Using (5) and (8) we get for $k$ large enough
\[ \|x' - \beta\| > (b_k - v - 1)d_k \geq \frac{k^3 - 3k^2v - 1}{(k^2 + 1)g_k} > \frac{1}{2k^2g_k} > \frac{1}{l}, \]
and, similarly, using (8) for $k$ large enough
\[ \|x' - \beta\| > (a_k - b_k)d_k \geq \frac{k}{(k^2 + 1)g_k} > \frac{1}{2k^2g_k} > \frac{1}{l}, \]
i.e. for $t_1 = a_{k-1} + 2k^2g_k$, $k = 2v + 1$, $v > n_1$, the inequality
\[ \|xv - \beta\| < \frac{1}{l} \]
has no solution with $0 < x < ct$.

In an analogous way it is possible to show that for $t = a_{k-1} + 2k^2g_k$, $k = 2v_1 + 1$, $v_1 = n_1$, inequality (1) has no solution with $-ct < x < 0$, which completes the proof.

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**Remarks on a Conjecture of Hanani in Additive Number Theory**

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In his paper [1] P. Erdős mentions the following conjecture due to H. Hanani:

(H$_0$) If $A(x) = \sum_{a_k \leq x} 1$ and $B(x) = \sum_{b_k \leq x} 1$, where $[a_k]$ and $[b_k]$ are both infinite sequences of increasing integers, and if every sufficiently great integer can be represented in the form $a_k + b_k$, then
\[ \lim_{x \to \infty} \frac{A(x)B(x)}{x} > 1. \]

This conjecture can be stated in the following equivalent form:

(H$_1$) If by $f(n)$ we denote the number of representations of the integer $n$ in the form $a_k + b_k$, $f(n) \geq 1$ for $n \geq n_1$, and
\[ \lim_{x \to \infty} \frac{A(x)B(x)}{x} \leq 1, \]
then one of the sequences $[a_k]$, $[b_k]$ must be finite.

It seems very probable that the following stronger conjecture holds:

(H$_2$) If $f(n) \geq k$ for $n \geq n_2$, and
\[ \lim_{x \to \infty} \frac{A(x)B(x)}{x} \leq k, \]
then one of the sequences $[a_k]$, $[b_k]$ must be finite.

The purpose of this paper is to prove the following theorem associated with the conjecture (H$_1$):

**Theorem.** If $f(n) \geq k$ for almost all integers, and
\[ \lim_{x \to \infty} \frac{A(x)B(x)}{x} \leq k, \]
then:

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(i) \( f(n) = k \) for almost all integers,

(ii) \( \lim_{x \to \infty} \frac{A(2x)}{A(x)} = 1 \), or \( \lim_{x \to \infty} \frac{B(2x)}{B(x)} = 1 \).

From (ii) and a result of G. Pólya [2] it follows that \( A(x) = o(x^z) \), or \( B(x) = o(x^z) \) for every positive \( z \).

Proof. (i) Let \( a_i \) be such a sequence that \( f(a_i) \geq 1 + k \). If

\[
N(x) = \sum_{a_i \leq x} 1,
\]

then from the remark that there are \( A(x)B(x) \) sums \( a_i + b_j \) with \( a_i, b_j \leq x \). It follows that

\[
A(x)B(x) \geq kx + N(x) + o(x).
\]

Now if \( \lim_{x \to \infty} \frac{A(x)B(x)}{x} \leq k \), then \( N(x) = o(x) \).

(ii) It is evident from the preceding that

\[
A(x)B(x) = kx + o(x).
\]

Let us denote by \( f_x(l) \) the number of representations of the integer \( l \) as \( a_i + b_j \) with \( a_i, b_j \leq x \). Let

\[
P(x) = \sum_{l \leq x} f_x(l),
\]

Then

\[
A(x)B(x) = \sum_{l \leq x} f_x(l) = N(x) + \sum_{l \leq x} f_x(l);
\]

therefore

\[
0 \leq P(x) = A(x)B(x) - \sum_{l \leq x} f_x(l).
\]

But

\[
\sum_{l \leq x} f_x(l) \geq k(x + o(x))
\]

and thus we have

\[
0 \leq P(x) \leq kx - kx + o(x) = o(x).
\]

Now let us show that the functions

\[
a(x) = \frac{A(\lfloor x \rfloor)}{A(x)}, \quad \beta(x) = \frac{B(\lfloor x \rfloor)}{B(x)}
\]

cannot have other points of accumulation than 1 and \( \frac{1}{2} \), when \( x \) tends to infinity.

