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FELL'S SUBGROUP ALGEBRA FOR LOCALLY COMPACT ABELIAN GROUPS AND L^1-COVARIANCE ALGEBRAS

BY

DETLEV POGUNTKE (BIELEFELD)

For any topological space X Fell has introduced (see [6]) a quasi-compact topology on the set $\Phi(X)$ of all closed subsets of X: For each quasi-compact subset C (the empty set is not excluded) and each finite family $\mathcal{F}$ (the empty family is not excluded) of nonempty open subsets of X let $Q(\mathcal{F}, C)$ be the set of all $Y \in \Phi(X)$ such that $Y \cap C = \emptyset$ and $Y \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The sets $Q(\mathcal{F}, C)$ form a basis of this topology. If X happens to be locally quasi-compact then $\Phi(X)$ is Hausdorff, hence compact. In this paper we shall be exclusively interested in locally compact spaces X, very often even in locally compact (abelian) groups.

Our first proposition says that a locally compact transformation group (G, X) gives rise to a continuous action of G on $\Phi(X)$. Then we specialize to $X = G$, where an alternative description of the above topology on $\Phi(G)$ was given by Bourbaki [3]. Next, two subspaces of $\Phi(G)$ are studied, namely the space $\Sigma(G)$ of closed subgroups and the space $\Lambda(G)$ of left cosets, $\Lambda(G) = \{gH \mid g \in G, H \in \Sigma(G)\}$. The space $\Sigma(G)$ is the basis for the construction of Fell's subgroup algebra $\mathcal{A}_{\mathrm{s}}(G)$ (see [7] and below). We show that the Banach algebra $\mathcal{A}_{\mathrm{s}}(G)$ has a bounded two-sided identity, and that the restriction maps $\mathcal{A}_{\mathrm{s}}(G) \to L^1(H)$ are surjective for all $H \in \Sigma(G)$.

In the second section the case of abelian groups G is treated. Then $\mathcal{A}_{\mathrm{s}}(G)$ is a commutative regular symmetric algebra, whose structure space is homeomorphic to $\Lambda(G^\wedge)$, where $G^\wedge$ denotes the Pontryagin dual. The arguments developed for those results also show that the map $\Sigma(G) \ni H \mapsto H^\perp \in \Sigma(G^\wedge)$ is a homeomorphism, which was proved by Williams [21]. Furthermore, it is shown that if the Haar measures on the various subgroups $H \in \Sigma(G)$ are chosen continuously then the associated Haar measures on the subgroups Δ of $G^\wedge$, via Poisson's summation formula, depend continuously on $\Delta \in \Sigma(G^\wedge)$.

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In the final section we extend some results of Pytlik [18], in particular we determine the $*$-primitive ideal spaces of covariance algebras $L^1(G,\mathcal{B})$, where G and $\mathcal{B}$ are commutative and some additional mild assumptions are satisfied. As an application we compute the support of the conjugation representation for semidirect products of abelian groups. This representation has recently drawn some attention (see [11, 12] and the references given there).

1. Some properties of $\Phi(X)$ and of the subgroup algebra. We start with an easy lemma, used several times in this article. For the notion of (sub)nets we refer to [13, p. 65ff].

LEMMA 1.1. *Let X be a locally compact space, and let $(S_i)_{i\in I}$ be a convergent net in $\Phi(X)$. Then the following subsets of X coincide with $S := \lim_{i\in I} S_i$:*

$$M_1 := \bigcap_J \Big(\bigcup_{j\in J} R_j\Big)^-,$$

where $(R_j)_{j\in J}$ is any subnet of $(S_i)_{i\in I}$,

$$M_2 := \bigcap_J \Big(\bigcup_{j\in J} S_j\Big)^-,$$

where J is any cofinal subset of I,

$$M_3 := \bigcap_{i_0\in I} \Big(\bigcup_{\substack{i\in I\\ i\ge i_0}} S_i\Big)^-,$$

and

$$M_4 := \{x\in X \mid \text{there exists a subnet } (R_j)_{j\in J} \text{ of } (S_i)_{i\in I} \text{ and points } x_j\in R_j \text{ such that } x=\lim_{j\in J} x_j\}.$$

P r o o f. The inclusions $M_1\subset M_2\subset M_3$ are obvious. To show $M_3\subset M_4$ let $x\in M_3$ be given. Let $\mathfrak{V}$ be a basis of the neighborhoods of x in X, and let $J := I\times\mathfrak{V}$ with the obvious ordering, i.e., $(i,V)\le(i',V')$ if $i\le i'$ and $V\supset V'$. For $j=(i,V)\in J$ choose $n=n(i,V)=n(j)\in I$ such that $S_n\cap V\ne\emptyset$ and $n(i,V)\ge i$, and choose a point $x_j\in S_n\cap V$. Clearly, the required subnet $(R_j)_{j\in J}$ is defined by $R_j=S_{n(j)}$, and the net $(x_j)_{j\in J}$ converges to x.

To see $M_4\subset S$, let $x\in M_4$, let $(R_j)_{j\in J}$ be a subnet of $(S_i)_{i\in I}$, let $x_j\in R_j$, and suppose that $x=\lim_{j\in J} x_j$ exists, but that $x\notin S=\lim_{i\in I} S_i = \lim_{j\in J} R_j$. Then there exists a compact neighborhood V of x with $V\cap S=\emptyset$. The set $Q(\emptyset,V)\subset\Phi(X)$ is a neighborhood of S in $\Phi(X)$. Hence there exists

$j_0 \in J$ such that $R_j \in Q(\emptyset, V)$ for all $j \geq j_0$, i.e., $R_j \cap V = \emptyset$. But this is absurd as (x_j) converges to x.

Finally, we have to show that S is contained in M_1, i.e., for any subnet $(R_j)_{j\in J}$ of $(S_i)_{i\in I}$ the set S has to be contained in $(\bigcup_{j\in J} R_j)^-$. Let $x \in S$ (if S is empty there is nothing to prove), and let F be any neighborhood of x in X. Then $Q(\{F\}, \emptyset) \subset \Phi(X)$ is a neighborhood of S. As $(R_j)_{j\in J}$ converges to S there exists $j_0 \in J$ such that $R_j \cap F \neq \emptyset$ for all $j \geq j_0$. In particular, $F \cap \bigcup_{j\in J} R_j \neq \emptyset$ for all F, whence $x \in (\bigcup_{j\in J} R_j)^-$. ■

PROPOSITION 1.2. *Let (G, X) be a locally compact transformation group. Then there is a natural action $G \times \Phi(X) \to \Phi(X)$, $(g, S) \mapsto gS$, which is continuous, i.e., $(G, \Phi(X))$ is a transformation group.*

Proof. Clearly, it suffices to check the continuity at a point (e, S). Let a typical neighborhood $Q(\mathcal{F}, C)$ of S be given. There exists a compact symmetric neighborhood V of e such that $S \cap VC = \emptyset$. For each $F \in \mathcal{F}$ choose a point $s_F \in F \cap S$, a neighborhood V_F of $s_F \in X$, and a symmetric neighborhood W_F of e in G such that $W_F V_F \subset F$. Then put $C' = VC$, $\mathcal{F}' = \{V_F \mid F \in \mathcal{F}\}$ and form the neighborhood $Q(\mathcal{F}', C')$ of S. It is easily verified that the neighborhood $W := V \cap \bigcap_{F\in\mathcal{F}} W_F$ has the property that $g \in W$, $S' \in Q(\mathcal{F}', C')$ implies $gS' \in Q(\mathcal{F}, C)$. ■

Now we consider $\Phi(G)$ for a locally compact group G. In this case Bourbaki has defined a topology on $\Phi(G)$ which is more closely related to the Hausdorff distance in the context of metric spaces. For $S \in \Phi(G)$, a neighborhood V of the identity in G and a compact set A in G let

$$P(S, V, A) = \{R \in \Phi(G) \mid S \cap A \subset VR,\ R \cap A \subset VS\}.$$

These sets $P(S, V, A)$, where V and A are varying, form a neighborhood basis of S for a certain topology on $\Phi(G)$, which we call the *Bourbaki topology*. As pointed out in [3] there is a natural uniform structure which gives this topology.

PROPOSITION 1.3. *For any locally compact group G the Bourbaki topology and the formerly introduced topology on $\Phi(G)$ coincide.*

Proof. Clearly, as both topologies are compact (for the Bourbaki topology see [3, pp. 188–189]), it would suffice to prove one inclusion of the topologies, but in order to clarify the relation it seems best to prove both inclusions.

First, let $P(S, V, A)$ be given. We have to construct a finite family $\mathcal{F}$ of open sets in G and a compact set C in G such that $S \in Q(\mathcal{F}, C) \subset P(S, V, A)$. Choose an open symmetric neighborhood U of e in G such that $U^2 \subset V$. There exist finitely many elements $s_1, \ldots, s_n$ in the compact set $A \cap S$ such that $A \cap S$ is covered by Us_j, $1 \leq j \leq n$. Put $\mathcal{F} := \{Us_j \mid$

$j = 1, \ldots, n\}$ and $C := A \setminus \bigcap_{j=1}^{n} Us_j$. Then one verifies $S \in Q(\mathcal{F}, C) \subset P(S, V, A)$.

Secondly, let $S \in Q(\mathcal{F}, C)$ be given. We have to construct a neighborhood V of the identity and a compact set A in G such that $P(S, V, A) \subset Q(\mathcal{F}, C)$. For each $F \in \mathcal{F}$ choose a point $s_F \in F \cap S$. Then choose a symmetric neighborhood V of the identity such that $S \cap VC = \emptyset$ and $Vs_F \subset F$ for all $F \in \mathcal{F}$. If $A := C \cup \{s_F \mid F \in \mathcal{F}\}$ then one checks that $P(S, V, A) \subset Q(\mathcal{F}, C)$. ∎

The subset $\Sigma(G)$ of $\Phi(G)$ consisting of all closed subgroups of G is closed (see [7]), hence $\Sigma(G)$ is a compact space. Next we consider the larger set $\Lambda(G)$ of all left cosets, i.e., $\Lambda(G)$ is the image of the obvious map $G \times \Sigma(G) \to \Phi(G)$. This map defines an equivalence relation $\sim$ on $G \times \Sigma(G)$.

PROPOSITION 1.4. *The subset $\Lambda(G) \cup \{\emptyset\}$ of $\Phi(G)$ is closed, hence compact, and the space $\Lambda(G)$ with the relativized topology is locally compact. The above equivalence relation $\sim$ on $G \times \Sigma(G)$ is open, i.e., the saturations of open sets are again open. The natural map from the space of equivalence classes in $G \times \Sigma(G)$ onto $\Lambda(G)$ is a homeomorphism.*

Proof. Let $(\lambda_i)_{i \in I}$ be a net in $\Lambda(G) \cup \{\emptyset\}$ which converges to a point $\lambda \in \Phi(G)$. We have to show that λ belongs to $\Lambda(G) \cup \{\emptyset\}$; of course, we may assume that $\lambda \neq \emptyset$. Let $x \in \lambda$. Passing to a subnet if necessary, and using the same letters I and λ_i again, by Lemma 1.1 we find $x_i \in \lambda_i$, $i \in I$, with $x = \lim x_i$. The λ_i define a net of subgroups $H_i := \{y \in G \mid \lambda_i y = \lambda_i\}$, $i \in I$, and without loss of generality we may assume that this net converges to $H \in \Sigma(G)$. Considering G as a G-transformation group for the left translations, Proposition 1.2 shows that G acts continuously by left translations on $\Phi(G)$. In particular, the convergence of (x_i) and of (H_i) implies that $\lambda_i = x_i H_i$ converges to xH, whence $\lambda = xH$ is a left coset.

To see the openness of the equivalence relation on $G \times \Sigma(G)$ we prove the (equivalent) "dual" version, namely that the closure of any saturated subset A of $G \times \Sigma(G)$ is again saturated. So, let $(g_i, H_i)_{i \in I}$ be a net in A which converges to $(g, H) \in G \times \Sigma(G)$, and let (g', H) be equivalent to (g, H), i.e., $g' = gh$ with $h \in H$. Without loss of generality we may assume by Lemma 1.1 that there exist $h_i \in H_i$ such that $\lim h_i = h$. The points $(g_i h_i, H_i)$, $i \in I$, are in A because A is saturated, and the net $(g_i h_i, H_i)_{i \in I}$ converges to (g', H). We have seen that $\overline{A}$ is saturated.

The homeomorphy of $(G \times \Sigma(G))/\!\sim$ with $\Lambda(G)$ follows from the continuity of $G \times \Sigma(G) \to \Lambda(G)$ by a similar reasoning as above, where we started with a convergent net (λ_i) in $\Lambda(G)$ and constructed x_i and H_i. ∎

Remark 1.5. Since the locally compact group G acts continuously on the whole (compact) space $\Phi(G)$, it acts in particular continuously on the locally compact space $\Lambda(G)$ (by left translations). Each closed subgroup of G can be realized as the stabilizer of some point in $\Lambda(G)$.

