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ON DITKIN SETS

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In the study of spectral synthesis S-sets and C-sets (see Rudin [3]; Reiter [2] uses the terminology Wiener sets and Wiener–Ditkin sets respectively) have been discussed extensively. A new concept of Ditkin sets was introduced and studied by Stegeman in [4] so that, in Reiter's terminology, Wiener–Ditkin sets are precisely sets which are both Wiener sets and Ditkin sets. The importance of such sets in spectral synthesis and their connection to the C-set–S-set problem (see Rudin [3]) are mentioned there.

In this paper we study local properties, unions and intersections of Ditkin sets. (Warning: Usually in the literature "Ditkin set" means "C-set", but we follow the terminology of Stegeman.) Our results include: (i) if each point of a closed set E has a closed relative Ditkin neighbourhood, then E is a Ditkin set; (ii) any closed countable union of Ditkin sets is a Ditkin set; (iii) if $E_1 \cap E_2$ is a Ditkin set, then $E_1 \cup E_2$ is a Ditkin set if and only if E_1 and E_2 are Ditkin sets; and (iv) if E_1, E_2 are Ditkin sets with disjoint boundaries then $E_1 \cap E_2$ is a Ditkin set.

Let G be a locally compact abelian group and let $A(G)$ denote the Fourier algebra of G (see Rudin [3]). Suppose $x \in G$, $f, g \in A(G)$ and J_1 and J_2 are ideals in $A(G)$. We write (i) $f =_x g$ if f and g agree in some neighbourhood of x; (ii) $f \in_x J_i$ if $f =_x g$ for some $g \in J_i$; (iii) $J_1 \subset_x J_2$ if $f \in_x J_2$ for every $f \in J_1$; (iv) $J_1 =_x J_2$ if $J_1 \subset_x J_2$ and $J_2 \subset_x J_1$. For a closed subset E of G, $j(E) = \{f \in A(G) : \operatorname{supp} f \cap E = \emptyset\}$ and $J(E) = \overline{j(E)}$. It is the smallest closed ideal with zero set E. We also have the well known Localisation Lemma: if $J \subset A(G)$ is a closed ideal and $f \in A(G)$, then $f \in_x J$ for each $x \in G$ implies $f \in J$.

Definition 1 ([4]). A closed subset $E \subset G$ is called a *Ditkin set* if $f \in \overline{fj(E)}$ for all $f \in J(E)$. Thus E is a Ditkin set if and only if $\Gamma(E) = \emptyset$, where

$$\Gamma(E) = \bigcup_{f \in J(E)} \{x : f \notin_x \overline{fj(E)}\}.$$

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We call $\Gamma(E)$ the *set of non-Ditkin points* of E.

DEFINITION 2. Let E be a closed subset of G. A closed subset F of E is *D-distinguished* (relative to E) if for every $f \in J(E)$ and $\varepsilon > 0$ there is a $v \in j(F)$ such that $\|f - vf\| < \varepsilon$. We call a point of E *D-regular* if it has a closed relative neighbourhood in E which is D-distinguished.

This definition and our first result are motivated by Reiter [2]. The proof of the following lemma is an adaptation of his arguments [2, p. 32].

LEMMA 3. *Every closed subset of E containing only D-regular points is D-distinguished.*

Proof. Suppose each point of the closed subset $F \subset E$ has a closed relative neighbourhood in E which is D-distinguished. Let $f \in J(E)$. To show $f \in \overline{fj(F)}$, it suffices to show that $f \in_x \overline{fj(F)}$ for all $x \in G$. Let $x \in G$ and let U be a compact neighbourhood of x in G. A finite number of D-distinguished closed relative neighbourhoods $U_1, \ldots, U_n$ cover $F \cap U$. Then choosing $k \in A(G)$ such that k=1 in a neighbourhood of x and k vanishes outside U, we see successively that $fk \in \overline{fkj(U_1)}, fk \in \overline{\overline{fkj(U_2)}j(U_1)} \subset \overline{fkj(U_1 \cup U_2)}, \ldots, fk \in \overline{fkj(F \cap U)} \subset \overline{fj(F)}$, so $f =_x fk \in \overline{fj(F)}$. ∎

