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ON PYTHAGOREAN ANGLES

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An angle φ will be called *Pythagorean* if it is congruent modulo $\pi/2$ to an angle of a Pythagorean triangle (rectangular triangle whose sides have integral lengths). In other words, φ is Pythagorean if and only if $\cos\varphi$ and $\sin\varphi$ are rational. Let $q_1, q_2, \ldots$ be the sequence of all primes of the form $4k+1$ (k natural). It is known that for every q_j there exists exactly one Pythagorean triangle (we identify isometric ones) the hypotenuse of which has the length q_j. Let φ_j be one of the acute angles of this triangle. We shall prove the following

THEOREM. *An angle φ is Pythagorean if and only if*

$$\varphi = k_0\frac{\pi}{2}+k_1\varphi_{j_1}+k_2\varphi_{j_2}+\ldots+k_n\varphi_{j_n} \tag{1}$$

where k_l are integers. The numbers k_l and j_l are uniquely determined by φ.

Notation. Let a, b be real numbers and $\gamma = a+bi$. We shall denote by $|\gamma|$ the module of γ and by $\arg\gamma$ any angle φ for which $\cos\varphi = a/|\gamma|$, $\sin\varphi = b/|\gamma|$ holds. If a and b are integers, then γ will be called a *complex integer*.

LEMMA. *If $\zeta = x+iy$, where x and y are relatively prime real integers and $|\zeta|$ is also an integer, then there exists a complex integer σ for which $\zeta = i^\varepsilon\sigma^2$ with $\varepsilon = 0$ or 1.*

Proof of the Lemma. Since $xy \neq 0$, from $x^2+y^2 = |\zeta|^2$ follows, as it is well known, that there exist real integers m, n such that one of the numbers x, y is m^2-n^2 and the other is $2mn$. If $x = m^2-n^2$, then we define $\sigma = m+in$ and $\zeta = \sigma^2$ follows. If $x = 2mn$, then we define $\sigma = -m+in$ and consequently $\zeta = i\sigma^2$.

Proof of the Theorem. Let φ be a Pythagorean angle. If $\varphi \equiv 0 \pmod{\pi/2}$, then there is a representation (1). In the opposite case φ is equal modulo $\pi/2$ to an acute angle of a Pythagorean triangle. We may assume that the lengths of the sides of that triangle are relatively prime numbers. Thus, if we denote by a the length of its hypotenuse, then evidently $\cos\varphi = x/a$ and $\sin\varphi = y/a$ where x, y are relatively prime integers. Let us consider the complex integer $\zeta = x+iy$. Since $|\zeta| = a$, it follows from our lemma that $\zeta = i^{\varepsilon}\sigma^2$ for some complex integer σ and for $\varepsilon = 0$ or 1. There exists a decomposition $\sigma = i^{c}\beta_1^{l_1}\beta_2^{l_2}\dots\beta_r^{l_r}$ where c is natural and β_k are complex prime (i. e. indecomposable) integers. It is known [1] that if β_k is not real, then $\beta_k = u+iv$ with $u, v = \pm 1$ or u^2+v^2 equal to one of the primes $q_1, q_2, \dots$. Thus for real β_k and also in the case when the first of the above possibilities holds we have $\arg\beta_k \equiv 0 \pmod{\pi/4}$ which implies $\arg\beta_k^2 \equiv 0 \pmod{\pi/2}$. In the other case $|\beta_k^2| = q_{j_k}$ and thus q_{j_k} is the hypotenuse of a Pythagorean triangle with one acute angle congruent modulo $\pi/2$ to $\arg\beta_k^2$. Consequently $\arg\beta_k^2 \equiv \pm\varphi_{j_k} \pmod{\pi/2}$. Now from $\zeta = i^{\varepsilon}\sigma^2 = i^{\varepsilon+2c}\beta_1^{2l_1}\dots\beta_r^{2l_r}$ follows

$$\arg\zeta \equiv (\varepsilon+2c)\frac{\pi}{2} + \sum_{k=1}^{r} l_k \arg\beta_k^2 \pmod{2\pi}.$$

Since

$$\varphi \equiv \arg\zeta \pmod{2\pi} \quad \text{and} \quad \arg\beta_k^2 \equiv 0 \text{ or } \pm\varphi_{j_k} \pmod{\pi/2}$$

we find that a representation (1) exists.

The uniqueness of this representaion follows if we prove that

$$k_0\frac{\pi}{2} + k_1\varphi_{j_1} + \dots + k_n\varphi_{j_n} = 0 \tag{2}$$

implies $k_0 = k_1 = \dots = k_n = 0$. Clearly $\cos\varphi_j = x_j/q_j$, $\sin\varphi_j = y_j/q_j$, where x_j, y_j are integers. Thus from the formulas

$$\begin{aligned} \cos(\varphi+\psi) &= \cos\varphi\cos\psi - \sin\varphi\sin\psi, \\ \sin(\varphi+\psi) &= \sin\varphi\cos\psi + \cos\varphi\sin\psi \end{aligned} \tag{3}$$

it follows that $\cos k\varphi_j$ and $\sin k\varphi_j$ can be represented by fractions whose denominator is a power of q_j. From (2) it follows that $\cos k_1\varphi_{j_1}$ or $\sin k_1\varphi_{j_1}$ is equal to $\pm\cos(k_2\varphi_{j_2}+\dots+k_n\varphi_{j_n})$, i. e. to a fraction whose denominator is by (3) relatively prime to q_{j_1}. But if two fractions with relatively prime denominators are equal, then they are integers. It follows that $k_1\varphi_{j_1} \equiv 0 \pmod{\pi/2}$. It is known [2] that if a Pythagorean angle $\varphi \not\equiv 0 \pmod{\pi/2}$, then φ is incommeasurable with π. Thus φ_{j_1} is incommeasurable with π and from $k_1\varphi_{j_1} \equiv 0 \pmod{\pi/2}$ follows $k_1 = 0$. By symmetry we also have $k_j = 0$ for $j = 2, \dots, n$. This implies $k_0 = 0$.

REFERENCES

[1] E. Landau, *Vorlesungen über Zahlentheorie*, Leipzig 1927, p. 12-15.

[2] J. M. H. Olmsted, *Rational values of trigonometric functions*, American Mathematical Monthly 52 (1945), p. 507-508.

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