_2}$ with the exception of a finite number of them, one can prove the existence of an irreducible generating system $\mathfrak{G}^*$ of the group G satisfying $\mathfrak{G}^*\subseteq\mathfrak{G}$.

Proof. It is easy to see that according to our assumptions the relation

$$g_0\in\{\mathfrak{G}\setminus(g_0)\} \text{ for a certain } g_0\in\mathfrak{G}$$

implies

$$\bar{g}_0\in\{\overline{\mathfrak{G}}\setminus(\bar{g}_0)\} \text{ with } \bar{g}_0\in\overline{\mathfrak{G}},$$

contradicting the hypothesis of $\overline{\mathfrak{G}}$ being irreducible.

LEMMA 2. *Let $G=G_1\times G_2$ and $\mathfrak{G}=(g_\delta)_{\delta\in\Delta}$ be a generating system of G. Let*

$$g_\delta = g_\delta^{(1)}g_\delta^{(2)} \text{ with } g_\delta^{(i)}\in G_i \text{ for } i=1,2.$$

Then $\mathfrak{G}^{(i)}=(g_\delta^{(i)})_{\delta\in\Delta}$ is a generating system of the group G_i $(i=1,2)$.

Proof. The proposition of Lemma 2 is obvious; it is sufficient to make use of the commutativity of the elements of groups G_1 and G_2.

Lemma 1 easily implies the following theorem, stating a general construction of groups with property P:

THEOREM 1. *Let G be a group and H its normal subgroup with the following properties:*

(I) *H has no irreducible generating system;*

(II) *G possesses a generating system $\mathfrak{G}$ satisfying* (1) *such that $\overline{\mathfrak{G}}$ is an irreducible generating system of the quotient group G/H.*

Then G has property P.

Now, using the preceding assertion we can formulate more special results.

COROLLARY 1. *Let $G=H\times A$ be the direct product of a group H satisfying* (I) *and a group A, and let the inequality*

$$m(H)\leqslant m(A) \tag{2}$$

be fulfilled. Let n_0 be such a number that

$$A^{n_0}=(e) \tag{3}$$

and

$$\{H^{n_0}\}=H. \tag{4}$$

Hence, especially, if $H^{n_0}=H$, (4) *is fulfilled.*

If, further, an infinite irreducible generating system $\mathfrak{A}$ of the group A exists, then G has property P.

Proof. Let $\mathfrak{H}$ be a generating system of the group H; in view of (I) it is necessarily infinite. Moreover, according to (2),

$$m(\mathfrak{H})\leqslant m(\mathfrak{A}) \tag{5}$$

holds. Let φ be a one-to-one correspondence between the set $\mathfrak{H}=(h_\delta)_{\delta\in\Delta}$ and $\mathfrak{A}'=(a_\delta)_{\delta\in\Delta}$, $\mathfrak{A}'\subseteq\mathfrak{A}$, $a_\delta=\varphi(h_\delta)$ for $\delta\in\Delta$. Let us define the set $\mathfrak{G}$ as follows:

$$\mathfrak{G}=(g_\delta)_{\delta\in\Delta}\cup(\mathfrak{A}\setminus\mathfrak{A}'), \tag{6}$$

where

$$g_\delta = h_\delta a_\delta \quad (\delta\in\Delta). \tag{7}$$

We are going to prove that $\mathfrak{G}$ is a generating system of G. If $h\in H^{n_0}$, then there exists an element $h_0\in H$ such that $h_0^{n_0}=h$. We have

$$h_0 = h_{\delta_1}h_{\delta_2}\dots h_{\delta_n} \text{ for suitable } h_{\delta_i}\in\mathfrak{H} \quad (i=1,2,\dots,n)$$

Hence for the element

$$g_0 = g_{\delta_1}g_{\delta_2}\dots g_{\delta_n} = h_0 a_{\delta_1}a_{\delta_2}\dots a_{\delta_n}$$

we obtain by (3)

$$h = h_0^{n_0} = g_0^{n_0}\in\{\mathfrak{G}\}.$$

Thus $\{\mathfrak{G}\}\supseteq H^{n_0}$ and, by (4), $\{\mathfrak{G}\}\supseteq H$. According to (6) and (7) we immediately deduce $\{\mathfrak{G}\} = G$. Since (1) and (II) is obviously valid for $\mathfrak{G}$ we are ready to apply Theorem 1 and we obtain the desired result.