THEOREM 4. *Let E be a closed subset of G such that each point of E has a closed relative neighbourhood in E which is a Ditkin set for $A(G)$. Then E itself is a Ditkin set for $A(G)$.*

Proof. A particular case of Lemma 3. ∎

COROLLARY 5. *If E is a closed subset of G such that each point on the boundary ∂E of E has a relative Ditkin neighbourhood, then E itself is a Ditkin set.*

Proof. This follows from the fact that if the boundary of E is a Ditkin set then E is a Ditkin set [4], and from the theorem above. ∎

Remark 6. Using local techniques we can also prove that the set of points which are not D-regular is a subset of the boundary.

We now come to results on unions and intersections of Ditkin sets. Our first result on unions includes the result of Stegeman [4] that the union of two Ditkin sets is again a Ditkin set.

THEOREM 7. *Any closed countable union of Ditkin sets is a Ditkin set.*

Proof. Let $\{E_i\}$ be a countable collection of Ditkin sets and suppose $E = \bigcup E_i$ is closed. Let $f \in J(E)$ and $x \in G$. Choose a compact neighbourhood U of x and choose $k \in A(G)$ such that $k = 1$ near x and $\operatorname{supp} k \subset U$. Then $f =_x fk$. Since $J(E) \subset J(E_1)$, we have $fk \in J(E_1)$. So $fk \in \overline{fkj(E_1)}$. Again $fk \in J(E_2)$ implies that $fk \in \overline{\overline{fkj(E_2)}j(E_1)} \subset \overline{fkj(E_1 \cup E_2)}$. Thus for every $n \geq 1$, $fk \in \overline{fkj(E_1 \cup \ldots \cup E_n)}$, that is, given $\varepsilon > 0$, there exists $g_n \in j(E_1 \cup \ldots \cup E_n)$ such that $\|fk - fkg_n\| < \varepsilon$. Put $C = E \cap U$; since C is a compact subset of E there exists N such that $C \subset \bigcup_{n=1}^{N} U_n$, where U_n is an open neighbourhood of E_n with $g_n = 0$ on U_n $(1 \leq n \leq N)$. Then $kg_N \in j(E)$. So $f \in_x \overline{fj(E)}$ for all x and $f \in \overline{fj(E)}$. Hence the result. ■

Remark 8. A similar argument also proves the following result: *If F_n is a D-distinguished subset of E_n, $n = 1, 2, \ldots$, and if $E = \bigcup E_n$ and $F = \bigcup F_n$ are closed, then F is D-distinguished in E.*

Corollary 9. *Let $\{E_i\}$ be a countable collection of Ditkin sets in G. Suppose that the closure F of $\overline{\bigcup E_i} \setminus \bigcup E_i$ is a Ditkin set; then $\overline{\bigcup E_i}$ is a Ditkin set.*

Proof. Apply the theorem for the sets $E_i, i \geq 0$, with $E_0 = F$. ■

In the remaining part of the paper we make use of the result of Stegeman that if E is a closed subset of G and $\Gamma(E) \subset D \subset E$ for some Ditkin set D, then E is a Ditkin set ([4]).

Lemma 10. *Let E_1, E_2 be two closed subsets of G. Then*

(i) $\Gamma(E_i) \subset \Gamma(E_1 \cup E_2) \cup (E_1 \cap E_2)$, $i = 1, 2$.

(ii) $\Gamma(E_1 \cap E_2) \subset \Gamma(E_1) \cup \Gamma(E_2) \cup (\partial(E_1) \cap \partial(E_2))$.

