Using (5) and (8) we get for k large enough

$$\|\xi' a-\beta\| > (b_k-r-1)|d_k| > \frac{k^4-3k^3c-1}{(k^4+1)q_k} > \frac{1}{2k^3 q_k} > \frac{1}{t_\nu}$$

and, similarly, using (8) for k large enough

$$\|\xi'' a-\beta\| > (a_k-b_k)|d_k| > \frac{k}{(k^4+1)q_k} > \frac{1}{2k^3 q_k} > \frac{1}{t_\nu},$$

i. e. for $t_\nu = c_{k-1}+2k^3 q_k$, $k = 2\nu+1$, $\nu > \nu_0$, the inequality

$$\|x\alpha-\beta\| < 1/t$$

has no solution with $0 < x < ct$.

In an analogous way it is possible to show that for $t = c'_{k-1}+2k^3 q_k$, $k = 2\nu$, $\nu > \nu_0$, inequality (1) has no solution with $-ct < x < 0$, which completes the proof.

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REMARKS ON A CONJECTURE OF HANANI IN ADDITIVE NUMBER THEORY

BY

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In his paper [1] P. Erdös mentions the following conjecture due to H. Hanani:

(H_0) *If* $A(x) = \sum_{a_k\leqslant x} 1$ *and* $B(x) = \sum_{b_k\leqslant x} 1$, *where* $\{a_k\}$ *and* $\{b_k\}$ *are both infinite sequences of increasing integers, and if every sufficiently great integer can be represented in the form* a_i+b_j, *then*

$$\overline{\lim_{x\to\infty}} \frac{A(x)B(x)}{x} > 1.$$

This conjecture can be stated in the following equivalent form:

(H_0') *If by* $f(n)$ *we denote the number of representations of the integer* n *in the form* a_i+b_j, $f(n) \geqslant 1$ *for* $n \geqslant n_0$, *and*

$$\overline{\lim_{x\to\infty}} \frac{A(x)B(x)}{x} \leqslant 1,$$

then one of the sequences $\{a_k\}$, $\{b_k\}$ *must be finite.*

It seems very probable that the following stronger conjecture holds:

(H_1) *If* $f(n) \geqslant k$ *for* $n \geqslant n_0$, *and*

$$\overline{\lim_{x\to\infty}} \frac{A(x)B(x)}{x} \leqslant k,$$

then one of the sequences $\{a_k\}$, $\{b_k\}$ *must be finite.*

The purpose of this paper is to prove the following theorem associated with the conjecture (H_1):

THEOREM. *If* $f(n) \geqslant k$ *for almost all integers, and*

$$\overline{\lim_{x\to\infty}} \frac{A(x)B(x)}{x} \leqslant k,$$

then:

(i) $f(n) = k$ *for almost all integers,*

(ii) $\lim\limits_{x\to\infty}\dfrac{A(2x)}{A(x)} = 1$, *or* $\lim\limits_{x\to\infty}\dfrac{B(2x)}{B(x)} = 1$.

From (ii) and a result of G. Pólya [2] it follows that $A(x) = o(x^{\varepsilon})$, or $B(x) = o(x^{\varepsilon})$ for every positive ε.

Proof. (i) Let n_i be such a sequence that $f(n_i) \geqslant 1+k$. If

$$N(x) = \sum_{n_i\leqslant x} 1,$$

then from the remark that there are $A(x)B(x)$ sums a_i+b_j with $a_i, b_j \leqslant x$ it follows that

$$A(x)B(x) \geqslant kx+N(x)+o(x).$$

Now if $\overline{\lim\limits_{x\to\infty}}\dfrac{A(x)B(x)}{x} \leqslant k$, then $N(x) = o(x)$.

(ii) It is evident from the preceding that

$$A(x)B(x) = kx+o(x). \tag{1}$$

Let us denote by $f_x(l)$ the number of representations of the integer l as a_i+b_j with $a_i, b_j \leqslant x$. Let

$$F(x) = \sum_{l>x} f_x(l).$$

Then

$$A(x)B(x) = \sum_{l\leqslant 2x} f_x(l) = F(x)+\sum_{l\leqslant x} f_x(l);$$

therefore

$$0 \leqslant F(x) = A(x)B(x)-\sum_{l\leqslant x} f_x(l).$$

But

$$\sum_{l\leqslant x} f_x(l) \geqslant k\big(x+o(x)\big)$$

and thus we have

$$0 \leqslant F(x) \leqslant kx-kx+o(x) = o(x). \tag{2}$$

Now let us show that the functions

$$a(x) = \frac{A(\frac{1}{2}x)}{A(x)}, \qquad \beta(x) = \frac{B(\frac{1}{2}x)}{B(x)}$$

cannot have other points of accumulation than 1 and $\frac{1}{2}$, when x tends to infinity.

From (2) we have

$$o(x) = F(x) \geqslant \{A(x)-A(\tfrac{1}{2}x)\}\cdot\{B(x)-B(\tfrac{1}{2}x)\} \geqslant 0,$$

for if $x/2 < a_i \leqslant x$, $x/2 < b_j \leqslant x$, then $a_i+b_j > x$. But from (1) we have

$$\{A(x)-A(\tfrac{1}{2}x)\}\cdot\{B(x)-B(\tfrac{1}{2}x)\} = \tfrac{3}{2}kx-A(\tfrac{1}{2}x)B(x)-A(x)B(\tfrac{1}{2}x)+o(x)$$

and it follows that

$$\lim_{x\to\infty}\frac{A(x)B(\frac{1}{2}x)+B(x)A(\frac{1}{2}x)x}{x} = \frac{3}{2}k.$$

From this and (1) we deduce that

$$\lim_{x\to\infty}\left\{\frac{A(\frac{1}{2}x)}{A(x)}+\frac{A(x)}{2A(\frac{1}{2}x)}\right\} = \frac{3}{2}.$$

If $\lim\limits_{i\to\infty} a(x_i) = g$ $(x_i\to\infty)$, then $g+1/2g = \frac{3}{2}$ and thus $g = 1$, or $g = \frac{1}{2}$.

