

*ESTIMATES FOR THE INTEGRAL MEANS  
OF HOLOMORPHIC FUNCTIONS ON BOUNDED DOMAINS IN  $\mathbb{C}^n$*

BY

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Let  $\mathcal{D} = \{z \in \mathbb{C}^n : \lambda(z) < 0\}$  be a bounded domain with  $C^\infty$  boundary. For  $f$  holomorphic in  $\mathcal{D}$ , let  $M_p(f, r)$  be the  $p$ th integral mean of  $f$  on  $\partial\mathcal{D}_r = \{z \in \mathcal{D} : \lambda(z) = -r\}$ . In this paper we prove that

$$\int_0^\varepsilon r^{s+|\alpha|q} M_p^q(D^\alpha f, r) dr \leq C \int_0^\varepsilon r^s M_p^q(f, r) dr$$

and

$$\int_0^\varepsilon r^s M_p^q(f, r) dr \leq C \left\{ \sum_{|\alpha| < m} |(D^\alpha f)(z_0)|^q + \sum_{|\alpha|=m} \int_0^\varepsilon r^{s+mq} M_p^q(D^\alpha f, r) dr \right\},$$

where  $z_0 \in \mathcal{D}$  is fixed,  $0 < p \leq \infty$ ,  $0 < q < \infty$ ,  $s > -1$ ,  $m \in \mathbb{N}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-index, and  $\varepsilon > 0$  is small enough. These inequalities generalize the known results in [9, 10] on the unit ball of  $\mathbb{C}^n$ . Two applications are given. The methods used in the proof of the inequalities also enable us to obtain some theorems about pluriharmonic functions on  $\mathcal{D}$ .

**1. Introduction.** Let  $\mathcal{D}$  be a bounded domain in  $\mathbb{C}^n = \mathbb{R}^{2n}$  with  $C^\infty$  boundary  $\partial\mathcal{D}$ , and  $\lambda(z)$  be a defining function of  $\mathcal{D}$ . That means,  $\lambda(z)$  is a  $C^\infty$  function,  $\mathcal{D} = \{z \in \mathbb{C}^n : \lambda(z) < 0\}$ , and  $|\nabla\lambda(z)| \neq 0$  on  $\partial\mathcal{D} = \{z \in \mathbb{C}^n : \lambda(z) = 0\}$ . A typical model of  $\mathcal{D}$  is the unit ball  $B = \{z \in \mathbb{C}^n : |z| < 1\}$  of  $\mathbb{C}^n$ . For  $r > 0$  and small enough, let  $\mathcal{D}_r = \{z : \lambda(z) < -r\}$ . Then  $\lambda(z) + r$  is a defining function of  $\mathcal{D}_r$ , and  $\partial\mathcal{D}_r$  is the level surface  $\{z : \lambda(z) = -r\}$ . Of course two different defining functions define two different systems of  $\{\mathcal{D}_r\}$ . We denote by  $d\sigma_r$  and  $d\sigma$  the induced surface measures on  $\partial\mathcal{D}_r$  and  $\partial\mathcal{D}$  respectively, and by  $dm$  the Lebesgue volume measure on  $\mathcal{D}$ . All this can

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be found in [6, 11].

Let  $H(\mathcal{D})$  be the space of all holomorphic functions in  $\mathcal{D}$ . For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\alpha_j$  being nonnegative integers, and  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ , we write  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ,  $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ . For  $f \in H(\mathcal{D})$ ,

$$(D^\alpha f)(z) = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}(z).$$

A continuous real function  $u$  on  $\mathcal{D}$  is called *pluriharmonic* if for every holomorphic mapping  $\phi$  of the unit disc  $D = \{z \in \mathbb{C} : |z| < 1\}$  into  $\mathcal{D}$ ,  $u \circ \phi$  is harmonic in  $D$ . If  $f = u + iv \in H(\mathcal{D})$ ,  $u = \operatorname{Re} f$ , then both  $u$  and  $v$  are pluriharmonic and  $v$  is the *pluriharmonic conjugate* of  $u$ .

For  $f$  continuous on  $\mathcal{D}$ , the *integral means*  $M_p(f, r)$ ,  $0 < p \leq \infty$ , are defined by

$$M_p(f, r) = \left\{ \int_{\partial \mathcal{D}_r} |f(z)|^p d\sigma_r(z) \right\}^{1/p} \quad \text{for } 0 < p < \infty$$

and

$$M_\infty(f, r) = \sup_{z \in \partial \mathcal{D}_r} |f(z)| \quad \text{for } p = \infty.$$

Here are the main results of the paper.

**THEOREM 1.** *Let  $0 < p \leq \infty$ ,  $0 < q < \infty$ ,  $s > -1$ , and  $\alpha = (\alpha_1, \dots, \alpha_n)$ . Then there exists  $\varepsilon > 0$  such that for  $f \in H(\mathcal{D})$ ,*

$$\int_0^\varepsilon r^{s+|\alpha|q} M_p^q(D^\alpha f, r) dr \leq C \int_0^\varepsilon r^s M_p^q(f, r) dr$$

and

$$(1.1) \quad \sup_{0 < r \leq \varepsilon} r^{s+|\alpha|} M_p(D^\alpha f, r) \leq C \sup_{0 < r \leq \varepsilon} r^s M_p(f, r).$$

**THEOREM 2.** *Let  $0 < p \leq \infty$ ,  $0 < q < \infty$ , and  $s > -1$ . Then for  $z_0 \in \mathcal{D}$  fixed and  $m \in \mathbb{N}$ , there exists  $\varepsilon > 0$  such that for  $f \in H(\mathcal{D})$ ,*

$$\int_0^\varepsilon r^s M_p^q(f, r) dr \leq C \left\{ \sum_{|\alpha| < m} |(D^\alpha f)(z_0)|^q + \sum_{|\alpha|=m} \int_0^\varepsilon r^{s+mq} M_p^q(D^\alpha f, r) dr \right\}.$$

**THEOREM 3.** *Let  $0 < p \leq \infty$ ,  $0 < q < \infty$ , and  $s > -1$ . Then for  $z_0 \in \mathcal{D}$  fixed, there exists  $\varepsilon > 0$  such that for  $f = u + iv \in H(\mathcal{D})$  with  $f(z_0)$  real*

and  $u(z) = \operatorname{Re} f(z)$ ,

$$\int_0^\varepsilon r^s M_p^q(v, r) dr \leq C \int_0^\varepsilon r^s M_p^q(u, r) dr.$$

Here, and later,  $C, C_1, C_2, \dots$  always denote positive constants, not necessarily the same at each occurrence; they are independent of the functions being considered.

