ou bien par la condition:

$$\delta \cdot H = \delta \quad ou \quad H \cdot \delta = 0$$

où H est la fonction de Heaviside.

TRAVAUX CITÉS

[1] A. César de Freitas, *Sur les distributions qui interviennent dans le calcul symbolique des électrotechniciens (cas des circuits à constantes concentrées)*, Universidade de Lisboa, Revista da Faculdade de Ciências, 2A série A, Ciências Matemáticas, 3 (1953-1954), p. 279-310.

[2] — *Un produit multiplicatif de distributions de Heaviside*, ibidem 5 (1955-1956), p. 135-146.

[3] L. Schwarz, *Théorie des distributions I*, Paris 1950.

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ON ABSOLUTE CONVERGENCE OF HAAR SERIES

BY

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1. Let $\chi_n^{(k)}(t)$ denote the Haar functions defined as follows

$$\chi_0^{(0)}(t) = 1 \quad \text{for} \quad 0 \leqslant t \leqslant 1$$

and

$$\chi_n^{(k)}(t) = \begin{cases} \sqrt{2^n} & \text{for} \quad \dfrac{2k-2}{2^{n+1}} \leqslant t < \dfrac{2k-1}{2^{n+1}}, \\ -\sqrt{2^n} & \text{for} \quad \dfrac{2k-1}{2^{n+1}} < t \leqslant \dfrac{2k}{2^{n+1}}, \\ 0 & \text{elsewhere in } \langle 0, 1 \rangle \end{cases}$$

for $n = 0, 1, \ldots; k = 1, 2, \ldots, 2^n$. Let $\chi_0^{(0)}(t) = \chi_1(t)$ and $\chi_n^{(k)}(t) = \chi_{2^n+k}(t)$ for $n = 0, 1, \ldots; k = 1, 2, \ldots, 2^n$ and $0 \leqslant t \leqslant 1$. Moreover, let a_n denote the Fourier coefficients of an integrable function $f(t)$, i. e.

$$a_n = \int_0^1 f(t)\chi_n(t)\,dt.$$

The problem of convergence of the series

$$\sum_{n=1}^{\infty} |a_n \chi_n(t)| \tag{1}$$

and

$$\sum_{n=1}^{\infty} |a_n| \tag{2}$$

will be considered. We obtain theorems analogous to those given by Bernstein and Zygmund for trigonometrical series. Questions concerning the divergence of series (2) and, more generally, of series (5) with $\beta = 0$, were considered by Orlicz ([1]).

2. First, let us remark that

$$a_n^{(k)} = \tfrac{1}{2} 2^{-n/2} \int_0^1 \{f[2^{-n-1}t + (2k-2)2^{-n-1}] - f[2^{-n-1}t + (2k-1)2^{-n-1}]\} dt$$

for $n = 0, 1, \ldots;\ k = 1, 2, \ldots, 2^n$, where $a_n^{(k)} = a_{2^n+k}$. We shall write

$$\omega(h) = \sup_{\substack{|\delta| \leqslant t \leqslant 1-|\delta| \\ |\delta| < h}} |f(t+\delta) - f(t)|$$

and

$$\omega_r(h) = \sup_{|\delta| \leqslant h} \Big[\int_0^1 |f(t+\delta) - f(t)|^r dt\Big]^{1/r},$$

wherein $r \geqslant 1$ and $f(t) = 0$ for $t \notin \langle 0, 1\rangle$.

3. THEOREM 1. *If $\beta \geqslant 0$, $\gamma, \lambda > 0$ and if the function $f(t)$ satisfies the condition*

$$\sum_{n=1}^{\infty} \frac{\omega^{\gamma}\left(\frac{1}{n}\right)}{n^{1-\beta-(\gamma-\lambda)/2}} < \infty, \tag{3}$$

then the series

$$\sum_{n=1}^{\infty} n^{\beta} |a_n|^{\gamma} |\chi_n(t)|^{\lambda}$$

is uniformly convergent in $\langle 0, 1\rangle$.

To prove this theorem, we choose for a given t a positive integer k_n such that $(k_n-1)/2^n \leqslant t < k_n/2^n$. Then,

$$\sum_{n=2}^{\infty} n^{\beta} |a_n|^{\gamma} |\chi_n(t)|^{\lambda} = \sum_{n=0}^{\infty} (2^n + k_n)^{\beta} |a_n^{(k_n)}|^{\gamma} |\chi_n^{(k_n)}(t)|^{\lambda}$$
$$\leqslant 2^{\beta-\gamma} \sum_{n=1}^{\infty} 2^{n[\beta+(\lambda-\gamma)/2]} \omega^{\gamma}(2^{-n})$$

and (3) imply theorem 1.

Theorem 1 implies the following

COROLLARY. *If $f(t)$ satisfies the Hölder condition with an exponent $\alpha > 0$ in $\langle 0, 1\rangle$, then the series (1) converges uniformly in $\langle 0, 1\rangle$.*

([1]) W. Orlicz, *Zur Theorie der Orthogonalreihen*, Bulletin de l'Académie Polonaise des Sciences 1927, p. 81-115, especially p. 105.

If we assume that $f(t)$ is only continuous in $\langle 0, 1\rangle$, then the statement in corollary 1 is not true (cf. ([1]), theorem 3, p. 99).