From (1) it is evident that if $a(x_i)\to\frac{1}{2}$ then $\beta(x_i)\to 1$, and if $\beta(x_i)\to\frac{1}{2}$ then $a(x_i)\to 1$.

It exists therefore a sequence x_i, tending to infinity, for which $a(x_i)\to 1$ or $\beta(x_i)\to 1$.

Let us assume that

$$a(x_i)\to 1. \tag{3}$$

We shall now prove the following

Lemma. *If for an increasing sequence t_k, tending to infinity, $a(t_k)\to 1$, then*

$$\lim_{k\to\infty}\frac{A(\frac{1}{4}t_k)}{A(t_k)} = 1.$$

From (1) and (2) we have

$$o(x) = F(x) \geqslant \{A(x)-A(\tfrac{1}{4}x)\}\{B(x)-B(\tfrac{3}{4}x)\}$$
$$= kx-B(x)A(\tfrac{1}{4}x)-B(\tfrac{3}{4}x)A(x)+B(\tfrac{3}{4}x)A(\tfrac{1}{4}x)+o(x) \geqslant 0.$$

Hence

$$\lim_{x\to\infty}\frac{B(x)A(\frac{1}{4}x)+A(x)B(\frac{3}{4}x)-A(\frac{1}{4}x)B(\frac{3}{4}x)}{x} = k,$$

and after applying (1) we see that

$$\lim_{x\to\infty}\left\{\frac{A(\frac{1}{4}x)}{A(x)}+\frac{3A(x)}{4A(\frac{1}{4}x)}-\frac{3A(\frac{1}{4}x)}{4A(\frac{3}{4}x)}\right\} = 1.$$

We have

$$\lim_{k\to\infty}\frac{A(t_k)}{A(\frac{3}{4}t_k)}=1,$$

for

$$1\leqslant\frac{A(t_k)}{A(\frac{3}{4}t_k)}\leqslant\frac{A(t_k)}{A(\frac{1}{2}t_k)}=\frac{1}{a(t_k)}\to 1.$$

Hence

$$1=\lim_{k\to\infty}\left\{\frac{A(\frac{1}{4}t_k)}{A(t_k)}+\frac{3A(t_k)}{4A(\frac{3}{4}t_k)}-\frac{3A(\frac{1}{4}t_k)\cdot A(t_k)}{A(t_k)\cdot A(\frac{3}{4}t_k)}\right\}$$

$$=\lim_{k\to\infty}\left\{\frac{A(\frac{1}{4}t_k)}{A(t_k)}+\frac{3}{4}-\frac{3A(\frac{1}{4}t_k)}{4A(t_k)}\right\},$$

and it follows that

$$\lim_{k\to\infty}\frac{A(\frac{1}{4}t_k)}{A(t_k)}=1.$$

The lemma is thus proved.

Now let y_n be a sequence monotonically tending to infinity. Let

$$X=\left\{x:\left|\frac{1}{a(x)}-1\right|<\frac{1}{4}\right\}.$$

It is evident that if $x\in X$, then also $[x]\in X$ for $[x/2]=[[x]/2]$ and $A(x)=A([x])$.

It follows that X is left-sidedly closed.

From (3) it follows also that X is unbounded.

Let $\xi_n=\inf\limits_{\substack{x\geqslant y_n\\ x\in X}} x$. Then $\xi_n\in X$ and so

$$\left|\frac{A(\xi_n)}{A(\frac{1}{2}\xi_n)}-1\right|<\frac{1}{4}.$$

From this inequality it follows that $a(\xi_n)\to 1$, since a subsequence ξ_{n_k} for which $a(\xi_{n_k})\to\frac{1}{2}$ cannot exist. From the lemma we immediately find

$$\lim_{n\to\infty}\frac{A(\xi_n)}{A(\frac{1}{4}\xi_n)}=1; \tag{4}$$

therefore

$$\lim_{n\to\infty}\frac{A(\frac{1}{2}\xi_n)}{A(\frac{1}{4}\xi_n)}=\lim_{n\to\infty}\frac{A(\frac{1}{2}\xi_n)\cdot A(\xi_n)}{A(\xi_n)A(\frac{1}{4}\xi_n)}=1. \tag{5}$$

For sufficiently great n we have $\frac{1}{2}\xi_n\leqslant y_n$, because if for an infinite sequence $\{n_k\}$ we had $\frac{1}{2}\xi_{n_k}>y_{n_k}$, we should have $\frac{1}{2}\xi_{n_k}\notin X$, and thus

$$\left|\frac{A(\frac{1}{2}\xi_{n_k})}{A(\frac{1}{4}\xi_{n_k})}-1\right|\geqslant\frac{1}{4},$$

which contradicts (5).

For sufficiently great n we thus have

$$\tfrac{1}{4}\xi_n\leqslant\tfrac{1}{2}y_n\leqslant\tfrac{1}{2}\xi_n\leqslant y_n\leqslant\xi_n.$$

From (4) we get

$$\lim_{n\to\infty}\frac{A(y_n)}{A(\frac{1}{2}y_n)}=1.$$

From the arbitrariness of the sequence y_n it follows that

$$\lim_{x\to\infty}a(x)=1,$$

which completes the proof.

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