The research leading to the results in this article was motivated by the results in [3, 12, 15] and especially in [9]. On the other hand, we can find that the results of those papers, together with their proofs, depend strongly on the homogeneity and the Bergman kernel of  $B$  (or  $D$ ). A bounded domain  $\mathcal{D}$  with  $C^\infty$  boundary need not be homogeneous and little is known about the Bergman kernel in this case. Therefore our theory will be more subtle.

This paper is organized as follows. In Section 2 some preliminaries are given. Theorems 1 and 2 will be proved in Section 3. In Section 4, we deal with pluriharmonic functions on  $\mathcal{D}$ , from which Theorem 3 follows. In the last section, Section 5, two applications of Theorems 1 and 2 are given, one of them to the Bloch functions on strongly pseudoconvex domains.

**2. Preliminaries.** Recall that  $\mathcal{D} = \{z \in \mathbb{C}^n : \lambda(z) < 0\}$  is a bounded domain with  $C^\infty$  boundary, and  $\mathcal{D}_r = \{z \in \mathcal{D} : \lambda(z) < -r\}$ . For  $r \geq 0$  small enough and  $\xi \in \partial\mathcal{D}_r$ , we write  $n_\xi$  for the unit inward normal vector of  $\partial\mathcal{D}_r$  at  $\xi$ . For  $z \in \mathcal{D}_r$  the Euclidean distance from  $z$  to  $\partial\mathcal{D}_r$  is denoted by  $\delta_r(z)$ .  $\delta_0(z)$  is often written as  $\delta(z)$  for short.

LEMMA 1. *There is a number  $\varepsilon > 0$  so that for each  $\xi \in \partial\mathcal{D}_r$  with  $0 \leq r \leq \varepsilon$ , there are balls  $B_\xi(\varepsilon) = \{z \in \mathbb{C}^n : |z - (\xi + \varepsilon n_\xi)| < \varepsilon\}$  and  $\widetilde{B}_\xi(\varepsilon) = \{z \in \mathbb{C}^n : |z - (\xi - \varepsilon n_\xi)| < \varepsilon\}$  that satisfy*

- (i)  $\overline{\widetilde{B}_\xi(\varepsilon)} \cap \overline{\mathcal{D}_r} = \{\xi\}$ ,
- (ii)  $\overline{B_\xi(\varepsilon)} \cap (\mathbb{C}^n \setminus \mathcal{D}_r) = \{\xi\}$ .

Lemma 1 is an improved version of the known result on p. 289 of [6]. Because  $\mathcal{D}$  has  $C^\infty$  boundary, we have

$$(2.1) \quad |\nabla \lambda(z)| \geq C > 0$$

for  $z$  in some neighborhood of  $\partial\mathcal{D}$ . Now Lemma 1 can be deduced from that geometric fact directly.

LEMMA 2. *Let  $\varepsilon$  be as in Lemma 1, and  $0 \leq r \leq \varepsilon$ . Then for  $\xi \in \partial\mathcal{D}_r$  and  $\zeta \in \mathcal{D}_r$ ,*

$$(2.2) \quad P_r(\zeta, \xi) \leq C \frac{\delta_r(\zeta)}{|\zeta - \xi|^{2n}},$$

where  $P_r(\cdot, \cdot)$  is the Poisson kernel of  $\mathcal{D}_r$  and  $C$  is independent of  $r$ ,  $\xi$ , and  $\zeta$ .

**Proof.** It is well known that

$$(2.3) \quad P_0(\zeta, \xi) \leq C \frac{\delta_0(\zeta)}{|\zeta - \xi|^{2n}} \quad (\xi \in \partial\mathcal{D}, \zeta \in \mathcal{D}).$$

This inequality is most conveniently obtained by comparing the explicitly known Poisson kernel for the exterior of a ball tangent to  $\partial\mathcal{D}$  at  $\xi$  (see [11, p. 2; 6, pp. 290–291] for the details). Meanwhile, a careful check shows that the constant  $C$  on the right side of (2.3) depends only on the radius of the ball which lies in the exterior of  $\mathcal{D}$  and is tangent to  $\partial\mathcal{D}$ . Hence we have (2.2) from Lemma 1.

**LEMMA 3.** *Let  $\varepsilon$  be as in Lemma 1. Then for  $0 < r < \varrho \leq \varepsilon$ ,*

$$\int_{\partial\mathcal{D}_\varrho} P_r(\zeta, \xi) d\sigma_\varrho(\zeta) \leq C \quad (\xi \in \partial\mathcal{D}_r).$$

**Proof.** By (2.1) we get

$$\frac{1}{C}(-\lambda(\zeta) - r) \leq \delta_r(\zeta) \leq C(-\lambda(\zeta) - r) \quad (\zeta \in \mathcal{D}_r).$$

Hence

$$\frac{1}{C}(\varrho - r) \leq \delta_r(\zeta) \leq C(\varrho - r) \quad (\zeta \in \partial\mathcal{D}_\varrho).$$

Since  $\mathcal{D}$  has smooth boundary, it is an elementary fact that

$$\int_{\partial\mathcal{D}_\varrho \cap \{\zeta: |\zeta - \xi| < t\}} d\sigma_\varrho(\zeta) \leq Ct^{2n-1},$$

where  $C$  is independent of  $\xi$  and  $\varrho$ . Then for  $\xi \in \partial\mathcal{D}_r$ , Lemma 2 gives

