nérale qui se prête à les définir systématiquement au moyen des propriétés (i)-(iv) de complexes abstraits.

Ajoutons, pour terminer, deux remarques:

1. Parmi les invariants de nature algébrique (c'est-à-dire qui s'expriment par des nombres, groupes, homomorphismes etc.), les invariants topologiques les plus importants sont, presque tous, en même temps des invariants du type d'homotopie. Tels sont les groupes d'homologie, ceux d'homotopie et autres. Il y a toutefois des exceptions, comme par exemple la dimension — et, plus généralement, les invariants de la forme (10) — qui sont des invariants topologiques des espaces de complexes abstraits sans en être des invariants du type d'homotopie. On a en effet pour le segment rectiligne I et le triangle Δ, qui sont de même type d'homotopie, $\chi_2^+(I) = 2$ et $\chi_2^+(\Delta) = 0$. Ainsi, ce sont des invariants topologiques d'un genre nouveau, promettant d'être applicables avec avantage dans des problèmes de nature topologique, mais non homotopique, donc tels que celui de la classification topologique des espaces, de leur isotopie, du plongement d'un espace dans un autre etc. Ces invariants paraissent en outre les plus simples dans leur genre (excepté la dimension).

2. La topologie combinatoire, qui concentrait sur elle l'attention des topologues pendant la première trentaine d'années de ce siècle, semble cesser peu à peu de susciter leur intérêt, malgré que *tous* les invariants topologiques des polyèdres finis se trouvent parmi les invariants combinatoires des complexes abstraits. Il est donc peut-être désirable de poursuivre les recherches dans ce domaine. Ce qui vient d'être exposé ici est déstiné à montrer qu'*il y a une possibilité de faire avancer ces recherches d'une façon méthodique*, en particulier dans l'ordre d'idées des théorèmes établis par Mayer et Lee pour des invariants numériques particuliers.

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ON CHAINS OF REGULAR TETRAHEDRA

BY

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In this paper we present the solution of a problem of Professor H. Steinhaus (conjecture (1) in [1]). We shall prove the following

THEOREM. *If $T_0, T_1, \ldots, T_n$ are regular tetrahedra such that*

1° T_i and T_{i+1} have strictly one face in common,

2° $T_i \neq T_{i+2}$,

then T_0 and T_n are not congruent by translation(1).

Proof. Given a rotation of the space, we may represent it by the pair $\langle a, \varphi\rangle$, where a is a vector situated on the rotation axis and φ is the rotation angle. We assume that for $\varphi > 0$ the rotation $\langle a, \varphi\rangle$ has a clockwise sense for an observer looking along its axis in the direction indicated by a.

Suppose that the vertices of T_0 are A, B, C, D. We write $\vec{AB} = a$, $\vec{CD} = b$. Let φ be the angle between two faces of T_0. We define

$$\Phi = \langle a, \varphi\rangle, \quad \Psi = \langle b, \varphi\rangle.$$

We shall prove first that there exists a sequence of rotations $\Theta_1, \ldots, \Theta_n$ satisfying $\Theta_1\Theta_2\ldots\Theta_i(T_0) = T_i$ $(i = 1, \ldots, n)$ and such that $\Theta_i^{\varepsilon_i} = \Phi$ or Ψ for $\varepsilon_i = 1$ or -1. Our proof is by induction. The assertion is obvious for $i = 1$. Suppose that it is true for some $i < n$. Write $\Theta = \Theta_1\Theta_2\ldots\Theta_i$. Observe that $T_{i+1} = \Omega(T_i)$, where $\Omega = \langle\Theta(c), \varepsilon\varphi\rangle$, $c = a$ or b, $\varepsilon = \pm 1$.

We define $\Theta_{i+1} = \langle c, \varepsilon\varphi\rangle$. Thus $\Theta_{i+1} = \Phi^{\pm 1}$ or $\Psi^{\pm 1}$.

Now observe the formula $\langle\Theta(c), \varepsilon\varphi\rangle = \Theta\langle c, \varepsilon\varphi\rangle\Theta^{-1}$. It implies $\Omega = \Theta\Theta_{i+1}\Theta^{-1}$. Thus $T_{i+1} = \Omega(T_i) = \Theta\Theta_{i+1}\Theta^{-1}\Theta(T_0) = \Theta_1\Theta_2\ldots\Theta_{i+1}(T_0)$.