Moreover, if another locally compact group M acts continuously and homomorphically on G then M acts continuously on $\Sigma(G)$. In particular, G acts by conjugation continuously on $\Sigma(G)$. This fact was used in [10]. ∎

It might be illuminating to see a simple example of a space $\Lambda(G)$. The closed subgroups of $G = \mathbb{R}$ can be parametrized by $\overline{\mathbb{R}}_+ = [0, \infty]$: to $0 < x < \infty$ corresponds the subgroup $x\mathbb{Z}$, to $x = 0$ the whole group $\mathbb{R}$ and to $x = \infty$ the trivial group. This map is a homeomorphism between $\Sigma(\mathbb{R})$ and $\overline{\mathbb{R}}_+$. The space $\Lambda(\mathbb{R})$ is homeomorphic to $(\mathbb{R} \times \overline{\mathbb{R}}_+)/\sim$, where (t, x) and (t', x') are called equivalent if either $0 < x = x' < \infty$ and $\frac{1}{x}(t - t') \in \mathbb{Z}$, or $x = x' = 0$, or $x = x' = \infty$ and $t = t'$.

As was shown in [3] and [8] there exists a continuous choice of Haar measures on the various closed subgroups of a locally compact group G. Actually, Bourbaki first topologized $\Sigma(G)$ by viewing it as the quotient $\mathfrak{M}/\mathbb{R}_+$, where $\mathfrak{M}$ is the set of all Haar measures on closed subgroups endowed with the weak convergence w.r.t. $\mathcal{C}_c(G)$, and then compared with the topology described above.

Proposition 1.6 (Glimm [8, appendix]). *For each locally compact group G there exists a choice of left Haar measures ν_H on the closed subgroups H of G such that for all $f \in \mathcal{C}_c(G)$ the function $H \mapsto \int_H f(x)\,d\nu_H(x)$ is continuous on $\Sigma(G)$. Moreover, for each such choice and each compact subset C of G there is a constant $E = E_C$ such that $\nu_H(xC \cap H) \le E$ for all $x \in G$ and all $H \in \Sigma(G)$. Therefore, the function $(f, H) \mapsto \int_H f(x)\,d\nu_H(x)$ on $\mathcal{C}_c(G) \times \Sigma(G)$ is continuous in both variables if $\mathcal{C}_c(G)$ is endowed with the usual inductive limit topology. Furthermore, the choice of ν_H is essentially unique: Two choices differ by a positive factor, which is continuous on $\Sigma(G)$ and hence in particular bounded and bounded away from zero.*

Glimm obtains the desired normalization of the Haar measures as follows. Fix $f_0 \in \mathcal{C}_c(G)$ with $f_0 \ge 0$ and $f_0(e) > 0$ and demand that

$$\int_H f_0(x)\,d\nu_H(x) = 1.$$

Moreover, Glimm shows that with this choice of ν_H one has $\nu_H(C \cap H) \le E_C$ for all $H \in \Sigma(G)$ and all compact subsets C. The above stated uniform version can be proved along the same lines.

From the essential uniqueness of the ν_H's, which was observed by Fell [7], it follows that the assertions remain true for other continuous choices. ∎

Motivated by the work of Glimm, Fell has associated with each locally compact group G the so-called *subgroup algebra* $\mathcal{A}_s(G)$ which is defined as follows. Consider the closed subspace Y of $G \times \Sigma(G)$ consisting of all pairs (x,H) such that $x \in H$. If Haar measures ν_H on $H \in \Sigma(G)$ are selected according to 1.6 then define a norm $\|\ \|_{\mathrm{s}}$, a multiplication and an involution on $\mathcal{C}_{\mathrm{c}}(Y)$ by

$$\|f\|_{\mathrm{s}} = \sup_{H\in\Sigma(G)} \int_H |f(x,H)|\, d\nu_H(x),$$

$$(f*g)(x,H) = \int_H f(xy,H)g(y^{-1},H)\, d\nu_H(y),$$

$$f^*(x,H) = \Delta_H(x)^{-1} f(x^{-1},H)^-,$$

where Δ_H denotes the modular function of H. In that way $\mathcal{C}_{\mathrm{c}}(Y)$ becomes an involutive normed algebra, and $\mathcal{A}_{\mathrm{s}}(G)$ denotes its completion. The next proposition says among other things that $L^1(H)$ is a quotient of $\mathcal{A}_{\mathrm{s}}(G)$ for each $H \in \Sigma(G)$.

PROPOSITION 1.7. *The algebra $\mathcal{A}_{\mathrm{s}}(G)$ has a two-sided bounded approximate identity which may be chosen in $\mathcal{C}_{\mathrm{c}}(Y)$. For each $H \in \Sigma(G)$ and each $f \in \mathcal{C}_{\mathrm{c}}(Y)$ define $R_H f \in \mathcal{C}_{\mathrm{c}}(H)$ by $(R_H f)(x) = f(x,H)$. The map R_H extends to a bounded $*$-morphism from $\mathcal{A}_{\mathrm{s}}(G)$ onto the involutive Banach algebra $L^1(H)$. Furthermore, the algebra $\mathcal{C}(\Sigma(G))$ acts in an obvious manner on $\mathcal{C}_{\mathrm{c}}(Y)$, and this action extends to $\mathcal{A}_{\mathrm{s}}(G)$. The kernel of the extended map $R_H : \mathcal{A}_{\mathrm{s}}(G) \to L^1(H)$ is just the closure of the span of $\mathcal{A}_{\mathrm{s}}(G)\{g \in \mathcal{C}(\Sigma(G)) \mid g(H) = 0\}$.*

Proof. An approximate identity can be constructed by the usual procedure. For each neighborhood U of the unit in G choose a function $\psi = \psi_U \in \mathcal{C}_{\mathrm{c}}(G)$ such that $\psi \ge 0$, $\psi(e) > 0$, and $\operatorname{supp}\psi \subset U$. Then define the continuous function I on $\Sigma(G)$ by $I(K) = \int_K \psi(x)\, d\nu_K(x)$, and $h = h_U$ on Y by $h(y,K) = \psi(y)I(K)^{-1}$. The family (h_U) is an approximate identity.

Clearly, R_H induces a bounded $*$-morphism from $\mathcal{A}_{\mathrm{s}}(G)$ into $L^1(H)$ for each $H \in \Sigma(G)$. To see the surjectivity it is enough to show that there is an $\varepsilon > 0$ such that for each $\varphi \in \mathcal{C}_{\mathrm{c}}(H)$ there exists $f \in \mathcal{C}_{\mathrm{c}}(Y)$ with $R_H f = \varphi$ and $\|f\|_{\mathcal{A}_{\mathrm{s}}(G)} \le (1+\varepsilon)\|\varphi\|_1$. Actually, we shall prove this claim for each ε. Choose any extension $\widetilde{\varphi} \in \mathcal{C}_{\mathrm{c}}(G)$ of φ. Without loss of generality we may assume that φ is different from zero. From the continuity of the family ν_K, $K \in \Sigma(G)$, follows the existence of a neighborhood V of H in $\Sigma(G)$ such that

$$\Big|\int_K |\widetilde{\varphi}(x)|\, d\nu_K(x) - \int_H |\varphi(x)|\, d\nu_H(x)\Big| \le \varepsilon \int_H |\varphi(x)|\, d\nu_H(x)$$

for all $K \in V$. Then choose $g \in \mathcal{C}_{\mathrm{c}}(\Sigma(G))$ such that $0 \le g \le 1$, $g(H) = 1$, and $g = 0$ outside V. The function $f \in \mathcal{C}_{\mathrm{c}}(Y)$ given by $f(y,K) = \widetilde{\varphi}(y)g(K)$ has the required properties.

Checking the final assertions is routine. ∎

2. Abstract abelian harmonic analysis, $\Sigma(G)$ and $\mathcal{A}_{\rm s}(G)$. For locally compact abelian groups G we first consider the canonical map $\Sigma(G) \ni H \mapsto H^{\perp} \in \Sigma(G^{\wedge})$, where $H^{\perp}$ denotes the annihilator of H in the Pontryagin dual $G^{\wedge}$.

PROPOSITION 2.1 (Williams [21]). *For each locally compact abelian group G the map $H \mapsto H^{\perp}$ is a homeomorphism from $\Sigma(G)$ onto $\Sigma(G^{\wedge})$.*

Proof. While Williams used L^2-spaces of G (and of quotients) our proof will be based on the duality between $L^1(G)$ and $L^\infty(G)$. This point of view is more in the spirit of this paper. Later we shall give still another proof using Poisson's summation formula and the existence of certain functions.

Let $(H_i)_{i\in I}$ be a convergent net in $\Sigma(G)$ with limit H_∞. As $\Sigma(G^{\wedge})$ is compact it suffices to show that each convergent subnet of $(H_i^{\perp})_{i\in I}$ converges to $H_\infty^{\perp}$. Hence we may assume from the beginning that $(H_i^{\perp})_{i\in I}$ converges to Δ, say. We have to prove that $\Delta = H_\infty^{\perp}$. As pointed out by Williams, the inclusion $\Delta \subset H_\infty^{\perp}$ is easy, indeed it readily follows from Lemma 1.1.

For the reverse inclusion we need another type of argument. According to Proposition 1.6 let Haar measures on the various subgroups of G be selected. In particular, this gives Haar measures ν_i on H_i for $i \in I \cup \{\infty\}$. For each $\varphi \in \mathcal{C}_{\rm c}(G)$ and each $i \in I \cup \{\infty\}$ define $T_i\varphi : G \to \mathbb{C}$ by $(T_i\varphi)(x) = \int_{H_i} \varphi(xh)\, d\nu_i(h)$. The $T_i\varphi$ are elements in $\mathcal{C}_{\rm c}(G/H_i)$, but we view them as members of $L^\infty(G/H_i) \subset L^\infty(G)$.

The norms $\|T_i\varphi\|_\infty$ are uniformly bounded, actually one has $\|T_i\varphi\|_\infty \le \|\varphi\|_\infty E_C$ where $C = \mathrm{supp}(\varphi)$ and E_C is as in 1.6. We claim that $(T_i\varphi)_{i\in I}$ converges to $T_\infty\varphi$ in the weak topology of $L^\infty(G)$. For $f \in \mathcal{C}_{\rm c}(G)$ and $i \in I \cup \{\infty\}$ one has

$$\langle T_i\varphi, f\rangle := \int\limits_G (T_i\varphi)(x) f(x)\, d\nu_G(x) = \int\limits_G \Big(\int\limits_{H_i} \varphi(xh)\, d\nu_i(h)\Big) f(x)\, d\nu_G(x)$$
$$= \int\limits_{H_i} \Big(\int\limits_G \varphi(xh) f(x)\, d\nu_G(x)\Big)\, d\nu_i(h) = \int\limits_{H_i} (\varphi * f^{\vee})(h)\, d\nu_i(h),$$

where $f^{\vee}$ is defined by $f^{\vee}(x) = f(x^{-1})$. Since $\varphi * f^{\vee} \in \mathcal{C}_{\rm c}(G)$ the continuity of the choice of the Haar measures gives

$$\lim_{i\in I} \langle T_i\varphi, f\rangle = \langle T_\infty\varphi, f\rangle.$$

As $\mathcal{C}_{\rm c}(G)$ is dense in $L^1(G)$ and as the norms $\|T_i\varphi\|_\infty$ are uniformly bounded it follows that indeed $(T_i\varphi)$ converges weakly to $T_\infty\varphi$.

Because $T_\infty(\mathcal{C}_{\rm c}(G))$ is weakly dense in $L^\infty(G/H_\infty)$ it follows that $L^\infty(G/H_\infty)$ is contained in the weak closure of $\bigcup_{i\ge i_0} L^\infty(G/H_i)$ for each $i_0 \in I$. Using the duality between $L^1(G)$ and $L^\infty(G)$ and the fact that the span of $H_i^{\perp}$, $i \in I \cup \{\infty\}$, is weakly dense in $L^\infty(G/H_i)$, one concludes that $k(H_\infty^\perp)$ contains $\bigcap_{i\ge i_0} k(H_i^\perp) = k((\bigcup_{i\ge i_0} H_i^\perp)^-)$, where k denotes the kernel in the "hull kernel sense". In view of the regularity of $L^1(G)$ this implies that $H_\infty^\perp$ is contained in $(\bigcup_{i\ge i_0} H_i^\perp)^-$. Hence we know that $H_\infty^\perp$ is contained in $\bigcap_{i_0\in I}(\bigcup_{i\ge i_0} H_i^\perp)^-$, which coincides with Δ by 1.1. Since we already observed that $\Delta \subset H_\infty^\perp$ the proof is finished. ∎

In order to prove the "continuity" of Poisson's summation formula and the regularity of $\mathcal{A}_{\mathrm{s}}(G)$ we need the following lemma.