$$\begin{aligned} & \int_{\partial\mathcal{D}_\varrho} P_r(\zeta, \xi) d\sigma_\varrho(\zeta) \\ & \leq C \left\{ \int_{\partial\mathcal{D}_\varrho \cap \{\zeta: |\zeta - \xi| < 2(\varrho - r)\}} \right. \\ & \quad \left. + \sum_{k=2}^{\infty} \int_{\partial\mathcal{D}_\varrho \cap \{\zeta: 2^{k-1}(\varrho - r) \leq |\zeta - \xi| < 2^k(\varrho - r)\}} \right\} \frac{\delta_r(\zeta)}{|\zeta - \xi|^{2n}} d\sigma_\varrho(\zeta) \\ & \leq C \left\{ \int_{\partial\mathcal{D}_\varrho \cap \{\zeta: |\zeta - \xi| < 2(\varrho - r)\}} \delta_r(\zeta)^{1-2n} d\sigma_\varrho(\zeta) \right. \\ & \quad \left. + \sum_{k=2}^{\infty} \int_{\partial\mathcal{D}_\varrho \cap \{\zeta: 2^{k-1}(\varrho - r) \leq |\zeta - \xi| < 2^k(\varrho - r)\}} \frac{\delta_r(\zeta) d\sigma_\varrho(\zeta)}{[2^{k-1}(\varrho - r)]^{2n}} \right\} \end{aligned}$$

$$\begin{aligned} &\leq C \left\{ (\varrho - r)^{1-2n} (\varrho - r)^{2n-1} \right. \\ &\quad \left. + \sum_{k=2}^{\infty} (\varrho - r) [2^{k-1} (\varrho - r)]^{-2n} [2^k (\varrho - r)]^{2n-1} \right\} \\ &\leq C \left\{ 1 + \sum_{k=2}^{\infty} \frac{1}{2^k} \right\}. \end{aligned}$$

This is the desired result.

LEMMA 4. *Let  $\varepsilon$  be as in Lemma 1. Then for  $0 < r < \varrho \leq \varepsilon$ ,  $0 < p \leq \infty$ , and  $f$  continuous with  $|f|^{\min(1,p)}$  subharmonic in  $\mathcal{D}$ ,*

$$(2.4) \quad M_p(f, \varrho) \leq CM_p(f, r).$$

PROOF. That (2.4) is valid for  $p = \infty$  is obvious. We assume  $0 < p < \infty$ . Since  $|f|^p$  is subharmonic in  $\mathcal{D}_r$  and continuous on  $\overline{\mathcal{D}_r}$ , by the reproducing property of the Poisson kernel, we get

$$|f(\zeta)|^p \leq \int_{\partial \mathcal{D}_r} |f(\xi)|^p P_r(\zeta, \xi) d\sigma_r(\xi) \quad (\zeta \in \mathcal{D}_r).$$

Now Lemma 3 gives

$$\begin{aligned} M_p^p(f, \varrho) &= \int_{\partial \mathcal{D}_\varrho} |f(\zeta)|^p d\sigma_\varrho(\zeta) \\ &\leq \int_{\partial \mathcal{D}_r} |f(\xi)|^p d\sigma_r(\xi) \int_{\partial \mathcal{D}_\varrho} P_r(\zeta, \xi) d\sigma_\varrho(\zeta) \\ &\leq C \int_{\partial \mathcal{D}_r} |f(\xi)|^p d\sigma_r(\xi) = CM_p^p(f, r). \end{aligned}$$

For  $r > 0$  sufficiently small,  $\alpha > 1$ , and  $z \in \partial \mathcal{D}_r$ , as in [6, p. 297] we set

$$\Gamma_{r,\alpha}(z) = \{w \in \mathcal{D}_r : |w - z| < \alpha \delta_r(w)\}.$$

For  $f \in H(\mathcal{D})$  and  $z \in \partial \mathcal{D}_r$ , define

$$f_{1,r}^{*,\alpha}(z) := \sup_{w \in \Gamma_{r,\alpha}(z)} |f(w)|.$$

LEMMA 5. *If  $\alpha > 1$  and  $r > 0$  is sufficiently small, then for  $0 < p < \infty$  and  $f \in H(\mathcal{D})$ ,*

$$(2.5) \quad \|f_{1,r}^{*,\alpha}\|_{L^p(\partial \mathcal{D}_r)}^p \leq CM_p^p(f, r),$$

where  $C$  depends on  $\alpha$  but not on  $r$ .

*Proof.* We know from [6, p. 304, (8.5.6)] that

$$\|f_{1,r}^{*,\alpha}\|_{L^p(\partial\mathcal{D}_r)}^p \leq C_{\alpha,r} M_p^p(f,r).$$

The only thing we should prove is that  $C_{\alpha,r}$  can be chosen so as not to depend on  $r$ . Analysing the corresponding results (8.4.4), (8.5.5), (8.5.6) in [6] carefully, one can find that  $C_{\alpha,r}$  depends only on the curvature of  $\partial\mathcal{D}_r$  and on the properties of the Poisson kernel  $P_r(\cdot, \cdot)$  (see [6, pp. 290–291]). Because  $\mathcal{D}$  is a bounded domain with smooth boundary, we see that (2.5) is exactly valid if  $r > 0$  is sufficiently small.

There are infinitely many defining functions of  $\mathcal{D}$ . For two defining functions  $\lambda_1(z)$  and  $\lambda_2(z)$ , we use  $\mathcal{D}_r^j$  and  $d\sigma_r^j$  to denote  $\{z : \lambda_j(z) < -r\}$  and the surface measure on  $\partial\mathcal{D}_r^j$ ,  $j = 1, 2$ .

**LEMMA 6.** *Let  $\lambda_1(z)$  and  $\lambda_2(z)$  be two defining functions of  $\mathcal{D}$ . There are positive constants  $\varepsilon$ ,  $c_1$  and  $c_2$  such that for  $0 < r \leq \varepsilon$ ,  $0 < p \leq \infty$ , and  $f \in H(\mathcal{D})$ ,*

$$\frac{1}{C} \int_{\partial\mathcal{D}_{c_1r}^2} |f(\xi)|^p d\sigma_{c_1r}^2(\xi) \leq \int_{\partial\mathcal{D}_r^1} |f(\xi)|^p d\sigma_r^1(\xi) \leq C \int_{\partial\mathcal{D}_{c_2r}^2} |f(\xi)|^p d\sigma_{c_2r}^2(\xi).$$