COROLLARY 2. *Let Π denote a fixed non-void set of primes. Let $G = H\times A$ be the direct product of a group H satisfying* (I) *and the relation*

$$H^p = H \text{ for every } p \in \Pi \tag{8}$$

and of a Π-group A with (2). *If, further, an infinite irreducible generating system $\mathfrak{A}$ of the group A exists, then G has property* P.

Proof. Following a similar line as in the proof of Corollary 1, we easily deduce inequality (5), where $\mathfrak{H} = (h_\delta)_{\delta\in\Delta}$ is a generating system of the group H. Let φ be again a one-to-one correspondence between $\mathfrak{H}$ and $\mathfrak{A}' = (a_\delta)_{\delta\in\Delta}$, $\mathfrak{A}' \subseteq \mathfrak{A}$. Let $O(a_\delta) = n_\delta$; by (8) we can choose an element $h_\delta^* \in H$ satisfying $h_\delta^{*n_\delta} = h_\delta$ (for every $\delta\in\Delta$). Let us define the set

$$\mathfrak{G} = (g_\delta)_{\delta\in\Delta} \cup (\mathfrak{A}\setminus\mathfrak{A}'),$$

where $g_\delta = h_\delta^* a_\delta$ $(\delta\in\Delta)$. Clearly $\mathfrak{G}$ is a generating system of the group G, which in view of Theorem 1 is irreducible.

Let us distinguish two special cases (remembering that a non-zero divisible abelian group has no irreducible generating system, see [1]).

COROLLARY 3. *Let H be a Π_1-group with property* (I) *and A a Π_2-group possessing an infinite irreducible generating system; let $\Pi_1 \cap \Pi_2 = \emptyset$ and* (2) *be fulfilled. Then $G = H\times A$ is a group with property* P.

COROLLARY 4. *Let H be a non-zero divisible abelian group and A a torsion group having an infinite irreducible generating system and satisfying inequality* (2). *Then $G = H\times A$ has property* P.

We shall conclude this section with the following

THEOREM 2. *Let $G = H\times A$ be the direct product of a group H satisfying* (I) *and a group A having a finite generating system. Then any generating system of G is reducible.*

Proof. Let $\mathfrak{G}$ be a generating system of the group G; let us denote by $\mathfrak{H}$ the set of components of the elements of $\mathfrak{G}$ in the subgroup H. According to property (I) and by Lemma 2 we easily deduce

$$m(\mathfrak{G}) \geqslant m(\mathfrak{H}) = m(H) \geqslant \aleph_0.$$

Since A has a finite generating system, there exists a finite subset $\mathfrak{G}_0 \subset \mathfrak{G}$ satisfying $\{\mathfrak{G}_0\} \supseteq A$. By (I) there exists necessarily an element $g\in\mathfrak{G}\setminus\mathfrak{G}_0$ such that the relation

$$h \in \{\mathfrak{H}\setminus(h)\}$$

holds for its component h in H. Now we easily obtain the following relations:

$$g = ha \in \{\mathfrak{H}\setminus(h)\}a \subseteq \{\mathfrak{G}\setminus(g)\}\{\mathfrak{G}_0\} \subseteq \{\mathfrak{G}\setminus(g)\};$$

this completes the proof of Theorem 2.

3. SOME REMARKS

1. As we know, any generating system of a non-zero divisible abelian group is reducible; moreover, any generating system of it is strongly reducible (see [1]). Let $G = H\times A$; supposing that every generating system of the group H is strongly reducible and that there exists a generating system $\mathfrak{A}$ of the group A satisfying $m(\mathfrak{A}) < m(H)$ one can easily prove the reducibility of any generating system of the group G (the proof follows the same lines as that of Theorem 2). Thus, the preceding assertion shows that assumption (2) in Corollaries 1-4 cannot be dispensed with.