From (2) we have

\[
o(x) = \frac{A(x) - A(\lfloor x \rfloor)}{\lfloor x \rfloor} \cdot \frac{B(\lfloor x \rfloor) - B(x)}{B(x)} \geq 0,
\]

for if \( x/2 < a_i \leq x, \ x/2 < b_j \leq x \), then \( a_i + b_j > x \). But from (1) we have

\[
A(x) - A(\lfloor x \rfloor) \cdot A(\lfloor x \rfloor) - B(\lfloor x \rfloor) + o(x)
\]

and it follows that

\[
\lim_{x \to \infty} \frac{A(x)B(\lfloor x \rfloor) + B(x)A(\lfloor x \rfloor)}{x} = \frac{3}{2}.
\]

From this and (1) we deduce that

\[
\lim_{x \to \infty} \left( \frac{A(\lfloor x \rfloor)}{A(x)} + \frac{A(\lfloor x \rfloor)}{A(x)} \right) = \frac{3}{2}.
\]

If \( \lim a(n) = g \) \( n \to \infty \), then \( g + 1/2g = \frac{3}{2} \) and thus \( g = 1 \), or \( g = \frac{1}{2} \).

From (1) it is evident that if \( a(n) \to \frac{1}{2} \) then \( \beta(n) \to 1 \), and if \( \beta(n) \to \frac{1}{2} \) then \( a(n) \to 1 \).

It exists therefore a sequence \( a_i \), tending to infinity, for which \( a(n) \to 1 \) or \( \beta(n) \to 1 \).

Let us assume that

\[
a(n) \to 1.
\]

We shall now prove the following

**Lemma:** If for an increasing sequence \( b_n \), tending to infinity, \( a(b_n) \to 1 \), then

\[
\lim_{x \to \infty} \frac{A(\lfloor x \rfloor)}{A(x)} = 1.
\]

From (1) and (2) we have

\[
o(x) = \frac{A(x) - A(\lfloor x \rfloor)}{\lfloor x \rfloor} \cdot \frac{B(\lfloor x \rfloor) - B(x)}{B(x)} \geq 0,
\]

Hence

\[
\lim_{x \to \infty} \frac{B(\lfloor x \rfloor)A(\lfloor x \rfloor) + A(x)B(\lfloor x \rfloor) - A(x)B(\lfloor x \rfloor)}{x} = k,
\]

and after applying (1) we see that

\[
\lim_{x \to \infty} \frac{A(\lfloor x \rfloor) + 3A(\lfloor x \rfloor)}{4A(\lfloor x \rfloor)} = 1.
\]
We have
\[
\lim_{k \to \infty} A(t_k) = 1,
\]
for
\[
1 \leq \frac{A(t_k)}{A(t_k)} = \frac{1}{a(t_k)} \to 1.
\]
Hence
\[
1 = \lim_{k \to \infty} \left\{ \frac{A(t_k)}{A(t_k)} + \frac{3.4(\xi_k)}{4.4(\xi_k)} - \frac{3.4(\xi_k)}{4.4(\xi_k)} \cdot A(t_k) \cdot A(t_k) \right\}
\]
\[
= \lim_{k \to \infty} \left\{ \frac{A(t_k)}{A(t_k)} + \frac{3}{4} \cdot \frac{3.4(\xi_k)}{4.4(\xi_k)} \right\},
\]
and it follows that
\[
\lim_{k \to \infty} A(t_k) = 1.
\]

The lemma is thus proved.

Now let \( y_n \) be a sequence monotonically tending to infinity. Let
\[
X = \left\{ x : \frac{1}{a(x)} - 1 \leq \frac{1}{4} \right\}.
\]

It is evident that if \( x \in X \), then also \([x] \in X \) for \([x]/2 = \lfloor x \rfloor/2\) and
\( A(x) = A([x]) \).

It follows that \( X \) is left-sidedly closed.

From (3) it follows also that \( X \) is unbounded.

Let \( \xi_n = \inf X \). Then \( \xi_n \in X \) and so
\[
\left| \frac{A(\xi_n)}{A(\xi_n)} - 1 \right| \leq \frac{1}{4}.
\]

From this inequality it follows that \( a(\xi_n) \to 1 \), since a subsequence \( \xi_{n_k} \) for which \( a(\xi_{n_k}) \to \frac{1}{4} \) cannot exist. From the lemma we immediately find
\[
\lim_{n \to \infty} A(\xi_n) = 1;
\]
therefore
\[
\lim_{n \to \infty} \frac{A(\xi_n)}{A(\xi_n)} = \lim_{n \to \infty} \frac{A(\xi_n)}{A(\xi_n) \cdot A(\xi_n)} = 1.
\]

For sufficiently great \( n \) we have \( \frac{1}{4} \xi_n \leq y_n \), because if for an infinite sequence \( \{u_k\} \) we had \( \frac{1}{4} \xi_n > y_n \), we should have \( \frac{1}{4} \xi_n \in X \), and thus
\[
\left| \frac{A(\xi_n)}{A(\xi_n)} - 1 \right| \geq \frac{1}{4},
\]
which contradicts (5).

For sufficiently great \( n \) we thus have
\[
\frac{1}{4} \xi_n \leq y_n \leq \frac{1}{4} \xi_n \leq y_n \leq \frac{1}{4} \xi_n.
\]

From (4) we get
\[
\lim_{n \to \infty} A(\xi_n) = 1.
\]

From the arbitrariness of the sequence \( y_n \) it follows that
\[
\lim_{x \to \infty} \alpha(x) = 1,
\]
which completes the proof.

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