*Proof.* As in the estimates of [6, p. 297], we have two positive constants  $\varepsilon$  and  $\lambda$  so that if  $0 < r < \varepsilon$ , then

$$(2.6) \quad \int_{\partial\mathcal{D}_r^2} |f(\xi)|^p d\sigma_r^2(\xi) \leq Cr^{-1} \int_{S(r)} |f(\xi)|^p dm(\xi),$$

where  $S(r) = \{z : -3r/\lambda^2 \leq \lambda_1(z) \leq -\lambda^2 r/3\}$ . By Lemma 4,

$$(2.7) \quad r^{-1} \int_{S(r)} |f(\xi)|^p dm(\xi) \leq Cr^{-1} \int_{\lambda^2 r/3}^{3r/\lambda^2} d\rho \int_{\partial\mathcal{D}_\rho^1} |f(\xi)|^p d\sigma_\rho^1(\xi) \\ \leq C \int_{\partial\mathcal{D}_{\lambda^2 r/3}^1} |f(\xi)|^p d\sigma_{\lambda^2 r/3}^1(\xi).$$

Now (2.6) and (2.7) imply

$$\int_{\partial\mathcal{D}_r^2} |f(\xi)|^p d\sigma_r^2(\xi) \leq C \int_{\partial\mathcal{D}_{\lambda^2 r/3}^1} |f(\xi)|^p d\sigma_{\lambda^2 r/3}^1(\xi).$$

This immediately gives the conclusion of the lemma.

**LEMMA 7.** *For  $z_0 \in \mathcal{D}$  fixed, there exists  $L > 0$  such that every  $z \in \overline{\mathcal{D}}$  can be connected with  $z_0$  by a broken line which lies in  $\overline{\mathcal{D}}$  and has length less than  $L$ .*

Proof. By Lemma 1, we have  $r > 0$  so that for  $\xi \in \partial\mathcal{D}$  and  $0 \leq t \leq 1$ ,

$$(2.7^*) \quad \xi + trn_\xi \in \overline{\mathcal{D}} \quad \text{and} \quad \delta(\xi + trn_\xi) = tr.$$

Since  $\mathcal{D}$  is a connected open subset of  $\mathbb{R}^{2n}$ , every  $z \in \mathcal{D}$  can be connected with  $z_0$  by some broken line  $\Gamma$  which lies in  $\mathcal{D}$ . Set

$$L(z) = \inf\{\text{length of } \Gamma : \Gamma \text{ lies in } \mathcal{D} \text{ and connects } z \text{ with } z_0\}.$$

Obviously,  $L(z)$  is upper-semicontinuous (for the definition, we refer to [6]). Hence  $L(z)$  is bounded above on the compact subset  $\{z \in \mathcal{D} : \delta(z) \geq r\}$ , say

$$(2.8) \quad L(z) < L - r \quad \text{whenever } \delta(z) \geq r.$$

For  $z \in \overline{\mathcal{D}}$ ,  $\delta(z) < r$ , let  $\xi$  be a point on  $\partial\mathcal{D}$  such that  $|\xi - z| = \delta(z)$ . Then by (2.7\*),  $z$  and  $\xi + rn_\xi$  can be connected by a segment of length not more than  $r$ . This and (2.8) give the conclusion of the lemma.

LEMMA 8. *If  $f$  is holomorphic in  $B(z, r) = \{w \in \mathbb{C}^n : |w - z| < r\}$ , then for  $0 < p < \infty$ ,*

$$|\nabla f(z)|^p \leq Cr^{-(2n+p)} \int_{B(z,r)} |f(w)|^p dm(w),$$

where  $C$  is independent of  $r$ ,  $f$  and  $\nabla f(z) = (\partial f/\partial z_1, \dots, \partial f/\partial z_n)$ .

Proof. For  $f \in H(B(z, r))$ , define  $g(w) = f(z + rw)$ ,  $w \in B(0, 1)$ . From Lemma 2 of [9],

$$|\nabla g(0)|^p \leq C \int_{B(0,1)} |g(w)|^p dm(w).$$

By a change of variables in the integral, we obtain

$$r^p |\nabla f(z)|^p \leq Cr^{-2n} \int_{B(z,r)} |f(w)|^p dm(w).$$

This is the desired inequality.

LEMMA 9. *Let  $1 \leq k < \infty$ ,  $s > -1$ ,  $l > 0$ , and let  $h : (0, \varepsilon) \rightarrow [0, \infty)$  be measurable. Then there exists a constant  $C$  independent of  $\varepsilon$  so that*

$$(2.9) \quad \int_0^\varepsilon r^s dr \left\{ \int_r^\varepsilon (\varrho - r)^{l-1} h(\varrho) d\varrho \right\}^k \leq C \int_0^\varepsilon r^{s+kl} h(r)^k dr.$$

Proof. The substitutions  $r = (1 - u)\varepsilon$  and  $\varrho = (1 - t)\varepsilon$  give

$$(2.10) \quad \int_0^\varepsilon r^s dr \left\{ \int_r^\varepsilon (\varrho - r)^{l-1} h(\varrho) d\varrho \right\}^k$$

$$\begin{aligned}
&= \int_0^1 \varepsilon^s (1-u)^s \varepsilon du \left\{ \int_{(1-u)\varepsilon}^{\varepsilon} (\varrho - (1-u)\varepsilon)^{l-1} h(\varrho) d\varrho \right\}^k \\
&= \varepsilon^{s+kl+1} \int_0^1 (1-u)^s du \left\{ \int_0^u (u-t)^{l-1} h((1-t)\varepsilon) dt \right\}^k.
\end{aligned}$$

Similarly,

$$(2.11) \quad \int_0^{\varepsilon} r^{s+kl} h(r)^k dr = \varepsilon^{s+kl+1} \int_0^1 (1-u)^{s+kl} h((1-u)\varepsilon)^k du.$$

By inequality (9.2) of [2, p. 758],

$$\begin{aligned}
\int_0^1 (1-u)^s du \left\{ \int_0^u (u-t)^{l-1} h((1-t)\varepsilon) dt \right\}^k \\
\leq C \int_0^1 (1-u)^{s+kl} h((1-u)\varepsilon)^k du,
\end{aligned}$$

which combined with (2.10) and (2.11) proves (2.9).