Let $\mathcal{T}$ be the group of translations and $\mathcal{H}$ the group of all sense-preserving isometries of E^3. We consider the quotient group $\mathcal{H}/\mathcal{T}$, i. e. the group of rotations of E^3 around a fixed point O. For $x \in \mathcal{H}$ let us denote

(1) Added in proof: We have been informed that T. J. Dekker generalized this result to simplices of more than three dimensions.

by x^* the coset in $\mathcal{H}/\mathcal{T}$ which contains x. For a set $\mathcal{S} \subset \mathcal{H}$ we define $\mathcal{S}^* = \{x^* : x \in \mathcal{S}\}$. In particular, we have $\mathcal{H}^* = \mathcal{H}/\mathcal{T}$. It is evident that x and x^* have parallel rotation axes (that of x^* passing through O) and equal rotation angles. Thus Φ^* and Ψ^* have the rotation angle φ and rotation axes perpendicular to each other. Such rotations are independent (cf. [2]). This means that if $\Gamma_1, \ldots, \Gamma_n \in \mathcal{H}^*$ satisfy $\Gamma_i^{\varepsilon_i} = \Phi^*$ or Ψ^*, where $\varepsilon_i = 1$ or -1 and $\Gamma_i \Gamma_{i+1} \neq e$ (e = the unity of $\mathcal{H}^*$), then $\Gamma_1 \Gamma_2 \ldots \Gamma_n \neq e$.

From assumptions 1° and 2° it follows that the face common to T_{i+1} and T_i is not parallel to any face of T_{i-1}. Hence T_{i+1} cannot be obtained from T_{i-1} by a translation. Since $T_{i+1} = \Theta_1 \ldots \Theta_{i-1} \Theta_i \Theta_{i+1} \Theta_{i-1}^{-1} \ldots \Theta_1^{-1}(T_{i-1})$, it follows that $\Theta_i \Theta_{i+1} \notin \mathcal{T}$. Thus $\Theta_i^* \Theta_{i+1}^* \neq e$ and we infer by $(\Theta_i^*)^{\varepsilon_i} = \Phi^*$ or Ψ^* and by the independence of Φ^* and Ψ^* that $\Theta_1^* \Theta_2^* \ldots \Theta_n^* \neq e$.

Let us denote by $\mathcal{S}$ the group of rotations which transform T_0 into itself. Since $\mathcal{S}^*$ is finite, we have, by the independence of Φ^* and Ψ^*, $\Theta_1^* \Theta_2^* \ldots \Theta_n^* \notin \mathcal{S}^*$. Consequently $\Theta_1 \Theta_2 \ldots \Theta_n$ is not a combination of a translation and a rotation belonging to $\mathcal{S}$. Thus $T_n = \Theta_1 \Theta_2 \ldots \Theta_n(T_0)$ and T_0 are not congruent by translation.

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ON MEASURES IN FIBRE BUNDLES

BY

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In the present paper I introduce the notion of a product measure in a fibre bundle. For the theory of fibre bundles I use the terminology and notation of Steenrod [2], and for the measure theory the terminology and notation of Halmos [1].

1. Let $\mathcal{B}(B, X, Y, G)$ be a fibre bundle with a locally compact base space X, a locally compact fibre Y, and thus a locally compact bundle space B. Consider Baire measures μ and ν given respectively in X and Y, and denote by $\mu \times \nu$ the product of those measures in the Cartesian product $X \times Y$.

A Baire measure λ in B is called the *product measure of μ and ν in the fibre bundle* $\mathcal{B}$ if for every representation of $\mathcal{B}$ as a coordinate bundle $\mathcal{B}(B, X, Y, G, V_j, \varphi_j)$ and for each Baire set $Z \subset V_j \times Y$ the equality

$$\lambda(\varphi_j(Z)) = (\mu \times \nu)(Z) \tag{1}$$

holds.

2. THEOREM 1. *A product measure λ of μ and ν in a fibre bundle $\mathcal{B}(B, X, Y, G)$ exists if and only if the measure ν in Y is invariant under transformations of the group G* ([1]).

Proof. a) Let us suppose that there exists in $\mathcal{B}$ a product measure λ. Consider any representation of the fibre bundle as a coordinate bundle $\mathcal{B}(B, X, Y, G, V_j, \varphi_j)$ and any fixed element g of G. The coordinate bundle $\mathcal{B}'(B, X, Y, G, V_j', \varphi_j')$ with $V_j' = V_j$, $\varphi_j'(x, y) = \varphi_j(x, g^{-1}y)$ is stricly equivalent to $\mathcal{B}$. In fact, the functions $\bar{g}_{jj}(x) = \varphi_{j,x}^{-1}\varphi_{j,x} = g$, $\bar{g}_{ji}(x) = g g_{ji}(x)$ are continuous.

Let A be a Baire subset of V_j of positive finite measure μ and E a measurable set in Y. We then have

$$\lambda(\varphi_j(A \times E)) = \mu(A)\nu(E). \tag{2}$$

([1]) Evidently, the measure λ is completely determined by μ and ν.