2. The following result proved in [3] is closely related to the above theorems: The abelian group

$$R^+ + \sum_{i=1}^{\infty}\{u_i\}$$

where R^+ is the additive group of all rational numbers and $\sum_{i=1}^{\infty}\{u_i\}$ the abelian free group of countable rank has an irreducible generating system, while the subgroup

$$R^+ + \sum_{i=1}^{n}\{u_i\}$$

for an arbitrary non-negative n has not this property (see also Theorem 2).

3. Let us show that the considered group H need not be a direct factor of the group G. Let us consider the additive abelian group of all rational numbers whose denominators are products of arbitrary powers of a fixed prime p_0 and of square-free numbers. Let H be the subgroup of G of all numbers whose denominators are powers of the prime p_0. Let $p_0, p_1, \ldots, p_i, \ldots$ be all primes. The set

$$\mathfrak{G} = (g_i)_{i=1,2,\ldots}$$

where $g_i = 1/p_0^i p_i$ is obviously an irreducible generating system of the group G (the cosets g_iH form an irreducible generating system of G/H). On the other hand, it is easy to see that any generating system of the subgroup H is reducible (and, moreover, H is not a direct factor of G).

It is easy to prove that the subgroups having property (I) are just the subgroups consisting of rational numbers whose denominators are products of powers of p_0 (those powers being not bounded) and primes of a fixed finite set Π.

4. The subgroup H need not even be normal in G. Let $G = H\times A$ be a group with property P, where H satisfies (I). Let B be a group posses-

sing an irreducible generating system and having a non-normal subgroup C with a finite generating system (e. g. the symmetric group S_n of degree $n \geqslant 3$). It is evident that the subgroup $H_0 = H \times C$ is not normal in the group $G_0 = H \times A \times B$ and according to Theorem 2 has no irreducible generating system; the group G_0 has, of course, property P.

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EMBEDDINGS IN GROUPS OF COUNTABLE PERMUTATIONS

BY

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The aim of this note is to answer a question put forward by J. Mycielski. The question is whether, given an arbitrary group G,

(*) *G is isomorphic to a group of permutations of a set X such that every permutation displaces not more than countably many elements of X.*

We shall prove

THEOREM 1 (1). (*) *is true for every abelian group G.*

THEOREM 2. *If F is a non-abelian free group with more than $2^{\aleph_0}$ free generators, F' is the commutator subgroup of F and F'' is the commutator subgroup of F', then the group $G = F/F''$ does not satisfy* (*).

If G is an abelian group of order $2^{\aleph_0}$, then Theorem 1 follows from a result of N. G. De Bruijn [1]: Every abelian group of order $2^{\mathfrak{n}}$, where $\mathfrak{n}$ is an arbitrary infinite cardinal, is isomorphic to a group of permutations of a set of $\mathfrak{n}$ elements.

Our proof of Theorem 2 can easily be generalized to a proof of the following result: *If $\mathfrak{n}$ is an arbitrary infinite cardinal and F is a non-abelian free group with more than $2^{\mathfrak{n}}$ free generators, then $G = F/F''$ is not isomorphic to a group of permutations of a set X such that every permutation displaces at most $\mathfrak{n}$ elements of X.*

PROOF OF THEOREM 1. We start with three lemmas:

(i) *If G is countable, then* (*) *is true.*

To see this it is enough to regard each $g \in G$ as the permutation $x \to gx$ on the set $X = G$.

(ii) *If $\{G_\tau : \tau \in T\}$ is a collection of groups and each G_τ satisfies* (*), *then the direct sum $G = \sum_{\tau \in T} G_\tau$ also satisfies* (*).

To prove this let us denote by X_τ disjoint sets such that (*) holds with G_τ, X_τ instead of G, X. Each $g \in G_\tau$ then acts as a permutation on

(1) We have been informed that A. Hulanicki found independently a proof of this theorem.