LEMMA 10. *If  $\beta > 0$ ,  $0 < p < q < \infty$ , and  $h(t)$  is a positive continuous function of  $t \in (0, 1)$  satisfying  $h(t_1) \leq Ch(t_2)$  whenever  $0 < t_1 < t_2 < 1$ , then*

$$(2.12) \quad \left\{ \int_0^1 (1-t)^{\beta q-1} h(t)^q dt \right\}^{1/q} \leq C \left\{ \int_0^1 (1-t)^{\beta p-1} h(t)^p dt \right\}^{1/p}.$$

PROOF. This lemma is an improved version of Lemma 8 of [9], with the hypothesis “ $h(t)$  is nondecreasing” replaced by “ $h(t_1) \leq Ch(t_2)$  whenever  $0 < t_1 < t_2 < 1$ ”. Imitating the proof of Lemma 5 of [8], we get

$$\begin{aligned}
\int_0^1 (1-t)^{\beta p-1} h(t)^p dt &\geq \int_t^1 (1-t)^{\beta p-1} h(t)^p dt \\
&\geq Ch(t)^p (1-t)^{\beta p} \quad (t \in (0, 1)).
\end{aligned}$$

Then

$$\begin{aligned}
&\left\{ \int_0^1 (1-t)^{\beta q-1} h(t)^q dt \right\}^{1/q} \\
&\leq \sup_{0 < t < 1} \{(1-t)^\beta h(t)\}^{(q-p)/q} \left\{ \int_0^1 (1-t)^{\beta p-1} h(t)^p dt \right\}^{1/q} \\
&\leq C \left\{ \int_0^1 (1-t)^{\beta p-1} h(t)^p dt \right\}^{1/p},
\end{aligned}$$

and (2.12) is proved.



**3. Proof of Theorems 1 and 2.** To prove Theorems 1 and 2 for any defining function, by Lemma 6 we need only prove they hold for some defining function. Therefore we can take the defining function to be

$$(3.1) \quad \lambda(z) = \begin{cases} -\delta(z), & z \in \overline{\mathcal{D}}, \\ \delta(z), & z \notin \overline{\mathcal{D}}. \end{cases}$$

**Proof of Theorem 1.** Let  $\varepsilon$  be as in Lemma 1. Suppose  $|\alpha| = 1$ . We have to prove

$$(3.2) \quad \int_0^\varepsilon r^{s+q} M_p^q(D^\alpha f, r) dr \leq C \int_0^\varepsilon r^s M_p^q(f, r) dr$$

and

$$(3.3) \quad \sup_{0 < r \leq \varepsilon} r^{s+1} M_p(D^\alpha f, r) \leq C \sup_{0 < r \leq \varepsilon} r^s M_p(f, r).$$

If  $0 < p < \infty$ , then by Lemma 8,

$$|\nabla f(z)|^p \leq C \delta(z)^{-(2n+p)} \int_{B(z, \delta(z)/2)} |f(w)|^p dm(w).$$

Let  $\chi_{B(z,r)}$  denote the characteristic function of  $B(z, r)$ . Then

$$\begin{aligned} M_p^p(\nabla f, r) &\leq Cr^{-(2n+p)} \int_{\partial \mathcal{D}_r} d\sigma_r(z) \int_{B(z, \delta(z)/2)} |f(w)|^p dm(w) \\ &= Cr^{-(2n+p)} \int_{\partial \mathcal{D}_r} d\sigma_r(z) \int_{\mathcal{D}} \chi_{B(z, \delta(z)/2)}(w) |f(w)|^p dm(w) \\ &= Cr^{-(2n+p)} \int_{\mathcal{D}} |f(w)|^p dm(w) \int_{\partial \mathcal{D}_r} \chi_{B(z, r/2)}(w) d\sigma_r(z) \\ &\leq Cr^{-(2n+p)} \int_{-3r/2 < \lambda(w) < -r/2} |f(w)|^p dm(w) \\ &\quad \times \int_{\partial \mathcal{D}_r} \chi_{B(w, r/2)}(z) d\sigma_r(z). \end{aligned}$$

But for  $-3r/2 < \lambda(w) < -r/2$ ,

$$\int_{\partial \mathcal{D}_r} \chi_{B(w, r/2)}(z) d\sigma_r(z) \leq Cr^{2n-1}.$$

Therefore

$$(3.4) \quad \begin{aligned} M_p^p(\nabla f, r) &\leq Cr^{-(p+1)} \int_{-3r/2 < \lambda(w) < -r/2} |f(w)|^p dm(w) \\ &\leq Cr^{-(p+1)} \int_{r/2}^{3r/2} d\rho \int_{\partial \mathcal{D}_\rho} |f(w)|^p d\sigma_\rho(w). \end{aligned}$$

To get the above inequality, we have used the “polar coordinates” (see also [6]). Now Lemma 4 gives

$$(3.4^*) \quad M_p^p(\nabla f, r) \leq Cr^{-p} \int_{\partial\mathcal{D}_{r/2}} |f(w)|^p d\sigma_{r/2}(w).$$

If  $p = \infty$ , then by Lemma 7 and the maximum modulus principle,

$$(3.5) \quad \begin{aligned} |\nabla f(z)| &\leq C\delta(z)^{-1} \sup_{w \in B(z, \delta(z)/2)} |f(w)| \\ &\leq Cr^{-1}M_\infty(f, r/2) \quad \text{for } z \in \partial\mathcal{D}_r. \end{aligned}$$

Combine (3.4\*) and (3.5) to obtain

$$(3.6) \quad rM_p(\nabla f, r) \leq CM_p(f, r/2) \quad \text{for } 0 < p \leq \infty.$$

Then we get

$$(3.7) \quad r^{s+q}M_p^q(\nabla f, r) \leq Cr^sM_p^q(f, r/2)$$

for  $0 < p \leq \infty$ ,  $0 < q < \infty$ , and  $s > -1$ . Therefore

$$\begin{aligned} \int_0^\varepsilon r^{s+q}M_p^q(\nabla f, r) dr &\leq C \int_0^\varepsilon r^sM_p^q(f, r/2) dr \\ &\leq C \int_0^\varepsilon r^sM_p^q(f, r) dr \end{aligned}$$

and

$$(3.8) \quad \begin{aligned} \sup_{0 < r \leq \varepsilon} r^{s+1}M_p(\nabla f, r) &\leq C \sup_{0 < r \leq \varepsilon} r^sM_p(f, r/2) \\ &\leq C \sup_{0 < r \leq \varepsilon} r^sM_p(f, r). \end{aligned}$$

This proves (3.2) and (3.3).

The general case can be proved by induction. Theorem 1 is proved.