LEMMA 2.2. *Let G be a locally compact abelian group, let $\chi_0 \in G^\wedge$, and let U be a neighborhood of χ_0 in $G^\wedge$. Then there exist a continuous function f on G, a compact set B in G and a sequence $x_1, x_2, \ldots$ in G such that*

(i) $\mathrm{supp}(f) \subset \bigcup_{n=1}^\infty x_n B$,

(ii) $\sum_{n=1}^\infty \varepsilon_n < \infty$, *where* $\varepsilon_n = \sup_{x\in x_n B} |f(x)|$ (*and hence* $f \in L^1(G)$),

(iii) *the Fourier transform* $\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\,dx$ *is nonnegative everywhere*, $\widehat{f}(\chi_0) \ne 0$,

(iv) *the support of* $\widehat{f}$ *is compact and contained in* U.

Proof. Clearly the lemma implies the regularity of $L^1(G)$, and our proof is a slight extension of the standard proof of this fact (compare e.g. [4]).

Without loss of generality we may assume that $\chi_0 = 1$. By the structure theory of locally compact abelian groups the group G can be identified with $\mathbb{R}^r \times H$, where H contains a compact open subgroup K. Accordingly, $G^\wedge$ splits into $\mathbb{R}^r \times H^\wedge$, and $K^\perp \cap H^\wedge$ is a compact open subgroup of $H^\wedge$. We choose (relatively compact) open symmetric neighborhoods V and W of the identity in $\mathbb{R}^r$ and in $K^\perp \cap H^\wedge$, respectively, such that $V^2 \times W^2 \subset U$. Then we choose nonnegative continuous functions α and β on $\mathbb{R}^r$ and on $H^\wedge$, respectively, which are nonzero at the identity and supported by V and W, respectively. In addition, we require that α is a Schwartz function. Then $\alpha \otimes \beta$ is in $L^2(G^\wedge)$, hence the inverse Fourier transform $g = \widetilde{a} \otimes \widetilde{\beta}$ of $\alpha \otimes \beta$, where

$$\widetilde{\alpha}(t) = \int_{\mathbb{R}^r} e^{itx}\alpha(x)\,dx \quad \text{and} \quad \widetilde{\beta}(h) = \int_{\hat{H}} \beta(\chi)\chi(h)\,d\chi,$$

is a continuous L^2-function on $G = \mathbb{R}^r \times H$.

The pointwise product $f = g\overline{g}$ is a continuous L^1-function on G whose Fourier transform is equal to $(\alpha * \alpha^*) \otimes (\beta * \beta^*)$, which is supported by $V^2 \times W^2$, nonnegative and nonzero at the origin.

The function $\widetilde{\beta} \in L^2(H)$ is constant on K-cosets because $\mathrm{supp}\,\beta \subset H^\wedge \cap K^\perp$. Hence there is a sequence (h_j) of elements in H and a sequence (η_j) of nonnegative real numbers such that $\mathrm{supp}(\widetilde{\beta}) \subset \bigcup_{j=1}^\infty h_j K$, $|\widetilde{\beta}| \le \eta_j$ on $h_j K$, and $\sum_{j=1}^\infty \eta_j^2 < \infty$. Let Q be the closed unit cube in $\mathbb{R}^r$. Since $\widetilde{\alpha}$ is a Schwartz function, for an appropriate sequence (t_l) of points in $\mathbb{R}^r$ the whole space $\mathbb{R}^r$ is covered by $t_l + Q$, and $|\widetilde{\alpha}|$ is bounded on $t_l + Q$ by ϱ_l with $\sum_{l=1}^{\infty} \varrho_l^2 < \infty$. Note that here $\mathbb{R}^r$ is written additively while other abelian groups are written multiplicatively.

Then put $B = Q \times K$, and enumerate the points $(t_l, h_j) \in G$ somehow to obtain the asserted sequence (x_n). ■

THEOREM 2.3. *Let G be a locally compact abelian group, and let ν_H, $H \in \Sigma(G)$, be a continuous selection of Haar measures in the sense of* 1.6. *Normalize the Haar measures μ_Δ on the various subgroups $\Delta \in \Sigma(G^\wedge)$ so that Poisson's summation formula*

$$\int_H f(x)\,d\nu_H(x) = \int_{H^\perp} \widehat{f}(\chi)\,d\mu_{H^\perp}(\chi)$$

holds for all, say, $f \in \mathcal{C}_c(G) * \mathcal{C}_c(G)$ (compare [19, p. 120] and [4, p. 127]). *Then (μ_Δ) is a continuous selection in the sense of* 1.6.

Proof. Fix a function f_0 on G with the properties (i)–(iv) of 2.2 corresponding to $\chi_0 = 1 \in G^\wedge$ and an arbitrary U. We shall use the notations (x_n), (ε_n), B in the meaning of 2.2. As f_0 is a continuous L^1-function Poisson's formula (see [19, p. 122])

$$I(H) := \int_H f_0(x)\,d\nu_H(x) = \int_{H^\perp} \widehat{f_0}(\chi)\,d\mu_{H^\perp}(\chi)$$

holds true for all $H \in \Sigma(G)$. We next claim that I is a continuous function on $\Sigma(G)$. To this end, for any (large) N choose a cut-off function $\varphi_N \in \mathcal{C}_c(G)$ such that $0 \le \varphi_N \le 1$ and $\varphi_N = 1$ on $\bigcup_{n=1}^N x_n B$. From the properties of f_0 we conclude that for all $H \in \Sigma(G)$,

$$\begin{aligned}\Big|I(H) - \int_H f_0(x)\varphi_N(x)\,d\nu_H(x)\Big| &\le \sum_{n=1}^{\infty} \int_{x_n B \cap H} |f_0(x)|(1-\varphi_N(x))\,d\nu_H(x)\\ &\le \sum_{n=N+1}^{\infty} \varepsilon_n \nu_H(x_n B \cap H) \le E_B \sum_{n=N+1}^{\infty} \varepsilon_n\end{aligned}$$

in view of 1.6. Therefore, for a given $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\Big|I(H) - \int_H f_0(x)\varphi_N(x)\,d\nu_H(x)\Big| \le \varepsilon$$

for all $H \in \Sigma(G)$ and $N \ge N_\varepsilon$. As (ν_K) is a continuous choice, for a given $H \in \Sigma(G)$ there is a neighborhood V of H in $\Sigma(G)$ such that

$$\Big|\int_H f_0(x)\varphi_{N_\varepsilon}(x)\,d\nu_H(x) - \int_K f_0(x)\varphi_{N_\varepsilon}(x)\,d\nu_K(x)\Big| \le \varepsilon$$

for all $K \in V$. Then clearly

$$|I(H) - I(K)| \le 3\varepsilon \quad \text{for all } K \in V.$$

If according to Glimm the Haar measures $d\varrho_\Delta$ on Δ, $\Delta \in \Sigma(G^\wedge)$, are normalized by

$$\int_\Delta \widehat{f}_0(\chi)\, d\varrho_\Delta(\chi) = 1$$

then (ϱ_Δ) is a continuous selection. But $d\mu_\Delta = I(\Delta^\perp) d\varrho_\Delta$, hence we are done if we use the fact that $\Delta \mapsto \Delta^\perp$ is continuous (see 2.1).

But from the present considerations one can also very easily deduce that $\Sigma(G)$ and $\Sigma(G^\wedge)$ are homeomorphic: Suppose that the net $(H_i)_{i\in I}$ in $\Sigma(G)$ converges to $H_\infty \in \Sigma(G)$ and that $(H_i^\perp)$ in $\Sigma(G^\wedge)$ converges to $\Delta \in \Sigma(G^\wedge)$. We claim $\Delta = H_\infty^\perp$. If $H_\infty^\perp$ is not contained in Δ (this is the more challenging case as was explained in the proof of 2.1) choose $\chi_0 \in H_\infty^\perp$, $\chi_0 \notin \Delta$, and a neighborhood U of χ_0 in $G^\wedge$ with $U \cap \Delta = \emptyset$. To χ_0 and U choose a function g on G as in 2.2. As above the net $J(H_i) := \int_{H_i} g(x)\, d\nu_{H_i}$, $i \in I$, converges in $\mathbb{R}$ to $J(H_\infty) := \int_{H_\infty} g(x)\, d\nu_{H_\infty}(x) = \int_{H_\infty^\perp} \widehat{g}(x)\, d\mu_{H_\infty^\perp}(x) > 0$ by Poisson's formula. On the other hand, again by Poisson's formula, one has

$$J(H_i) = \int_{H_i^\perp} \widehat{g}(x)\, d\mu_{H_i^\perp}(x) = I(H_i) \int_{H_i^\perp} \widehat{g}(x)\, d\varrho_{H_i^\perp}(x).$$

As $I(H_i)$ stays bounded and as $\int_{H_i^\perp} \widehat{g}(x)\, d\varrho_{H_i^\perp}(x)$ converges by Glimm's result to $\int_\Delta \widehat{g}(x)\, d\varrho_\Delta(x) = 0$ we conclude that $J(H_i)$ converges to zero, a contradiction. ■

For illustration let us consider the space $\Sigma(\mathbb{R}^2)$. The set $\Sigma(\mathbb{R}^2)$ decomposes into six $GL_2(\mathbb{R})$-orbits, namely into the two one-point sets $\Sigma_{0,0}$ and $\Sigma_{2,0}$, consisting of the trivial and the whole subgroup $\mathbb{R}^2$, respectively, $\Sigma_{1,0} := \{\mathbb{R}b \mid b \in \mathbb{R}^2,\ b \neq 0\}$, $\Sigma_{0,1} := \{\mathbb{Z}a \mid a \in \mathbb{R}^2,\ a \neq 0\}$, $\Sigma_{1,1} = \{\mathbb{Z}a + \mathbb{R}b \mid a, b \in \mathbb{R}^2 \text{ are linearly independent}\}$, and the set $\Sigma_{0,2}$ of all lattices in $\mathbb{R}^2$. The sets $\Sigma_{0,0}$, $\Sigma_{2,0}$ and $\Sigma_{1,0}$ are closed in $\Sigma(\mathbb{R}^2)$, the latter being homeomorphic to the real projective line. The closure of $\Sigma_{0,1}$ is $\Sigma_{0,0} \cup \Sigma_{0,1} \cup \Sigma_{1,0}$. Actually, a given net $a_j\mathbb{Z}$ in $\Sigma_{0,1}$ converges to $\{0\}$ iff $\lim_j |a_j| = \infty$, it converges to $a\mathbb{Z}$, $a \neq 0$, iff $\lim_j a_j = a$ after possibly changing the signs of the a_j's, and it converges to $b\mathbb{R} \in \Sigma_{1,0}$, $|b| = 1$, iff $\lim_j a_j = 0$ and $\lim_j a_j/|a_j| = b$ after a possible change of signs. In all other cases the net $a_j\mathbb{Z}$ does not converge. By duality (the set $\Sigma_{0,1}$ is mapped onto $\Sigma_{1,1}(\mathbb{R}^{2\wedge})$) the closure of $\Sigma_{1,1}(\mathbb{R}^2)$ is equal to $\Sigma_{2,0} \cup \Sigma_{1,1} \cup \Sigma_{1,0}$, and one has a similar description of convergence of nets in $\Sigma_{1,1}(\mathbb{R}^2)$. It follows that $\Sigma_{0,2}$ is open (and dense) in $\Sigma(\mathbb{R}^2)$. In particular, $\Sigma_{0,2}$ is locally closed as are all six $GL_2(\mathbb{R})$-orbits, and hence all are homeomorphic to homogeneous spaces (compare also [3, p. 187]). We do not consider the more subtle question which nets of lattices converge to boundary points.

The next theorem contains some basic properties of the commutative Banach algebra $\mathcal{A}_s(G)$.

THEOREM 2.4. *For each locally compact abelian group G the involutive Banach algebra $\mathcal{A}_s(G)$ is symmetric and regular. Its structure space $\mathcal{A}_s(G)^\wedge$ is homeomorphic to the coset space $\Lambda(G^\wedge)$, which by* 1.4 *is homeomorphic to a certain quotient of $G^\wedge \times \Sigma(G^\wedge)$ and, by* 2.1, *to a quotient of $G^\wedge \times \Sigma(G)$ as well.*

Proof. To prove symmetry we must show that each nonzero (bounded) multiplicative linear functional η on $\mathcal{A}_s(G)$ is hermitean. At the same time we shall determine the set $\mathcal{A}_s(G)^\wedge$. As $\mathcal{C}(\Sigma(G))$ acts on $\mathcal{C}_c(Y)$ (and on $\mathcal{A}_s(G)$) we find a multiplicative linear functional η' on $\mathcal{C}(\Sigma(G))$ such that

$$\eta(\varphi f) = \eta'(\varphi)\eta(f)$$

for all $\varphi \in \mathcal{C}(\Sigma(G))$ and $f \in \mathcal{A}_s(G)$. The multiplicative linear functionals of $\mathcal{C}(\Sigma(G))$ are known: there is a unique $H \in \Sigma(G)$ such that $\eta'(\varphi) = \varphi(H)$. Then using 1.7 we conclude that η factors through the (extended) morphism $R_H : \mathcal{A}_s(G) \to L^1(H)$, and yields a multiplicative linear functional on $L^1(H)$. As those are known, there exists $\chi \in H^\wedge$ such that

$$\eta(f) = \int\limits_H \chi(x) f(x, H)\, d\nu_H(x)$$

for $f \in \mathcal{C}_c(Y)$. Clearly η is hermitean. On the other hand, each such pair (H, χ) gives rise to a multiplicative linear functional of $\mathcal{A}_s(G)$. Moreover, this set of pairs can be identified with $\Lambda(G^\wedge)$: to $\gamma\Delta \in \Lambda(G^\wedge)$ corresponds the pair $(\gamma|_{\Delta^\perp}, \Delta^\perp)$.