THEOREM 2. *Given $\beta \geqslant 0$ and $\gamma > 0$, let us assume the function $f(x)$ to satisfy the condition*

$$\sum_{n=1}^{\infty} \frac{\omega_{\mu}^{\gamma}\left(\frac{1}{n}\right)}{n^{\gamma/2-\beta}} < \infty, \tag{4}$$

where $\mu = \max(\gamma, 1)$. Then the series

$$\sum_{n=1}^{\infty} n^{\beta} |a_n|^{\gamma} \tag{5}$$

is convergent.

Proof. It is easily verified that

$$a_n^{(k)} = -\sqrt{2^n} \int_{(2k-2)/2^{n+1}}^{(2k-1)/2^{n+1}} [f(t+2^{-n-1}) - f(t)] dt$$

for $n = 0, 1, \ldots;\ k = 1, 2, \ldots, 2^n$. Let $\gamma > 1$. Since

$$\sum_{k=1}^{2^n} |a_n^{(k)}|^{\gamma} \leqslant 2^{1-\gamma} 2^{n(1-\gamma/2)} \omega_{\gamma}^{\gamma}(2^{-n}),$$

we have

$$\sum_{n=2}^{\infty} n^{\beta} |a_n|^{\gamma} = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} (2^n + k)^{\beta} |a_n^{(k)}|^{\gamma}$$
$$\leqslant 2^{1+\beta-\gamma} \sum_{n=0}^{\infty} 2^{n(1+\beta-\gamma/2)} \omega_{\gamma}^{\gamma}(2^{-n}).$$

This together with (4) imply the convergence of series (5). If $0 < \gamma \leqslant 1$, then

$$\sum_{k=1}^{2^n} |a_n^{(k)}|^{\gamma} \leqslant 2^{n(1-\gamma/2)} \omega_1^{\gamma}(2^{-n});$$

hence

$$\sum_{n=2}^{\infty} n^{\beta} |a_n|^{\gamma} \leqslant 2^{\beta} \sum_{n=0}^{\infty} 2^{n(1+\beta-\gamma/2)} \omega_1^{\gamma}(2^{-n}).$$

By (4) and by the last inequality we obtain the convergence of series (5).

Theorem 2 implies

COROLLARY 2. *If $f(t)$ satisfies the Hölder condition with an exponent $a > 1/2$ in $\langle 0, 1\rangle$, then the series* (2) *is convergent.*

It can be easily seen that the uniform convergence of the series (1) does not imply the convergence of series (2) and conversely. We do not know whether the convergence of (2) implies the uniform convergence of (1), a_n being the coefficients of a continuous function (**P 279**).

We shall now apply the notion of the p-th variation of a function $f(t)$ as defined by the formula

$$V_p(f) = \Big[\sup_{\Pi} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p\Big]^{1/p},$$

where Π is an arbitrary partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of the interval $\langle 0, 1\rangle$.

THEOREM 3. *If $V_p(f) < \infty$ for a function $f(t)$ and for a number $1 \leqslant p < 2$, and if*

$$\sum_{n=1}^{\infty} \frac{1}{n}\, \omega^{1-p/2}\Big(\frac{1}{n}\Big) < \infty,$$

then series (2) *is convergent.*

Proof. Since

$$|a_n^{(k)}|^2 \leqslant 2^{-n-2}\omega^{2-p}(2^{-n-1}) \int_0^1 |f[2^{-n-1}t + (2k-2)2^{-n-1}] - f[2^{-n-1}t + (2k-1)2^{-n-1}]|^p\, dt,$$

we have

$$\sum_{k=1}^{2^n} |a_n^{(k)}| \leqslant \Big(\sum_{k=1}^{2^n} |a_n^{(k)}|^2\Big)^{1/2} \cdot 2^{n/2}$$

$$\leqslant \tfrac{1}{2}\omega^{1-p/2}(2^{-n}) \Big[\int_0^1 \Big\{\sum_{k=1}^{2^n} |f[2^{-n-1}t + (2k-2)2^{-n-1}] - f[2^{-n-1}t + (2k-1)2^{-n-1}]|^p\Big\} dt\Big]^{1/2}$$

$$\leqslant \tfrac{1}{2} V_p^{p/2}(f)\, \omega^{1-p/2}(2^{-n});$$

hence the series (2) is majorized by

$$\frac{1}{2} V_p^{p/2}(f) \sum_{n=0}^{\infty} \omega^{1-p/2}(2^{-n}),$$

which proves our theorem.

COROLLARY 3. *If $f(t)$ satisfies the Hölder condition with an exponent $a > 0$, and if $V_p(f) < \infty$ for a number $1 \leqslant p < 2$, then series* (2) *is convergent.*

Theorem 3 may also be extended to the case of series (5).

4. Let $r_n(t)$ denote the n-th Rademacher function, i. e. $r_n(t) = \operatorname{sgn}\sin 2^n \pi t$ for $n = 1, 2, \ldots$ and for $0 \leqslant t \leqslant 1$. By the method similar to that used above we establish the two following theorems:

THEOREM 4. *If $\gamma > 0$ and if*

$$\sum_{n=1}^{\infty} \frac{1}{n}\, \omega_1^{\gamma}\Big(\frac{1}{n}\Big) < \infty,$$

then

$$\sum_{n=1}^{\infty} |a_n|^{\gamma} < \infty.$$

THEOREM 5. *If the function $f(t)$ is of bounded p-th variation in $\langle 0, 1\rangle$ for a $p \geqslant 1$, then the series $\sum_{1}^{\infty} |a_n|^{\gamma}$ is convergent for every $\gamma > 0$.*

Here, $a_n = \int_0^1 f(t) r_n(t)\, dt$.

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