Recall that we have chosen the defining function (3.1). By Lemma 1, we have  $\varepsilon > 0$  so that  $\xi + rn_\xi \in \partial\mathcal{D}_r$  for  $\xi \in \partial\mathcal{D}$  and  $0 < r \leq \varepsilon$ . Now we define the mapping

$$(3.9) \quad \tau_{r,t} : \partial\mathcal{D}_r \rightarrow \partial\mathcal{D}_t, \quad z \mapsto \pi(z) + tn_{\pi(z)},$$

for  $0 < t \leq \varepsilon$ , where  $\pi(z)$  is the unique point on  $\partial\mathcal{D}$  closest to  $z$ . Then  $\tau_{r,t}$  is one-to-one and the corresponding Jacobian  $J_{r,t}(z)$  satisfies

$$(3.10) \quad C_1 \leq J_{r,t}(z) \leq C_2 \quad (r, t \in (0, \varepsilon], z \in \partial\mathcal{D}_r).$$

The following lemma will be needed in the proof of Theorem 2.

LEMMA 11. *There exists  $\varepsilon > 0$  so that for  $0 < r \leq \varepsilon$  and  $f \in H(\mathcal{D})$ :*

(a) *If  $1 \leq p \leq \infty$ , then*

$$(3.11) \quad M_p(f - f \circ \tau_{r,\varepsilon}, r) \leq C(\varepsilon - r) \int_0^1 M_p(\nabla f, tr + (1-t)\varepsilon) dt.$$

(b) *If  $0 < p < 1$ , then*

$$(3.12) \quad M_p^p(f - f \circ \tau_{r,\varepsilon}, r) \leq C(\varepsilon - r)^p \int_0^1 (1-t)^{p-1} M_p^p(\nabla f, tr + (1-t)\varepsilon) dt.$$

Proof. Let  $\varepsilon$  be as in Lemma 1. For  $z \in \partial\mathcal{D}_r$ ,

$$\begin{aligned} |f(z) - f \circ \tau_{r,\varepsilon}(z)| &\leq \int_0^1 |\nabla f(tz + (1-t)\tau_{r,\varepsilon}(z))| \cdot |z - \tau_{r,\varepsilon}(z)| dt \\ &\leq (\varepsilon - r) \int_0^1 \sup_{z \in \partial\mathcal{D}_r} |\nabla f(tz + (1-t)\tau_{r,\varepsilon}(z))| dt. \end{aligned}$$

(a) If  $p = \infty$ , then

$$\begin{aligned} (3.13) \quad M_\infty(f - f \circ \tau_{r,\varepsilon}, r) &= \sup_{z \in \partial\mathcal{D}_r} |f(z) - f \circ \tau_{r,\varepsilon}(z)| \\ &\leq (\varepsilon - r) \int_0^1 \sup_{z \in \partial\mathcal{D}_r} |\nabla f(tz + (1-t)\tau_{r,\varepsilon}(z))| dt \\ &= (\varepsilon - r) \int_0^1 M_\infty(\nabla f, tr + (1-t)\varepsilon) dt. \end{aligned}$$

If  $1 \leq p < \infty$ , apply Minkowski's inequality to get

$$\begin{aligned} M_p(f - f \circ \tau_{r,\varepsilon}, r) &\leq (\varepsilon - r) \left\{ \int_{\partial\mathcal{D}_r} d\sigma_r(z) \left[ \int_0^1 |\nabla f(tz + (1-t)\tau_{r,\varepsilon}(z))|^p dt \right]^p \right\}^{1/p} \\ &\leq (\varepsilon - r) \int_0^1 dt \left[ \int_{\partial\mathcal{D}_r} |\nabla f(tz + (1-t)\tau_{r,\varepsilon}(z))|^p d\sigma_r(z) \right]^{1/p}. \end{aligned}$$

Setting  $w = tz + (1-t)\tau_{r,\varepsilon}(z)$  in the inner integral, by (3.10) we obtain

$$(3.14) \quad M_p(f - f \circ \tau_{r,\varepsilon}, r) \leq C(\varepsilon - r) \int_0^1 \left[ \int_{\partial\mathcal{D}_{tr+(1-t)\varepsilon}} |\nabla f(w)|^p d\sigma_{tr+(1-t)\varepsilon}(w) \right]^{1/p} dt.$$

Now (3.11) follows from (3.13) and (3.14).

(b) If  $0 < p < 1$ , set  $t_k = 1 - 2^{-k}$ . Applying the lemma on p. 57 of [1], we get

$$\begin{aligned}
& \int_{\partial \mathcal{D}_r} |f(z) - f \circ \tau_{r,\varepsilon}(z)|^p d\sigma_r(z) \\
& \leq (\varepsilon - r)^p \int_{\partial \mathcal{D}_r} \left[ \sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} |\nabla f(tz + (1-t)\tau_{r,\varepsilon}(z))| dt \right]^p d\sigma_r(z) \\
& \leq (\varepsilon - r)^p \int_{\partial \mathcal{D}_r} \left[ \sum_{j=1}^{\infty} \sup_{t_{j-1} \leq t \leq t_j} |\nabla f(tz + (1-t)\tau_{r,\varepsilon}(z))| (t_j - t_{j-1}) \right]^p d\sigma_r(z) \\
& \leq (\varepsilon - r)^p \int_{\partial \mathcal{D}_r} \sum_{j=1}^{\infty} \sup_{t_{j-1} \leq t \leq t_j} |\nabla f(tz + (1-t)\tau_{r,\varepsilon}(z))|^p 2^{-pj} d\sigma_r(z) \\
& \leq C(\varepsilon - r)^p \int_{\partial \mathcal{D}_r} \sum_{j=1}^{\infty} |(\nabla f)_{1,t_j r + (1-t_j)\varepsilon}^{*,10}(t_j z + (1-t_j)\tau_{r,\varepsilon}(z))|^p 2^{-pj} d\sigma_r(z) \\
& \leq C(\varepsilon - r)^p \sum_{j=1}^{\infty} 2^{-pj} \int_{\partial \mathcal{D}_{t_j r + (1-t_j)\varepsilon}} |(\nabla f)_{1,t_j r + (1-t_j)\varepsilon}^{*,10}(w)|^p d\sigma_{t_j r + (1-t_j)\varepsilon}(w).
\end{aligned}$$

Applying Lemma 5 to  $\partial f / \partial z_j$  ( $j = 1, \dots, n$ ), we find that

$$M_p^p(f - f \circ \tau_{r,\varepsilon}, r) \leq C(\varepsilon - r)^p \sum_{j=1}^{\infty} 2^{-pj} M_p^p(\nabla f, t_j r + (1-t_j)\varepsilon).$$

Using the same method as that on p. 628 of [9], and Lemma 4, we get (3.12). The lemma is proved.