Next we show that $\mathcal{A}_s(G)^\wedge$, according to Gelfand equipped with the weak topology, is indeed homeomorphic to $\Lambda(G^\wedge)$. More precisely, we show that the canonical map from the compact space $\mathcal{A}_s(G)^\wedge \cup \{0\}$ into $\Lambda(G^\wedge) \cup \{\emptyset\}$ is continuous. Let $(\eta_i)_{i\in I}$ be a convergent net in $\mathcal{A}_s(G)^\wedge \cup \{0\}$ with limit η_∞ and denote by λ_i, $i \in I$, the corresponding points in $\Lambda(G^\wedge) \cup \{\emptyset\}$. Without loss of generality we may assume that $\eta_i \neq 0$ for all $i \in I$, i.e., $\lambda_i = \gamma_i H_i^\perp$ for some $\gamma_i \in G^\wedge$, $H_i \in \Sigma(G)$, and that $(\lambda_i)_{i\in I}$ converges to, say, $\lambda_\infty \in \Lambda(G^\wedge) \cup \{\emptyset\}$. We have to show that η_∞ corresponds to λ_∞. To this end, we distinguish two cases.

Case 1: $\eta_\infty \neq 0$, i.e., η_∞ corresponds to a point $\chi K^\perp \in \Lambda(G^\wedge)$, $K \in \Sigma(G)$. For each $\varphi \in \mathcal{C}_c(G)$ and each $a \in G$ define $\varphi^a \in \mathcal{C}_c(Y)$ by $\varphi^a(x, H) = \varphi(ax)$, and put $\varphi_i(a) := \eta_i(\varphi^a)$ for $i \in I$ as well as $\varphi_\infty(a) = \eta_\infty(\varphi^a)$, i.e., $\varphi_i(a) = \int_{H_i} \varphi(ah)\gamma_i(h)\, d\nu_i(h)$ and $\varphi_\infty(a) = \int_K \varphi(ak)\chi(k)\, d\nu(k)$, where $d\nu_i$ and $d\nu$ denote the chosen Haar measures on H_i and K, respectively. By assumption the functions φ_i on G converge pointwise to φ_∞. Once more for all $a\in G$ and all $i\in I\cup\{\infty\}$ one has $|\varphi_i(a)|\le\|\varphi\|_\infty E_C$, where $C=\mathrm{supp}(\varphi)$.

Moreover, for all $f\in\mathcal{C}_c(G)$ and all $i\in I$,

$$\langle\varphi_i,f\rangle=\int\limits_G\varphi_i(x)f(x)\,d\nu_G(x)=\int\limits_G\int\limits_{H_i}\varphi(xh)\gamma_i(h)\,d\nu_i(h)f(x)\,d\nu_G(x)$$
$$=\int\limits_{H_i}(\varphi*f^\vee)(h)\gamma_i(h)\,d\nu_i(h)=\eta_i(\varphi*f^\vee\otimes 1|_Y),$$

and similarly $\langle\varphi_\infty,f\rangle=\eta_\infty(\varphi*f\otimes 1|_Y)$. Hence $\langle\varphi_i,f\rangle$ converges to $\langle\varphi_\infty,f\rangle$. As $\|\varphi_i\|_\infty$ is bounded in i, it follows that the net (φ_i) in $L^\infty(G)$ converges weakly to φ_∞. Then also the net $(\check\varphi_i)$ converges to $\check\varphi_\infty$ for all $\varphi\in\mathcal{C}_c(G)$. The functions $\check\varphi_i$ are contained in $L^\infty(G,H_i,\gamma_i)$ consisting of all functions $\psi\in L^\infty(G)$ with $\psi(xh)=\gamma_i(h)\psi(x)$ for $h\in H_i$. And $\check\varphi_\infty$ is contained in $L^\infty(G,K,\chi)$ which is defined accordingly. Using the regularity of $L^1(G)$, the fact that $L^\infty(G,H_i,\gamma_i)$ is "weakly spanned" by the set $\lambda_i=\gamma_iH_i^\perp$, and the fact that $\{\check\varphi_\infty\mid\varphi\in\mathcal{C}_c(G)\}$ is weakly dense in $L^\infty(G,K,\chi)$ we conclude as in the proof of 2.1 that for each $i_0\in I$ the coset $\chi K^\perp$ is contained in the closure of $\bigcup_{i\ge i_0}\lambda_i$. Hence $\chi K^\perp$ is contained in $\bigcap_{i_0\in I}(\bigcup_{i\ge i_0}\lambda_i)^-=\lambda_\infty$.

In particular, λ_∞ is not empty. Passing to a subnet and changing the γ_i inside λ_i if necessary, we may assume that (γ_i) converges to χ and that (H_i) converges to H_∞. Then $\lambda_\infty=\chi H_\infty^\perp$ by 1.4 and 2.1, and $\chi K^\perp\subset\chi H_\infty^\perp$.

As we observed above, for each $\varphi\in\mathcal{C}_c(G)$ the numbers $\varphi_i(e)=\int_{H_i}\varphi(h)\gamma_i(h)\,d\nu_i(h)$ converge to $\int_K\varphi(k)\chi(k)\,d\nu(k)$. On the other hand, as (γ_i) converges to χ uniformly on compacta and (H_i) converges to H_∞, from 1.6 it follows that the integrals $\int_{H_i}\varphi(h)\gamma_i(h)\,d\nu_i(h)$ converge to $\int_{H_\infty}\varphi(h)\chi(h)\,d\nu_\infty(h)$, where $d\nu_\infty$ is the chosen Haar measure on H_∞. Hence

$$\int\limits_K\varphi(k)\chi(k)\,d\nu(k)=\int\limits_{H_\infty}\varphi(h)\chi(h)\,d\nu_\infty(h)$$

for all $\varphi\in\mathcal{C}_c(G)$. Clearly this implies $H_\infty=K$, whence $\chi K^\perp=\chi H_\infty^\perp=\lambda_\infty$.

Case 2: $\eta_\infty=0$. We have to show $\lambda_\infty=\emptyset$. Suppose to the contrary that $\lambda_\infty\ne\emptyset$. Then we may assume that the $\gamma_i\in\lambda_i$ converge to $\gamma_\infty\in\lambda_\infty$, that (H_i) converges to H_∞, and that $\lambda_\infty=\gamma_\infty H_\infty^\perp$. Now for each $\varphi\in\mathcal{C}_c(G)$ the integrals $\varphi_i(e)=\int_{H_i}\varphi(h)\gamma_i(h)\,d\nu_i(h)=\eta_i(\varphi\otimes 1|_Y)$ converge to zero. On the other hand, these integrals converge to $\int_{H_\infty}\varphi(h)\gamma_\infty(h)\,d\nu_\infty(h)$. Hence the latter integral is zero for all $\varphi\in\mathcal{C}_c(G)$, which is impossible.

To prove the regularity of $\mathcal{A}_s(G)$ take a point η_0 in $\mathcal{A}_s(G)^\wedge$ and a neighborhood V of η_0. We have to show the existence of an element $a\in\mathcal{A}_s(G)$ such that $\eta_0(a)\ne 0$, but $\eta(a)=0$ for $\eta\notin V$. Since $G^\wedge$ acts on $\mathcal{A}_s(G)$ and on $\mathcal{A}_s(G)^\wedge\cong\Lambda(G^\wedge)$ we may assume that η_0 corresponds to a subgroup, say $H_0^\perp$, in $\Lambda(G^\wedge)$. Furthermore, we may assume that V corresponds to the subset $Q(\mathcal{F},C)\cap\Lambda(G^\wedge)$ of $\Lambda(G^\wedge)$, where $\mathcal{F}$ is a finite collection of open nonempty subsets in $G^\wedge$, and C is a compact subset of $G^\wedge$ (for the definition of $Q(\mathcal{F},C)$ see the introduction). Since $\mathcal{A}_{\rm s}(G)$ is an algebra and since $Q(\mathcal{F},C)=Q(\emptyset,C)\cap\bigcap_{F\in\mathcal{F}}Q(\{F\},\emptyset)$ it is good enough to solve our problem for each of the cases $Q(\emptyset,C)$ or $Q(\{F\},\emptyset)$ separately.

Case 1: Suppose that V corresponds to $Q(\{F\},\emptyset)\cap\Lambda(G^\wedge)$ for some nonempty open subset F of $G^\wedge$. Then $F\cap H_0^\perp$ is not empty; pick $\chi_0\in F\cap H_0^\perp$. Apply 2.2 to χ_0 and $U=F$ in order to obtain a function f on G with the properties stated there. In the following we shall use the notations B, x_n,ε_n as in 2.2. For each $N\in\mathbb{N}$ choose a cut-off function $\varphi_N\in\mathcal{C}_{\rm c}(G)$ with $0\le\varphi_N\le 1$ and $\varphi_N=1$ on $\bigcup_{n=1}^N x_nB$. Then define $a_N\in\mathcal{C}_{\rm c}(Y)\subset\mathcal{A}_{\rm s}(G)$ by $a_N(x,H)=\varphi_N(x^{-1})f(x^{-1})$. As in the proof of Theorem 2.3 one sees that the a_N form a Cauchy sequence in $\mathcal{A}_{\rm s}(G)$. Let $a:=\lim_{N\to\infty}a_N\in\mathcal{A}_{\rm s}(G)$.

If $\eta\in\mathcal{A}_{\rm s}(G)^\wedge$ corresponds to the coset $\chi H^\perp$ in $\Lambda(G^\wedge)$ then

$$\eta(a)=\lim_{N\to\infty}\int_H \chi(h)f(h^{-1})\varphi_N(h^{-1})\,d\nu_H(h)=\int_H \chi(h)^{-1}f(h)\,d\nu_H(h).$$

Poisson's summation formula yields

$$\eta(a)=\int_{H^\perp}\widehat{f}(\chi\gamma)\,d\mu_{H^\perp}(\gamma).$$

In particular, we have $\eta_0(a)=\int_{H_0^\perp}\widehat{f}(\gamma)\,d\mu_{H_0^\perp}(\gamma)>0$. But if $\eta\notin V$, i.e., if the corresponding coset $\chi H^\perp$ is disjoint from F then $\eta(a)=0$.

Case 2: Suppose that V corresponds to $Q(\emptyset,C)\cap\Lambda(G^\wedge)$ for some compact subset C of $G^\wedge$. Then $H_0^\perp\cap C=\emptyset$ and there exists a compact symmetric neighborhood U of the identity χ_0 in $G^\wedge$ such that $H_0^\perp\cap CU=\emptyset$. To χ_0 and U choose a function f on G according to 2.2. Moreover, choose a continuous function g on $\Sigma(G^\wedge)$ such that $g(H_0^\perp)\neq 0$, but $g(H^\perp)=0$ if $H^\perp\notin Q(\emptyset,CU)$, i.e., $H^\perp\cap CU\neq\emptyset$. If the cut-off functions φ_N are as above then define $a_N\in\mathcal{C}_{\rm c}(Y)\subset\mathcal{A}_{\rm s}(G)$ by

$$a_N(x,H)=g(H^\perp)\varphi_N(x^{-1})f(x^{-1}).$$

Observe that $\Sigma(G)\ni H\mapsto g(H^\perp)$ is continuous as $\Sigma(G)$ and $\Sigma(G^\wedge)$ are homeomorphic.

Again (a_N) converges to an element $a\in\mathcal{A}_{\rm s}(G)$. And if $\eta\in\mathcal{A}_{\rm s}(G)^\wedge$ corresponds to $\chi H^\perp\in\Lambda(G^\wedge)$ then

$$\eta(a)=g(H^\perp)\int_H\chi(h)^{-1}f(h)\,d\nu_H(h)=g(H^\perp)\int_{H^\perp}\widehat{f}(\chi\gamma)\,d\mu_{H^\perp}(\gamma).$$

Again $\eta_0(a)=g(H_0^\perp)\int_{H_0^\perp}\widehat{f}(\gamma)\,d\mu_{H_0^\perp}(\gamma)\neq 0$, but if $\eta\notin V$, i.e., $\chi H^\perp\cap C\neq\emptyset$, then $\eta(a)=0$: In case $H^\perp\cap CU\neq\emptyset$ the factor $g(H^\perp)$ vanishes, in case $H^\perp\cap CU=\emptyset$ the sets $U=U^{-1}$ and $\chi H^\perp$ are disjoint, hence the integral $\int_{H^\perp}\widehat{f}(\chi\gamma)\,d\mu_{H^\perp}(\gamma)$ vanishes. ∎

In the proof of the theorem the multiplicative linear functionals on $\mathcal{A}_s(G)$ were first parametrized by pairs (H,χ), $H\in\Sigma(G)$, $\chi\in H^\wedge$. Identifying the set of all such pairs, say $\mathfrak{X}$, in a canonical manner with $\Lambda(G^\wedge)$ we introduced a topology. This raises the question if the topology on $\mathfrak{X}$ can be described in a more internal fashion. We conclude this section by giving an answer to this question.