Proof. To prove (i), let $x \in \Gamma(E_1)$ and $x \notin E_1 \cap E_2$. Then there exists $f \in J(E_1)$ such that $f \notin_x \overline{fj(E_1)}$. Suppose that $x \notin \Gamma(E_1 \cup E_2)$. Choose a neighbourhood U of x such that $\overline{U} \cap E_2 = \emptyset$. Choose $k \in A(G)$ such that $k = 1$ in a neighbourhood of x and $\operatorname{supp} k \subset U$. Then $fk \in J(E_1 \cup E_2)$, so $f =_x fk \in_x \overline{fkj(E_1 \cup E_2)} \subset \overline{fkj(E_1)}$, a contradiction. A similar proof holds for $\Gamma(E_2)$.

To prove (ii), let $x \in \Gamma(E_1 \cap E_2)$ and $x \notin \partial(E_1) \cap \partial(E_2)$. Then there exists $f \in J(E_1 \cap E_2)$ such that $f \notin_x \overline{fj(E_1 \cap E_2)}$. Moreover, $x \in \partial(E_1)$ or $x \in \partial(E_2)$ but not both. Assume $x \in \partial(E_1)$. We prove that $x \in \Gamma(E_1)$.

Suppose that $x \notin \Gamma(E_1)$. Since $x \in E_2^0$, the interior of E_2, there is a neighbourhood V of x such that $\overline{V} \subset E_2^0$. Choose $k \in A(G)$ such that $k = 1$ in a neighbourhood of x and $\operatorname{supp} k \subset V$. Then $fk \in J(E_1)$. But $x \notin \Gamma(E_1)$ implies that $fk \in_x \overline{fkj(E_1)} \subset \overline{fkj(E_1 \cap E_2)}$, a contradiction.

Similarly if $x \in \partial(E_2)$ then $x \in \Gamma(E_2)$. ■

We now deduce some consequences of Lemma 10.

Theorem 11. *Let E_1, E_2 be two closed subsets of G. Let $E = E_1 \cup E_2$. Suppose there are two Ditkin sets D_1, D_2 such that $E_1 \cap E_2 \subset D_1 \subset E_1$ and $E_1 \cap E_2 \subset D_2 \subset E_2$. Then E is a Ditkin set if and only if E_1 and E_2 are Ditkin sets.*

Proof. Since the union of two Ditkin sets is a Ditkin set, one part is trivial. The other part follows from Lemma 10(i) and Stegeman's result (stated after Corollary 9). ■

COROLLARY 12. *Let E be a Ditkin set in G, let F be a closed subset of E and let $E_1 = E \setminus F^0$. If there exists a Ditkin set D such that $\partial F \subset D \subset E_1$ then E_1 is a Ditkin set. In particular, if E and ∂F are Ditkin sets then E_1 is a Ditkin set.*

Proof. If $E_1 = E \setminus F^0$ and $E_2 = F$, then $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \partial F$, so we can apply Theorem 11. ■

THEOREM 13. *Let E_1, E_2 be Ditkin sets in G. If there is a Ditkin set D such that $\partial(E_1) \cap \partial(E_2) \subset D \subset E_1 \cap E_2$ then $E_1 \cap E_2$ is a Ditkin set. In particular, the conclusion holds if $\partial(E_1) \cap \partial(E_2) = \emptyset$.*

Proof. This is a consequence of Lemma 10(ii) and Stegeman's result. ■

The fundamental, but probably very difficult, question about Ditkin sets is the one raised in [4]: is every closed set a Ditkin set?

A study of S-sets using local techniques can be found in [1].

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REFERENCES

[1] T. K. Muraleedharan and K. Parthasarathy, *On unions and intersections of sets of synthesis*, Proc. Amer. Math. Soc., to appear.

[2] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Oxford University Press, Oxford, 1968.

[3] W. Rudin, *Fourier Analysis on Groups*, Interscience, New York, 1962.

[4] J. D. Stegeman, *Some problems on spectral synthesis*, in: Proc. Harmonic Analysis (Iraklion, 1978), Lecture Notes in Math. 781, Springer, Berlin, 1980, 194–203.

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