**Proof of Theorem 2.** Let us first deal with  $m = 1$ . For  $\varepsilon > 0$  small enough, it is sufficient to prove

$$(3.15) \quad \int_0^\varepsilon r^s M_p^q(f, r) dr \leq C \left\{ |f(z_0)|^q + \int_0^\varepsilon r^{s+q} M_p^q(\nabla f, r) dr \right\}.$$

This will be a trivial consequence of the following two inequalities:

$$(3.16) \quad \int_0^\varepsilon r^s M_p^q(f - f \circ \tau_{r,\varepsilon}, r) dr \leq C \int_0^\varepsilon r^{s+q} M_p^q(\nabla f, r) dr$$

and

$$(3.17) \quad \sup_{z \in \overline{\mathcal{D}_\varepsilon}} |f(z) - f(z_0)|^q \leq C \int_0^\varepsilon r^{s+q} M_p^q(\nabla f, r) dr.$$

In fact,

$$\begin{aligned} \int_0^\varepsilon r^s M_p^q(f, r) dr &\leq C \left\{ \int_0^\varepsilon r^s M_p^q(f - f \circ \tau_{r,\varepsilon}, r) dr \right. \\ &\quad \left. + \sup_{z \in \mathcal{D} \setminus \mathcal{D}_\varepsilon} |f \circ \tau_{r,\varepsilon}(z) - f(z_0)|^q + |f(z_0)|^q \right\} \\ &\leq C \left\{ |f(z_0)|^q + \int_0^\varepsilon r^{s+q} M_p^q(\nabla f, r) dr \right\}. \end{aligned}$$

The proof of (3.16) will be divided into four steps.

Case 1:  $1 \leq p \leq \infty$  and  $q \geq 1$ . By Lemma 11, we have  $\varepsilon > 0$  such that

$$\int_0^\varepsilon r^s M_p^q(f - f \circ \tau_{r,\varepsilon}, r) dr \leq C \int_0^\varepsilon r^s \left[ \int_0^1 (\varepsilon - r) M_p(\nabla f, tr + (1-t)\varepsilon) dt \right]^q dr.$$

Setting  $\varrho = tr + (1-t)\varepsilon$  in the inner integral gives

$$\int_0^\varepsilon r^s M_p^q(f - f \circ \tau_{r,\varepsilon}, r) dr \leq C \int_0^\varepsilon r^s \left[ \int_r^\varepsilon M_p(\nabla f, \varrho) d\varrho \right]^q dr.$$

Taking  $l = 1$ ,  $k = q$ , and  $h(r) = M_p(\nabla f, r)$  in Lemma 9 gives (3.16).

Case 2:  $1 \leq p \leq \infty$  and  $0 < q < 1$ . Taking  $\beta = 1$  and  $h(t) = M_p(\nabla f, tr + (1-t)\varepsilon)$  in Lemma 10 gives

$$\begin{aligned} &\left\{ \int_0^1 (1-t)^{p-1} M_p^p(\nabla f, tr + (1-t)\varepsilon) dt \right\}^{1/p} \\ &\leq C \left\{ \int_0^1 (1-t)^{q-1} M_p^q(\nabla f, tr + (1-t)\varepsilon) dt \right\}^{1/q}. \end{aligned}$$

Hence Lemma 11 yields

$$M_p^q(f - f \circ \tau_{r,\varepsilon}, r) \leq C(\varepsilon - r)^q \left\{ \int_0^1 (1-t)^{q-1} M_p^q(\nabla f, tr + (1-t)\varepsilon) dt \right\}.$$

Setting  $\varrho = tr + (1-t)\varepsilon$  in the integral shows that

$$\int_0^1 (1-t)^{q-1} M_p^q(\nabla f, tr + (1-t)\varepsilon) dt = (\varepsilon - r)^{-q} \int_r^\varepsilon (\varrho - r)^{q-1} M_p^q(\nabla f, \varrho) d\varrho.$$

Thus

$$M_p^q(f - f \circ \tau_{r,\varepsilon}, r) \leq C \int_r^\varepsilon (\varrho - r)^{q-1} M_p^q(\nabla f, \varrho) d\varrho.$$

Now by Lemma 9 again we get

$$\begin{aligned} \int_0^\varepsilon r^s M_p^q(f - f \circ \tau_{r,\varepsilon}, r) dr &\leq C \int_0^\varepsilon r^s \left[ \int_r^\varepsilon (\varrho - r)^{q-1} M_p^q(\nabla f, \varrho) d\varrho \right] dr \\ &\leq C \int_0^\varepsilon r^{s+q} M_p^q(\nabla f, r) dr. \end{aligned}$$

Case 3:  $0 < p < 1$  and  $p \geq q$ . Lemma 11 gives

$$M_p^q(f - f \circ \tau_{r,\varepsilon}, r) \leq C(\varepsilon - r)^q \left[ \int_0^1 (1-t)^{p-1} M_p^p(\nabla f, tr + (1-t)\varepsilon) dt \right]^{q/p}.$$

As in Case 2, by Lemma 10 we get

$$\begin{aligned} M_p^q(f - f \circ \tau_{r,\varepsilon}, r) &\leq C(\varepsilon - r)^q \int_0^1 (1-t)^{pq/p-1} M_p^{pq/p}(\nabla f, tr + (1-t)\varepsilon) dt \\ &= C \int_r^\varepsilon (\varrho - r)^{q-1} M_p^q(\nabla f, \varrho) d\varrho. \end{aligned}$$

Now (3.16) follows as in Case 2.

Case 4:  $0 < p < 1$  and  $p < q$ . By Lemma 11 and Lemma 9,

$$\begin{aligned} \int_0^\varepsilon r^s M_p^q(f - f \circ \tau_{r,\varepsilon}, r) dr &\leq C \int_0^\varepsilon r^s (\varepsilon - r)^q \left[ \int_0^1 (1-t)^{p-1} M_p^p(\nabla f, tr + (1-t)\varepsilon) dt \right]^{q/p} dr \\ &= C \int_0^\varepsilon r^s \left[ \int_r^\varepsilon (\varrho - r)^{p-1} M_p^p(\nabla f, \varrho) d\varrho \right]^{q/p} dr \\ &\leq C \int_0^\varepsilon r^{s+pq/p} M_p^{pq/p}(\nabla f, r) dr. \end{aligned}$$

Thus (3.16) is proved.