Let C be a compact subset of G, let $\mathcal{F}$ be a finite family of nonempty open sets in G, and let $\mathfrak{A}$ be a finite family of pairs (A,U), where A is a compact subset of G, and U is an open subset of the torus $\mathbb{T}$. For each such triple $C,\mathcal{F},\mathfrak{A}$ let $W(\mathcal{F},C,\mathfrak{A})$ be the set of all pairs (H,χ) in $\mathfrak{X}$ such that $H\in Q(\mathcal{F},C)$ and $\chi(H\cap A)\subset U$ for all $(A,U)\in\mathfrak{A}$. Clearly, the collection of all those $W(\mathcal{F},C,\mathfrak{A})$ is the basis of a topology on $\mathfrak{X}$. Henceforth we shall view $\mathfrak{X}$ as being topologized that way. Evidently, for each fixed $H\in\Sigma(G)$ the relative topology on $\{(H,\chi)\mid\chi\in H^\wedge\}\subset\mathfrak{X}$ coincides with the usual topology on the Pontryagin dual. Furthermore, $\mathfrak{X}$ is a Hausdorff space: Let (H_1,χ_1), (H_2,χ_2) be given. If $H_1\neq H_2$ then these points can be separated by means of the Hausdorff property of $\Sigma(G)$. If $H=H_1=H_2$, but $\chi_1\neq\chi_2$ then choose $x\in H$ with $\chi_1(x)\neq\chi_2(x)$, choose disjoint open neighborhoods U_1 and U_2 of $\chi_1(x)$ and $\chi_2(x)$, respectively, and choose a relatively compact open neighborhood F of x in G with closure A such that $\chi_j(A\cap H)\subset U_j$ for $j=1,2$. The two sets $W(\{F\},\emptyset,\{(A,U_j)\})$ are disjoint neighborhoods of (H,χ_1) and (H,χ_2), respectively.

THEOREM 2.5. *A net $(H_i,\chi_i)_{i\in I}$ converges in $\mathfrak{X}=\{(H,\chi)\mid H\in\Sigma(G),\ \chi\in H^\wedge\}$ to (H_∞,χ_∞) if and only if $(H_i)_{i\in I}$ converges to H_∞ in $\Sigma(G)$ and if for any subnet $(K_j,\omega_j)_{j\in J}$ of $(H_i,\chi_i)_{i\in I}$ and any convergent net $(x_j)_{j\in J}$ with $x_j\in K_j$ the net $(\omega_j(x_j))_{j\in J}$ converges to $\chi_\infty(x_\infty)$, where $x_\infty=\lim_{j\in J}x_j$. The canonical map $\Lambda(G^\wedge)\to\mathfrak{X}$, $\gamma\Delta\mapsto(\Delta^\perp,\gamma|_{\Delta^\perp})$, is a homeomorphism.*

Remark 2.6. In view of the characterization of limits in $\Lambda(G^\wedge)$ (compare Lemma 1.1), the theorem gives a criterion when a net of partially defined characters can be extended continuously to the whole of G. More precisely, let $(H_i,\chi_i)_{i\in I}$ be a convergent net in $\mathfrak{X}$ with limit (H_∞,χ_∞), and let an extension $\gamma_\infty\in G^\wedge$ of χ_∞ be given. Then there exists a subnet $(M_j,\zeta_j)_{j\in J}$ of $(H_i,\chi_i)_{i\in I}$ and a net $(\gamma_j)_{j\in J}$ in $G^\wedge$ such that $\gamma_j|_{M_j}=\zeta_j$ and $\lim_{j\in J}\gamma_j=\gamma_\infty$. This consideration also shows that Theorem 2.5 improves Theorem 2.1.

P r o o f o f T h e o r e m 2.5. Suppose that $(H_i,\chi_i)_{i\in I}$ is a convergent net in $\mathfrak{X}$ with limit (H_∞,χ_∞). The definition of the topology of $\mathfrak{X}$ clearly shows that (H_i) must converge to H_∞ in the topology of $\Sigma(G)$. Suppose further that the subnet $(K_j,\omega_j)_{j\in J}$ and the net $(x_j)_{j\in J}$ with limit x_∞ are as in the theorem. To a given open neighborhood U of $\chi_\infty(x_\infty)$ in $\mathbb{T}$ choose an open relatively compact neighborhood V of x_∞ in G with $\chi_\infty(\overline{V}\cap H_\infty)\subset U$. Since (x_j) converges to x_∞ and (K_j,ω_j) converges to (H_∞,χ_∞) there exists $j_0\in J$ such that $x_j\in V$ and $(K_j,\omega_j)\in W(\emptyset,\emptyset,\{(\overline{V},U)\})$ for $j\geq j_0$. In particular, $\chi_j(x_j)\in U$ for $j\geq j_0$.

Next, suppose that a net $(H_i,\chi_i)_{i\in I}$ and a point $(H_\infty,\chi_\infty)\in\mathfrak{X}$ fulfill the criterion of the theorem, and that a neighborhood $W(\mathcal{F},C,\mathfrak{A})$ of (H_∞,χ_∞) is given. We have to show that there is an $i_0\in I$ such that $(H_i,\chi_i)\in W(\mathcal{F},C,\mathfrak{A})$ for $i\geq i_0$. Without loss of generality we may assume that $\mathfrak{A}$ consists of one element (A,U). By assumption there exists $i_1\in I$ such that $H_i\in Q(\mathcal{F},C)$ for $i\geq i_1$. Furthermore, either there exists an $i_0\geq i_1$ such that $i\geq i_0$ implies $\chi_i(H_i\cap A)\subset U$, or the set $I':=\{i\in I\mid i\geq i_1$ and $\chi_i(A\cap H_i)\not\subset U\}$ is cofinal in I. In the first case we are done, in the second case for $i\in I'$ we choose $a_i\in A\cap H_i$ with $\chi_i(a_i)\notin U$. As A is compact the net $(a_i)_{i\in I'}$ has a convergent subnet. Therefore, we find a subnet $(K_j,\omega_j)_{j\in J}$ of $(H_i,\chi_i)_{i\in I}$ and a convergent net $(x_j)_{j\in J}$ with $x_j\in K_j\cap A$ and $\omega_j(x_j)\notin U$. By the criterion, $(\omega_j(x_j))$ converges to $\chi_\infty(x_\infty)$, where $x_\infty=\lim_{j\in J}x_j\in A\cap H_\infty$. From $(H_\infty,\chi_\infty)\in W(\mathcal{F},C,\mathfrak{A})$ it follows that $\chi_\infty(x_\infty)\in U$, which leads to a contradiction. Therefore, it is impossible that the above set I' is cofinal in I.

Next, suppose that the above map $\Lambda(G^\wedge)\to\mathfrak{X}$ is not continuous. Then there exists a convergent net $\lambda_i=\gamma_i\Delta_i$, $i\in I$, with limit $\lambda_\infty=\gamma_\infty\Delta_\infty$ and a neighborhood W of (H_∞,χ_∞) such that $(H_i,\chi_i)\notin W$ for all $i\in I$, where $H_i=\Delta_i^\perp$ and $\chi_i=\gamma_i|H_i$ for $i\in I\cup\{\infty\}$. Passing to a subnet and changing the γ_i inside λ_i if necessary, by 1.1 we may assume in addition that $(\gamma_i)_{i\in I}$ converges to γ_∞ in $G^\wedge$. Since the canonical map $\Lambda(G^\wedge)\to\Sigma(G^\wedge)$, $\gamma\Delta\mapsto\Delta$, is continuous by 1.4, from 2.1 it follows that $(H_i)_{i\in I}$ converges to H_∞. Now the convergence of the net (γ_i) readily implies that the above criterion for the convergence of (H_i,χ_i) in $\mathfrak{X}$ is satisfied, which leads to a contradiction to the assumption that $\Lambda(G^\wedge)\to\mathfrak{X}$ is not continuous.

To see that the inverse map $\mathfrak{X}\to\Lambda(G^\wedge)$ is continuous, in view of the Hausdorff property of $\mathfrak{X}$ and of the just established continuity we only have to exclude that there exists a net $(\lambda_i)_{i\in I}$, $\lambda_i=\gamma_i\Delta_i$, in $\Lambda(G^\wedge)$ converging in $\Phi(G^\wedge)$ to the empty set while (H_i,χ_i), where $H_i=\Delta_i^\perp$ and $\chi_i=\gamma_i|_{H_i}$, converges in $\mathfrak{X}$ to some point (H_∞,χ_∞). Suppose that (λ_i) is such a net. Without loss of generality we may assume in addition that $\chi_\infty=1$: Take any extension $\gamma\in G^\wedge$ of χ_∞ and consider the net $(\lambda_i')_{i\in I}$, $\lambda_i'=\gamma^{-1}\gamma_i\Delta_i$, which still converges to the empty set (compare 1.5), while $(H_i,\gamma^{-1}|_{H_i}\chi_i)_{i\in I}$

converges to $(H_\infty, 1) \in \mathfrak{X}$. Observe that each $\gamma \in G^\wedge$ yields by multiplication a homeomorphism of $\mathfrak{X}$. Actually, as soon as the claimed homeomorphy is established it will be clear that $(G^\wedge, \mathfrak{X})$ is a topological transformation group. The weaker statement (already applied here) follows for instance from the convergence criterion.

Once more take Haar measures ν_i, $i \in I \cup \{\infty\}$, on H_i according to 1.6. We claim that for each $\varphi \in \mathcal{C}_{\rm c}(G)$ the net

$$\int\limits_{H_i} \varphi(x)\chi_i(x)\, d\nu_i(x), \qquad i \in I,$$

converges to $\int_{H_\infty} \varphi(x)\, d\nu_\infty(x)$. Denote by A the support of φ. By definition of the topology on $\mathfrak{X}$ the net $(\varepsilon_i)_{i\in I}$, where $\varepsilon_i := \sup_{x\in H_i\cap A} |\chi_i(x) - 1|$, converges to zero. Using Tietze's extension theorem we find continuous functions $\psi_i : G \to \mathbb{C}$ such that $\psi_i = \chi_i$ on $H_i \cap A$ and $|\psi_i(x) - 1| \le \varepsilon_i$ for all $x \in G$. Clearly, $(\varphi\psi_i)_{i\in I}$ converges to φ in the inductive limit topology of $\mathcal{C}_{\rm c}(G)$, hence by 1.6 the integrals

$$\int\limits_{H_i} \varphi(x)\psi_i(x)\, d\nu_i(x) = \int\limits_{H_i} \varphi(x)\chi_i(x)\, d\nu_i(x)$$

converge to

$$\int\limits_{H_\infty} \varphi(x)\, d\nu_\infty(x).$$

With this information at hand one may argue exactly as in the proof of 2.4: For each $\varphi \in \mathcal{C}_{\rm c}(G)$ and $i \in I \cup \{\infty\}$ define $\varphi_i : G \to \mathbb{C}$ by

$$\varphi_i(a) = \int\limits_{H_i} \varphi(ax)\chi_i(x)\, d\nu_i(x).$$

As there, one shows that $(\varphi_i)_{i\in I}$ converges to φ_∞ in the weak topology of $L^\infty(G)$, and one concludes that $H_\infty^\perp$ is contained in $\bigcap_{i_0\in I}(\bigcup_{i\ge i_0} \lambda_i)^-$, which contradicts the assumption that (λ_i) converges to the empty set. ∎

3. Covariance algebras. Throughout this section let G be a second countable locally compact abelian group, and let $\mathcal{B}$ be a separable commutative symmetric regular Banach $*$-algebra with a bounded approximate identity. Moreover, we assume that a strongly continuous action T of G on $\mathcal{B}$ is given with the usual properties so that we can form the covariance algebra $L^1(G, \mathcal{B}, T)$:

$$(f * g)(x) = \int\limits_G f(xy)^{y^{-1}} g(y^{-1})\, dy, \qquad f^*(x) = f(x^{-1})^{*x},$$

where we put $b^x := T_{x^{-1}} b$ for $b \in \mathcal{B}$ and $x \in G$. The separability conditions are only imposed for the later application of the (ungeneralized) Effros–Hahn conjecture to the C^*-completion of $L^1(G,\mathcal{B},T)$. The basic idea is to reduce questions on the ideal theory of $L^1(G,\mathcal{B},T)$ to questions in terms of the commutative algebra $L^1(G,\mathcal{B})$, which is nothing but the projective tensor product of $L^1(G)$ and $\mathcal{B}$. In the group case this idea, which traces back to Leptin [15, 2], was exploited by Pytlik [18]. Our first results are easy translations of Pytlik's.

Since $\mathcal{B}$ has a bounded approximate identity each closed two-sided ideal $\mathcal{I}$ in $L^1(G,\mathcal{B},T)$ is also an ideal in the adjoint algebra [14, p. 196], in particular if $f\in\mathcal{I}$ then the following functions are in $\mathcal{I}$ as well:

(1) $x\mapsto f(x)^a$,
(2) $x\mapsto f(ax)$ for all $a\in G$,
(3) $x\mapsto f(x)b$,
(4) $x\mapsto b^x f(x)$ for all $b\in\mathcal{B}$.

Properties (2) and (3) tell that $\mathcal{I}$ is an ideal in $L^1(G,\mathcal{B})$. Properties (1) and (4) have consequences for the hull $h(\mathcal{I}_{\rm c})$, where $\mathcal{I}_{\rm c}$ means the set $\mathcal{I}$ considered as an ideal in the commutative algebra $L^1(G,\mathcal{B})$. Clearly, the structure space of $L^1(G,\mathcal{B})$ is just $G^\wedge\times\mathcal{B}^\wedge$ where $\mathcal{B}^\wedge$ is the Gelfand structure space of $\mathcal{B}$ consisting of all nonzero multiplicative (hermitean, by assumption) linear functionals on $\mathcal{B}$.

LEMMA 3.1. *Let $\mathcal{I}$ be a closed two-sided ideal in $L^1(G,\mathcal{B},T)$. If $(\gamma,\beta)\in G^\wedge\times\mathcal{B}^\wedge$ is in the hull $h(\mathcal{I}_{\rm c})$ of $\mathcal{I}_{\rm c}$ then (γ',β') is in $h(\mathcal{I}_{\rm c})$ as well, provided that β' is in the closure of the G-orbit through $\beta\in\widehat{\mathcal{B}}$ ($x\in G$ acts on $\beta\in\mathcal{B}^\wedge$ by $(x\beta)(b)=\beta(b^x)$ for $b\in\mathcal{B}$), and that γ and γ' agree on the stabilizer G_β of β.*

Proof. The assumption $(\gamma,\beta)\in h(\mathcal{I}_{\rm c})$ means $\int_G\gamma(x)\beta(f(x))\,dx=0$ for all $f\in\mathcal{I}$. From (4) we conclude that

$$0=\int\limits_G\gamma(x)\beta(b^xf(x))\,dx=\int\limits_G\gamma(x)(x\beta)(b)\beta(f(x))\,dx$$

for all $b\in\mathcal{B}$ and $f\in\mathcal{I}$. Using Weil's formula we obtain

$$\int\limits_{G/G_\beta}(x\beta)(b)\Big(\int\limits_{G_\beta}\gamma(xh)\beta(f(xh))\,dh\Big)\,d\dot{x}=0.$$

Since by [17, Corollary] the collection of functions $x\mapsto(x\beta)(b)$, $b\in\mathcal{B}$, is weakly dense in $L^\infty(G/G_\beta)$, in the above identity we may replace the function $x\mapsto(x\beta)(b)$ by any $\alpha\in(G/G_\beta)^\wedge$. Therefore,

$$0=\int\limits_{G/G_\beta}\alpha(x)\Big(\int\limits_{G_\beta}\gamma(xh)\beta(f(xh))\,dh\Big)\,d\dot{x}=\int\limits_G\alpha(x)\gamma(x)\beta(f(x))\,dx$$

for all $f \in \mathcal{I}$ and all $\alpha \in (G/G_\beta)^\wedge$. But this means that $(\gamma', \beta) \in h(\mathcal{I}_c)$ if $\gamma' = \gamma$ on G_β.

That β may be replaced by anything in its G-orbit (without leaving $h(\mathcal{I}_c)$) follows immediately from (1). ∎

It is not hard to write down a collection of irreducible representations of $L^1(G, \mathcal{B}, T)$. One simply induces from the stabilizer G_β of $\beta \in \mathcal{B}^\wedge$ using a character on G_β: Given $(\gamma, \beta) \in G^\wedge \times \mathcal{B}^\wedge$ define representations of G and $\mathcal{B}$ in $L^2(G/G_\beta)$ by

$$\{\pi_{\gamma,\beta}(x)\xi\}(t) = \gamma(x)\xi(x^{-1}t), \qquad \{\pi_{\gamma,\beta}(b)\xi\}(t) = \beta(b^t)\xi(t),$$

for $t \in G/G_\beta$, $\xi \in L^2(G/G_\beta)$, $x \in G$ and $b \in \mathcal{B}$. Note that $\beta(b^t) = (t\beta)(b)$ only depends on the coset $t \in G/G_\beta$. This covariant pair of representations defines an irreducible involutive representation $\pi_{\gamma,\beta}$ of $L^1(G, \mathcal{B}, T)$ by

$$\pi_{\gamma,\beta}(f)\xi = \int_G \pi_{\gamma,\beta}(x)\pi_{\gamma,\beta}(f(x))\xi \, dx.$$

Observe that we have not given directly the induced representation in its usual form, which acts in a space of functions on G with some transformation property with respect to $\gamma|_{G_\beta}$. However, multiplying the functions in the latter space by γ one ends up with the above picture.

Of course, it is very easy to compute the kernel of $\pi_{\gamma,\beta}$, and one obtains the expected result, whose proof here is omitted (compare also the proof of Proposition 3.5).

LEMMA 3.2. *The ideal* $(\ker \pi_{\gamma,\beta})_c$ *in* $L^1(G, \mathcal{B})$ *is the kernel in the hull-kernel sense of the subset*

$$\{(\gamma', \beta') \mid \gamma = \gamma' \text{ on } G_\beta,\ \beta' \in (G\beta)^-\}$$

of $G^\wedge \times \mathcal{B}^\wedge$. ∎

The representations $\pi_{\gamma,\beta}$ extend to the C^*-hull of $L^1(G, \mathcal{B}, T)$, which is nothing but the transformation group C^*-algebra $C^*(G, C_\infty(\mathcal{B}^\wedge), T^\wedge)$ associated with the G-space $\mathcal{B}^\wedge$, the action being given in 3.1. The Effros–Hahn conjecture [9] gives that the map $(\gamma, \beta) \mapsto \ker_{C^*} \pi_{\gamma,\beta}$ from $G^\wedge \times \widehat{\mathcal{B}}$ into the primitive ideal space $\mathrm{Priv}(G, C_\infty(\mathcal{B}^\wedge))$ is surjective. Therefore, also $(\gamma, \beta) \mapsto \ker_{L^1(G,\mathcal{B},T)} \pi_{\gamma,\beta}$ from $G^\wedge \times \mathcal{B}^\wedge$ into $\mathrm{Priv}_* L^1(G, \mathcal{B}, T)$ is surjective, where $\mathrm{Priv}_* L^1(G, \mathcal{B}, T)$ denotes the set of kernels of irreducible involutive representations of $L^1(G, \mathcal{B}, T)$. Trivially, the canonical map $\mathrm{Priv}(G, C_\infty(\mathcal{B}^\wedge)) \to \mathrm{Priv}_* L^1(G, \mathcal{B}, T)$ is continuous if both spaces are equipped with the Jacobson topology. Moreover, the above map $G^\wedge \times \mathcal{B}^\wedge \to \mathrm{Priv}(G, C_\infty(\mathcal{B}^\wedge))$ is continuous (see [20]). The next theorem gives more precise information.

THEOREM 3.3. *The canonical map* $\mathrm{Priv}(G, C_\infty(\mathcal{B}^\wedge)) \to \mathrm{Priv}_* L^1(G, \mathcal{B}, T)$ *is a homeomorphism*, *i.e.*, $L^1(G, \mathcal{B}, T)$ *is a* $*$*-regular algebra in the sense of*

[1, 2]. *Both spaces are homeomorphic to the quotient space* $(G^\wedge \times \mathcal{B}^\wedge)/\sim$, *where the equivalence relation* $\sim$ *is defined by*: $(\gamma,\beta) \sim (\gamma',\beta')$ *if* $(G\beta)^- = (G\beta')^-$ *and* $\gamma = \gamma'$ *on* G_β $(= G_{\beta'})$. *Saturations* (*according to this equivalence relation*) *of open subsets of* $G^\wedge \times \mathcal{B}^\wedge$ *are again open.*

Remark 3.4. The C^*-part of the theorem was obtained by Williams [20]. While we shall use the continuity of the map $(\gamma,\beta) \mapsto \ker_{C^*} \pi_{\gamma,\beta}$, the openness of this map will be an easy consequence of the regularity of the commutative Banach algebra $L^1(G,\mathcal{B})$, which is an object not to be seen in the context of C^*-algebras.

Proof of Theorem 3.3. First we show the openness of the equivalence relation $\sim$, or rather as in 1.4 its dual version. Let A be a saturated subset in $G^\wedge \times \mathcal{B}^\wedge$, $(\gamma,\beta) \in \overline{A}$ and $(\gamma',\beta') \sim (\gamma,\beta)$. We claim that $(\gamma',\beta') \in \overline{A}$. Let (γ_n,β_n) be a sequence in $G^\wedge \times \mathcal{B}^\wedge$ converging to (γ,β). Without loss of generality we may assume that the stabilizer groups G_{β_n} converge in $\Sigma(G)$, say $G' = \lim_n G_{\beta_n}$. From 1.1 and the fact that $\mathcal{B}^\wedge$ is a G-space it follows that G' is contained in G_β. By definition of the equivalence relation the difference $\gamma^{-1}\gamma'$ is in $G_\beta^\perp \subset (G')^\perp$. Since $(G_{\beta_n}^\perp)$ converges by 2.1 to $(G')^\perp$ there exist a subsequence (β_{n_k}) and characters α_k in $G_{\beta_{n_k}}^\perp$ with $\lim_k \alpha_k = \gamma^{-1}\gamma'$. The pairs $(\alpha_k\gamma_{n_k},\beta_{n_k})$ are in A, but this sequence converges to $(\gamma',\beta) \in \overline{A}$. Since A (and hence $\overline{A}$) is G-invariant, where G acts only on the second component, we conclude that $(\gamma',\beta') \in \overline{A}$.

There is a commutative diagram

$$\begin{array}{ccc} G^\wedge \times \mathcal{B}^\wedge & \stackrel{\psi}{\rightarrow} & \mathrm{Priv}(G, C_\infty(\mathcal{B}^\wedge)) \\ \varphi\downarrow & & \downarrow \\ \mathrm{Priv}_* L^1(G,\mathcal{B},T) & \stackrel{\mathrm{Id}}{\longrightarrow} & \mathrm{Priv}_* L^1(G,\mathcal{B},T) \end{array}$$

of surjective continuous maps, where ψ is defined by $\psi(\gamma,\beta) = \ker_{C^*} \pi_{\gamma,\beta}$, and φ, $\varphi(\gamma,\beta) = \ker_{L^1(G,\mathcal{B},T)} \pi_{\gamma,\beta}$, is the composition of ψ with the canonical map $\mathrm{Priv}(G, C_\infty(\mathcal{B}^\wedge)) \to \mathrm{Priv}_* L^1(G,\mathcal{B},T)$. The φ-images $\varphi(\gamma,\beta)$ and $\varphi(\gamma',\beta')$ coincide if and only if $(\gamma,\beta) \sim (\gamma',\beta')$. This follows at once from the description of $\ker_{L^1(G,\mathcal{B},T)} \pi_{\gamma,\beta}$ given in 3.2 and the regularity of $L^1(G,\mathcal{B})$. Consequently, the equation $\psi(\gamma,\beta) = \psi(\gamma',\beta')$ implies $(\gamma,\beta) \sim (\gamma',\beta')$. On the other hand, if equivalent pairs (γ,β) and (γ',β') are given then $\psi(\gamma,\beta) = \psi(\gamma',x\beta)$ for all $x \in G$ as the representations $\pi_{\gamma,\beta}$ and $\pi_{\gamma',x\beta}$ are equivalent. Since β' is in the closure of $G\beta$ the continuity of ψ implies $\psi(\gamma,\beta) \subset \psi(\gamma',\beta')$, whence equality by interchanging the pairs. Altogether, in the above diagram we get only bijective maps if we replace $G^\wedge \times \mathcal{B}^\wedge$ by $(G^\wedge \times \mathcal{B}^\wedge)/\sim$.

In order to see that all three maps are homeomorphisms it is enough to show that the φ-image of a closed saturated subset A of $G^\wedge \times \mathcal{B}^\wedge$ is closed in

$\mathrm{Priv}_* L^1(G,\mathcal{B},T)$. Let $P=\varphi(\gamma_0,\beta_0)$ be a point in the closure of $\varphi(A)$, i.e.,

$$\ker_{L^1(G,\mathcal{B},T)}\pi_{\gamma_0,\beta_0}\supset\bigcap_{(\gamma,\beta)\in A}\ker_{L^1(G,\mathcal{B},T)}\pi_{\gamma,\beta}.$$

Transferring this inclusion into $L^1(G,\mathcal{B})$ yields by 3.2 and the regularity of $L^1(G,\mathcal{B})$ that $[\gamma_0,\beta_0]$ is contained in A, where $[\gamma_0,\beta_0]$ denotes the equivalence class of (γ_0,β_0). It follows that P is in $\varphi(A)$, hence $\varphi(A)$ is closed. ∎

For the needs of a forthcoming paper on representations of so-called diamond groups we include the following proposition. Here it may be considered as an exercise to Theorem 3.3. In addition to $(G,\mathcal{B},T)$ let another second countable locally compact abelian group H and a continuous homomorphism $\varrho:H\to G^\wedge$ be given. This homomorphism defines an action R of H on $L^1(G,\mathcal{B},T)$ by

$$(R_af)(x)=\varrho(a)(x)^{-1}f(x)$$

for $a\in H$, $f\in L^1(G,\mathcal{B})$ and $x\in G$. For any triple $(\alpha,\gamma,\beta)\in H^\wedge\times G^\wedge\times\mathcal{B}^\wedge$ we define a representation $\tau=\tau_{\alpha,\gamma,\beta}$ of the associated covariance algebra $L^1(H,L^1(G,\mathcal{B},T),R)$ in $L^2(G/G_\beta\cap\ker\widehat{\varrho})$, where $\widehat{\varrho}:G\to H^\wedge$ denotes the dual homomorphism, $\widehat{\varrho}(x)(a)=\varrho(a)(x)$. The unitary representations τ_H and τ_G of H and G, respectively, are defined by

$$(\tau_H(a)\xi)(t)=\alpha(a)\varrho(a)(t)^{-1}\xi(t),\qquad(\tau_G(x)\xi)(t)=\gamma(x)\xi(x^{-1}t),$$

and an involutive representation $\tau_{\mathcal{B}}$ of $\mathcal{B}$ is defined by

$$(\tau_{\mathcal{B}}(b)\xi)(t)=\beta(b^t)\xi(t)$$

for $a\in H$, $x\in G$, $\xi\in L^2(G/G_\beta\cap\ker\widehat{\varrho})$, $t\in G/G_\beta\cap\ker\widehat{\varrho}$ and $b\in\mathcal{B}$. Observe that $(\tau_G,\tau_{\mathcal{B}})$ looks similar to the above $\pi_{\gamma,\beta}$, but that these representations act in different spaces. The representations τ_G and $\tau_{\mathcal{B}}$ form a covariant pair for $(G,\mathcal{B},T)$, hence they yield a representation of $L^1(G,\mathcal{B},T)$, which together with τ_H forms a covariant pair of $(H,L^1(G,\mathcal{B},T),R)$. The latter pair yields $\tau_{\alpha,\gamma,\beta}$, explicitly

$$(\tau_{\alpha,\gamma,\beta}(g)\xi)(t)=\int_H\int_G\alpha(a)\varrho(a)(t)^{-1}\gamma(x)\beta(g(a,x)^{x^{-1}t})\xi(x^{-1}t)\,dx\,da$$

for $g\in L^1(H\times G,\mathcal{B})$.

PROPOSITION 3.5. *The algebra $L^1(H,L^1(G,\mathcal{B},T),R)$ is of the type studied in this section, namely it is isomorphic to $L^1(G,\mathcal{B}',T')$, where $\mathcal{B}'=L^1(H,\mathcal{B})=L^1(H)\otimes\mathcal{B}$, and the action T' is given by*

$$(T'_xh)(a)=\widehat{\varrho}(x)(a)T_x(h(a))$$

for $x\in G$, $a\in H$ and $h\in L^1(H,\mathcal{B})$. In particular, in view of 3.3 the algebra $L^1(H,L^1(G,\mathcal{B},T),R)$ is $*$-regular. All the above representations $\tau_{\alpha,\gamma,\beta}$ are *irreducible. The map*

$$H^\wedge \times G^\wedge \times \mathcal{B}^\wedge \ni (\alpha,\gamma,\beta) \mapsto \ker \tau_{\alpha,\gamma,\beta} \in \mathrm{Priv}_* L^1(H, L^1(G,\mathcal{B},T), R)$$

is surjective and induces a homeomorphism from $(H^\wedge \times G^\wedge \times \mathcal{B}^\wedge)/\approx$ *onto* $\mathrm{Priv}_* L^1(H, L^1(G,\mathcal{B},T), R)$, *where the* (*open*) *equivalence relation* $\approx$ *on* $H^\wedge \times G^\wedge \times \mathcal{B}^\wedge$ *is defined as follows*: $(\alpha,\gamma,\beta) \approx (\alpha',\gamma',\beta')$ *if the* G-*quasi-orbits through* (α,β) *and* (α',β') *coincide*, *where* $x \in G$ *acts on* $(\alpha,\beta) \in H^\wedge \times \mathcal{B}^\wedge$ *by* $x(\alpha,\beta) = (\widehat{\varrho}(x)^{-1}\alpha, x\beta)$, *and if* γ *and* γ' *coincide on the stabilizer* $G_{(\alpha,\beta)} = \ker\widehat{\varrho} \cap G_\beta$.

Moreover, *the action of* H *on* $L^1(G,\mathcal{B},T)$ *induces an action of* H *on* $\mathrm{Priv}_* L^1(G,\mathcal{B},T)$. *In terms of the above parametrization of the latter space the action is given by* $a\cdot\ker\pi_{\gamma,\beta} = \ker_{\gamma\varrho(a),\beta}$ *for* $a \in H$ *and* $(\gamma,\beta) \in G^\wedge \times \mathcal{B}^\wedge$. *Each ideal* P *in* $\mathrm{Priv}_* L^1(H, L^1(G,\mathcal{B},T), R)$ *defines by restriction an* H-*quasi-orbit in* $\mathrm{Priv}_* L^1(G,\mathcal{B},T)$. *If* $P = \ker\tau_{\alpha,\gamma,\beta}$ *then the corresponding* H-*quasi-orbit is parametrized by*

$$\{(\gamma',\beta') \in G^\wedge \times \mathcal{B}^\wedge \mid \gamma' = \gamma \text{ on } G_\beta \cap \ker\widehat{\varrho},\ (G\beta)^- = (G\beta')^-\}.$$

P r o o f. Define $J : L^1(H \times G, \mathcal{B}) \to L^1(G \times H, \mathcal{B})$ by

$$(Jg)(x,a) = \varrho(a)(x)^{-1} g(a,x),$$

which is clearly an isometric isomorphism of Banach spaces. A simple computation shows that J defines a $*$-isomorphism from $L^1(H, L^1(G,\mathcal{B},T), R)$ onto $L^1(G, L^1(H,\mathcal{B}), T') = L^1(G,\mathcal{B}',T')$. The space $\mathrm{Priv}_* L^1(G,\mathcal{B}',T')$ can be parametrized by $G^\wedge \times (\mathcal{B}')^\wedge$ according to Theorem 3.3. Identifying $(\mathcal{B}')^\wedge$ with $H^\wedge \times \mathcal{B}^\wedge$ for each $\gamma \in G^\wedge$ and each $(\alpha,\beta) \in H^\wedge \times \mathcal{B}^\wedge$ we have an irreducible representation $\pi_{\gamma,(\alpha,\beta)}$ of $L^1(G,\mathcal{B}',T')$. Again it is easily verified that the transferred representation $\pi_{\gamma,(\alpha,\beta)} \circ J$ of $L^1(H, L^1(G,\mathcal{B},T), R)$ is nothing but $\tau_{\alpha,\gamma,\beta}$ of the proposition. Hence the $\tau_{\alpha,\gamma,\beta}$ are irreducible, and they exhaust the dual of $L^1(H, L^1(G,\mathcal{B},T), R)$ up to weak equivalence. Transferring the equivalence relation $\sim$ of 3.3 on $G^\wedge \times (H^\wedge \times \mathcal{B}^\wedge)$ clearly gives the equivalence relation $\approx$ on $H^\wedge \times G^\wedge \times \mathcal{B}^\wedge$.

Finally, we consider the kernel of the restriction τ' of $\tau_{\alpha,\gamma,\beta}$ to $L^1(G,\mathcal{B},T)$. This kernel could be computed by writing τ' as a direct integral over irreducibles, but we prefer a more direct way, whose arguments also provide us with a proof of Lemma 3.2. Put $S := G_\beta \cap \ker\widehat{\varrho}$ for short; an L^1-function $f : G \to \mathcal{B}$ is in the kernel of τ' if and only if

$$\int\limits_{G/S} \int\limits_{G} \beta(f(x)^{x^{-1}t})\gamma(x)\xi(x^{-1}t)\eta(t)\,dx\,dt = 0$$

or

$$\int\limits_{G/S} \int\limits_{G} (t\beta)(f(x))\gamma(x)\xi(t)\eta(xt)\,dx\,dt = 0$$

for all $\xi,\eta\in\mathcal{C}_{\rm c}(G/S)$. Defining $F:G/S\times G/S\to\mathbb{C}$ by

$$F(t,u)=\int\limits_S (t\beta)(f(xs))\gamma(xs)\,ds,$$

where x is any point in the coset $u\in G/S$, this condition may be written as

$$\int\limits_{G/S}\int\limits_{G/S} F(t,u)\xi(t)\eta(ut)\,dt\,du=0.$$

Since each $\varphi\in\mathcal{C}_{\rm c}(G/S\times G/S)$ can be uniformly approximated by functions of the form $(t,u)\mapsto\sum_{j=1}^n\xi_j(t)\eta_j(ut)$, where the $\xi_j,\eta_j\in\mathcal{C}_{\rm c}(G/S)$ have their support in a fixed compact subset of G/S depending only on φ, the latter condition is equivalent to

$$\int\limits_{G/S}\int\limits_{G/S} F(t,u)\xi(t)\eta(u)\,dt\,du=0$$

for all $\xi,\eta\in\mathcal{C}_{\rm c}(G/S)$. But this property of F is equivalent to

$$\int\limits_{G/S} F(t,u)\eta(u)\,du=0$$

for all $t\in G/S$ and all $\eta\in\mathcal{C}_{\rm c}(G/S)$.

Inserting the expression for F we find that $f\in L^1(G,\mathcal{B},T)$ is in the kernel of τ' if and only if

$$\int\limits_G (t\beta)(f(x))\gamma(x)\eta(x)\,dx=0$$

for all $t\in G$ and all $\eta\in\mathcal{C}_{\rm c}(G/S)$, which is equivalent to

$$\int\limits_G (t\beta)(f(x))\gamma(x)\zeta(x)\,dx=0$$

for all $t\in G$ and all $\zeta\in(G/S)^\wedge$.

This shows that the "commutative" kernel $(\ker\tau')_{\rm c}$ in $L^1(G)\otimes\mathcal{B}$ is the kernel of the subset $C:=\gamma(G/S)^\wedge\times(G\beta)^-$ of $G^\wedge\times\mathcal{B}^\wedge$. The (unproved) Lemma 3.2 gives $\ker\tau'=\bigcap_{(\gamma',\beta')\in C}\ker\pi_{\gamma',\beta'}$. Conversely, specializing to $H=\{1\}$ from our above considerations one can immediately deduce Lemma 3.2.

Using our knowledge, Theorem 3.3, about the topology of the space $\mathrm{Priv}_*\,L^1(G,\mathcal{B},T)$ and the H-action we easily conclude that the H-quasi-orbit associated with $\{H\ker\pi_{\gamma,\beta}\}^-=\{\ker\pi_{\gamma',\beta'}\mid(\gamma',\beta')\in C\}$ is just $\{\ker\pi_{\gamma',\beta'}\mid(\gamma',\beta')\in C,\ (G\beta')^-=(G\beta)^-\}$; observe that the closure of $\varrho(H)$ is equal to $(\ker\widehat{\varrho})^\perp\subset G^\wedge$. ∎

Next, we turn to the study of the Wiener property of $L^1(G,\mathcal{B},T)$ in the sense of [16]. Recall that an involutive Banach algebra has this property if each proper two-sided ideal is annihilated by an involutive nondegenerate (irreducible) representation. For commutative symmetric regular Banach $*$-algebras $\mathcal{A}$ this is equivalent to saying that the ideal $\mathcal{A}_0$ of all elements in $\mathcal{A}$ with compactly supported Gelfand transform is dense in $\mathcal{A}$ or that the empty subset of $\mathcal{A}^\wedge$ is a set of synthesis (compare [19, Chap. 2]).

THEOREM 3.6. *If in addition to our general assumptions $\mathcal{B}$ has the Wiener property then $L^1(G,\mathcal{B},T)$ has the Wiener property as well.*

Proof. The algebra $L^1(G)$ has the Wiener property (see for instance [19, Chap. 6]). Hence the algebraic tensor product $L^1(G)_0 \otimes \mathcal{B}_0$ (in the above terminology) is dense in the projective tensor product $L^1(G) \otimes \mathcal{B} = L^1(G,\mathcal{B})$, which shows that the commutative algebra $L^1(G,\mathcal{B})$ has the Wiener property. Let a proper two-sided ideal $\mathcal{I}$ in $L^1(G,\mathcal{B},T)$ be given. Then the hull $h(\mathcal{I}_c) \subset G^\wedge \times \mathcal{B}^\wedge$ of the corresponding ideal $\mathcal{I}_c$ in $L^1(G,\mathcal{B})$ is not empty, say $(\gamma_0,\beta_0) \in h(\mathcal{I}_c)$. By Lemma 3.1 the whole equivalence class $[\gamma_0,\beta_0]$ is contained in $h(\mathcal{I}_c)$, from which we conclude by means of Lemma 3.2 that $\ker_{L^1(G,\mathcal{B},T)} \pi_{\gamma_0,\beta_0}$ contains $\mathcal{I}$. ∎

Finally, we consider semidirect products $M = G \ltimes B$ of second countable locally compact abelian groups. Their L^1-algebras $L^1(M)$ with respect to the left invariant measure obtained as the tensor product of invariant measures on G and B may be written as $L^1(G,\mathcal{B},T)$, where $\mathcal{B} = L^1(B)$ and the action T is given by $(T_x\varphi)(b) = \delta(x)\varphi(x^{-1}bx)$ for $\varphi \in L^1(B)$, $x \in G$ and $b \in B$; here δ is the modular function of the action of G on B, which coincides with the modular function Δ of M in the sense that $\Delta(xb) = \delta(x)$.

Hence all our previous results apply to $L^1(M)$ because $L^1(B)$ is a symmetric regular Banach $*$-algebra satisfying the Wiener property. Note that the $*$-regularity of $L^1(M)$ was proved in [1] without separability conditions (by reducing to this case). Identifying the structure space $\mathcal{B}^\wedge = L^1(B)^\wedge$ with the Pontryagin dual $B^\wedge$ gives a parametrization of the homeomorphic spaces $\operatorname{Priv} C^*(M)$ and $\operatorname{Priv}_* L^1(M)$ by $G^\wedge \times B^\wedge$. The G-action on $L^1(B)^\wedge$ corresponds to the action of G on $B^\wedge$ given by $(x\beta)(b) = \beta(x^{-1}bx)$ for $\beta \in B^\wedge$, $b \in B$ and $x \in G$. The above equivalence relation can be interpreted as follows. Each pair $(\gamma,\beta) \in G^\wedge \times B^\wedge$ defines a character $\chi = \chi_{\gamma,\beta}$ on $G_\beta B$ by $\chi(xb) = \gamma(x)\beta(b)$. Two pairs (γ,β) and (γ',β') are equivalent iff the corresponding characters χ and χ' have the same domain and lie on the same G-quasi-orbit. The above considered representations $\pi_{\gamma,\beta}$ correspond to unitary representations of M, denoted by the same letters. Explicitly, one has

$$(\pi_{\gamma,\beta}(xb)\xi)(t) = \gamma(x)\beta(t^{-1}xbx^{-1}t)\xi(x^{-1}t)$$

for $\xi \in L^2(G/G_\beta)$, $x \in G$, $b \in B$ and $t \in G/G_\beta$.

In our next theorem we determine the support of the tensor product of two such representations in $\operatorname{Priv} C^*(M) = \operatorname{Priv}_* L^1(M)$.

THEOREM 3.7. *For* $(\gamma,\beta),(\gamma',\beta')\in G^\wedge\times B^\wedge$ *the support of* $\pi_{\gamma,\beta}\otimes\pi_{\gamma',\beta'}$ *in* $\mathrm{Priv}\,C^*(M)$ (*or in* $\mathrm{Priv}_*\,L^1(M)$), *i.e., the set of all ideals* P *in* $\mathrm{Priv}\,C^*(M)$ (*or in* $\mathrm{Priv}_*\,L^1(M)$) *with* $P\supset\ker_{C^*(M)}\pi_{\gamma,\beta}\otimes\pi_{\gamma',\beta'}$ (*or* $P\supset\ker_{L^1(M)}\pi_{\gamma,\beta}\otimes\pi_{\gamma',\beta'}$) *is parametrized by the subset*

$$\gamma\gamma'(G_\beta\cap G_{\beta'})^\perp\times[(G\beta)(G\beta')]^-$$

of $G^\wedge\times B^\wedge$.

Proof. In view of the $*$-regularity of $L^1(M)$ it suffices to prove the L^1-version of this theorem. The C^*-version was only formulated in order to exhibit the relation to the more common notion of weak containment.

To compute the L^1-kernel of $\pi_{\gamma,\beta}\otimes\pi_{\gamma',\beta'}$ we apply the usual trick, namely we consider first the outer tensor product $\pi_{\gamma,\beta}\times\pi_{\gamma',\beta'}$, which is an irreducible representation of $M\times M$. Since $M\times M$ is also a semidirect product of abelian groups, Lemma 3.2 applies, and we conclude that the kernel of $\pi_{\gamma,\beta}\times\pi_{\gamma',\beta'}$ consists of all $g\in L^1(M\times M)$ such that

$$\int\limits_G\int\limits_B\int\limits_G\int\limits_B g(x_1b_1,x_2b_2)\gamma_1(x_1)\beta_1(b_1)\gamma_2(x_2)\beta_2(b_2)\,dx_1\,db_1\,dx_2\,db_2=0$$

for all $\gamma_1\in\gamma(G_\beta)^\perp$, $\gamma_2\in\gamma'(G_{\beta'})^\perp$, $\beta_1\in(G\beta)^-$ and $\beta_2\in(G\beta')^-$.

Embedding M diagonally in $M\times M$ one sees that $f\in L^1(M)$ belongs to $\ker_{L^1(M)}\pi_{\gamma,\beta}\otimes\pi_{\gamma',\beta'}$ iff $f*h\in L^1(M\times M)$, given by

$$(f*h)(x_1b_1,x_2b_2)=\int\limits_G\int\limits_B f(xb)h(b^{-1}x^{-1}x_1b_1,b^{-1}x^{-1}x_2b_2)\,dx\,db,$$

belongs to $\ker\pi_{\gamma,\beta}\times\pi_{\gamma',\beta'}$ for all, say, $h\in\mathcal{C}_c(M\times M)$.

Combining these two equations yields that $f\in L^1(M)$ belongs to $\ker_{L^1(M)}\pi_{\gamma,\beta}\otimes\pi_{\gamma',\beta'}$ iff

$$\begin{aligned}0&=\int\limits_G\int\limits_B\int\limits_G\int\limits_B\int\limits_G\int\limits_B f(xb)h(b^{-1}x^{-1}x_1b_1,b^{-1}x^{-1}x_2b_2)\\&\qquad\gamma_1(x_1)\beta_1(b_1)\gamma_2(x_2)\beta_2(b_2)\,dx\,db\,dx_1\,db_1\,dx_2\,db_2\\&=\int\limits_G\int\limits_B\int\limits_G\int\limits_B h(x_1b_1,x_2b_2)\Big(\int\limits_G\int\limits_B f(xb)\gamma_1(xx_1)\beta_1(x_1^{-1}bx_1b_1)\\&\qquad\gamma_2(xx_2)\beta_2(x_2^{-1}bx_2b_2)\,dx\,db\Big)dx_1\,db_1\,dx_2\,db_2\end{aligned}$$

for all $h\in\mathcal{C}_c(M\times M)$ and all $\gamma_1,\gamma_2,\beta_1,\beta_2$ as above.

Since h is arbitrary this means that the inner integral has to be identically zero, in other words,

$$0=\int\limits_G\int\limits_B f(xb)\alpha(x)\zeta(b)\,dx\,db$$

for all $\alpha \in \gamma\gamma'(G_\beta)^\perp(G_{\beta'})^\perp$ and $\zeta \in (G\beta)^-(G\beta')^-$. But as Fourier transforms are continuous and $(G_\beta)^\perp(G_{\beta'})^\perp$ is dense in $(G_\beta \cap G_{\beta'})^\perp$, in view of Lemma 3.2 this is precisely what we wanted to show. ■

Theorem 3.7 can be used to determine the support of the so-called conjugation representation κ_M in $L^2(M)$ given by

$$(\kappa_M(m)\xi)(t) = \Delta(m)^{1/2}\xi(m^{-1}tm)$$

for $t, m \in M$ and $\xi \in L^2(M)$ (compare [11, 12]). To this end we introduce some notation. Denote by

$$[\ ,\ ] : G \times B \to B$$

the commutator, i.e., $[x, b] = xbx^{-1}b^{-1}$. This map is multiplicative in the second variable. For $x \in G$ and $\beta \in B^\wedge$ define $\beta_x \in B^\wedge$ by $x\beta = \beta\beta_x$, i.e., $\beta_x(b) = \beta([x^{-1}, b])$. One easily verifies that

(i) $(\beta\beta')_x = \beta_x\beta'_x$,
(ii) $\beta_x\beta_y^{-1} = (y\beta)_{xy^{-1}}$, and
(iii) $y(\beta_x) = (y\beta)_x$

hold for all $x, y \in G$ and $\beta, \beta' \in B^\wedge$. Moreover, put

$$X_\beta = \{(\gamma, \beta_x) \in G^\wedge \times B^\wedge \mid x \in G, \gamma \in G_\beta^\perp\} \quad \text{and}$$
$$X'_\beta = \{(\gamma, \beta'_x) \in G^\wedge \times B^\wedge \mid \gamma \in G_\beta^\perp,\ x \in G,\ \beta' \in (G\beta)^-\}.$$

COROLLARY 3.8. *The support of* κ_M, $M = G \ltimes B$, *in* $\operatorname{Priv}_* L^1(M) = \operatorname{Priv} C^*(M)$ *is parametrized by the subset* $(\bigcup_{\beta\in B^\wedge} X_\beta)^- = (\bigcup_{\beta\in B^\wedge} X'_\beta)^-$ *of* $G^\wedge \times B^\wedge$.

Remark 3.9. Everything in this description rests on the map $[\ ,\ ]$ (and its dual version $(x, \beta) \mapsto \beta_x$), because $x \in G_\beta$ iff $\beta_x = 1$. A more explicit description of $(\bigcup_{\beta\in B^\wedge} X_\beta)^-$ in special cases requires a more concrete knowledge of this map $[\ ,\ ]$.

Proof of Corollary 3.8. It is easily verified that X'_β is contained in $(\bigcup_{\beta\in B^\wedge} X_\beta)^-$ from which one concludes that the latter set coincides with $(\bigcup_{\beta\in B^\wedge} X'_\beta)^-$. Obviously, each individual X'_β is saturated with respect to the equivalence relation $\sim$, whence $(\bigcup_{\beta\in B^\wedge} X'_\beta)^-$ is saturated. By [11, Corollary 1] the support of κ_M is the smallest closed subset of $\operatorname{Priv} C^*(M)$ containing the supports of all the $\pi \otimes \overline{\pi}$, $\pi \in M^\wedge$, where $\overline{\pi}$ denotes the conjugate representation. Since each π is weakly equivalent to one of the $\pi_{\gamma,\beta}$ and since evidently $\overline{\pi}_{\gamma,\beta} = \pi_{\gamma^{-1},\beta^{-1}}$ one only has to consider the supports of the various $\pi_{\gamma,\beta} \otimes \pi_{\gamma^{-1},\beta^{-1}}$ which are given in 3.7. Then one quickly finds the claimed structure of the support of κ_M.

We just remark that one can also compute directly, i.e., without using [11], the kernel of κ_M in $L^1(M)$ by means of the Fourier transform of the abelian group $G \times B$. ∎

Similarly, one can also treat the two-sided translation representation τ of $M \times M$ on $L^2(M)$, i.e., $(\tau(m,n)\xi)(t) = \Delta(n)^{1/2}\xi(m^{-1}tn)$ for $m,n,t \in M$ and $\xi \in L^2(M)$ (compare also [11]). If one parametrizes $\mathrm{Priv}\, C^*(M \times M) = \mathrm{Priv}_* L^1(M \times M)$ in the obvious way by $G^\wedge \times B^\wedge \times G^\wedge \times B^\wedge$ then the support of τ is the closure of the union of all the $Y_{\gamma,\beta}$, $(\gamma,\beta) \in G^\wedge \times B^\wedge$, where $Y_{\gamma,\beta}$ is defined as

$$Y_{\gamma,\beta} = \{(\gamma, \beta, \alpha\gamma^{-1}, (x\beta)^{-1}) \mid \alpha \in G_\beta^\perp,\ x \in G\}.$$

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FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT BIELEFELD
POSTFACH 100 131
33501 BIELEFELD, GERMANY

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