Now for fixed  $z_0$  we prove (3.17). Without loss of generality we may assume  $z_0 \in \mathcal{D}_\varepsilon$ . Applying Lemma 7 to the domain  $\mathcal{D}_\varepsilon$  (if  $\varepsilon > 0$  is small enough,  $\mathcal{D}_\varepsilon$  has  $C^\infty$  boundary), we have  $L > 0$  so that any  $z$  and  $z_0$  in  $\mathcal{D}_\varepsilon$  can be connected by some broken line  $\Gamma(t)$  ( $0 \leq t \leq 1$ ,  $\Gamma(0) = z_0$  and  $\Gamma(1) = z$ ) in  $\overline{\mathcal{D}_\varepsilon}$ , with length  $\leq L$ . Then

$$(3.18) \quad |f(z) - f(z_0)| = \left| \int_0^1 (\nabla f)(\Gamma(t)) \cdot \Gamma'(t) dt \right| \leq L \sup_{z \in \overline{\mathcal{D}_\varepsilon}} |\nabla f(z)|.$$

Since  $|\nabla f|^p$  is “plurisubharmonic”, we have some  $z' \in \partial\mathcal{D}_{\varepsilon/2}$  so that

$$(3.19) \quad \sup_{z \in \overline{\mathcal{D}}_\varepsilon} |\nabla f(z)| \leq C |\nabla f(z')|$$

$$\leq C \left\{ |B(z', \varepsilon/4)|^{-1} \int_{B(z', \varepsilon/4)} |\nabla f(w)| dm(w) \right\}^{1/p}$$

$$\leq CM_p(\nabla f, \varepsilon/4) \leq C \left\{ \int_{\varepsilon/8}^{\varepsilon/4} r^{s+q} M_p^p(f, r) dr \right\}^{1/q}.$$

To derive the last two inequalities above we have applied Lemma 4 to functions  $\partial f / \partial z_j$  for  $j = 1, \dots, n$ . Inequality (3.17) now follows from (3.18) and (3.19).

For  $m = 2$ , applying (3.15) twice, we have

$$\int_0^\varepsilon r^s M_p^q(f, r) dr \leq C \left\{ |f(z_0)|^q + \int_0^\varepsilon r^{s+q} M_p^q(\nabla f, r) dr \right\}$$

$$\leq C \left\{ |f(z_0)|^q + \sum_{j=1}^n \int_0^\varepsilon r^{s+q} M_p^q \left( \frac{\partial f}{\partial z_j}, r \right) dr \right\}$$

$$\leq C \left\{ |f(z_0)|^q + \sum_{j=1}^n \left| \frac{\partial f}{\partial z_j}(z_0) \right|^q \right.$$

$$\left. + \sum_{j=1}^n \int_0^\varepsilon r^{s+2q} M_p^q \left( \nabla \left( \frac{\partial f}{\partial z_j} \right), r \right) dr \right\}$$

$$\leq C \left\{ \sum_{|\alpha| < 2} |D^\alpha f(z_0)|^q + \sum_{|\alpha|=2} \int_0^\varepsilon r^{s+2q} M_p^q(D^\alpha f, r) dr \right\}.$$

As in Theorem 1, the general case can be proved by induction. The proof of Theorem 2 is complete.

Relating to (1.1) in Theorem 1 we have the following:

**THEOREM 4.** *Let  $0 < p \leq \infty$  and  $s > 0$ . Then for  $z_0 \in \mathcal{D}$  fixed and  $m \in \mathbb{N}$ , there exists  $\varepsilon > 0$  such that for  $f \in H(\mathcal{D})$ ,*

$$\sup_{0 < r \leq \varepsilon} r^s M_p(f, r) \leq C \left\{ \sum_{|\alpha| < m} |D^\alpha f(z_0)| + \sum_{|\alpha|=m} \sup_{0 < r \leq \varepsilon} r^{s+m} M_p(D^\alpha f, r) \right\}.$$

**Proof.** For  $m = 1$  it suffices to prove

$$(3.20) \quad \sup_{0 < r \leq \varepsilon} r^s M_p(f - f \circ \tau_{r, \varepsilon}, r) \leq C \sup_{0 < r \leq \varepsilon} r^{s+1} M_p(\nabla f, r),$$

and

$$(3.21) \quad \sup_{z \in \mathcal{D}_\varepsilon} |f(z) - f(z_0)| \leq C \sup_{0 < r \leq \varepsilon} r^{s+1} M_p(\nabla f, r)$$

for  $\varepsilon > 0$  small enough. (3.21) is almost trivial (see also (3.18)). To prove (3.20), we consider two cases.

Case 1:  $1 \leq p \leq \infty$ . By Lemma 11, we have  $\varepsilon > 0$  such that for  $f \in H(\mathcal{D})$ ,

$$\begin{aligned} M_p(f - f \circ \tau_{r,\varepsilon}, r) &\leq C \int_r^\varepsilon M_p(\nabla f, t) dt \\ &\leq C \left( \sup_{0 < r \leq \varepsilon} r^{s+1} M_p(\nabla f, r) \right) \int_r^\varepsilon t^{-(s+1)} dt \\ &\leq C \left( \sup_{0 < r \leq \varepsilon} r^{s+1} M_p(\nabla f, r) \right) r^{-s}. \end{aligned}$$

This gives (3.20).

Case 2:  $0 < p < 1$ . By Lemma 11 again,

$$\begin{aligned} (3.22) \quad M_p^p(f - f \circ \tau_{r,\varepsilon}, r) &\leq C \int_r^\varepsilon (\varrho - r)^{p-1} M_p^p(\nabla f, \varrho) d\varrho \\ &\leq C \left( \sup_{0 < r \leq \varepsilon} r^{s+1} M_p(\nabla f, r) \right)^p \int_r^\varepsilon (\varrho - r)^{p-1} \varrho^{-(s+1)p} d\varrho. \end{aligned}$$

Integration by parts gives

$$(3.23) \quad \int_r^\varepsilon (\varrho - r)^{p-1} \varrho^{-(s+1)p} d\varrho \leq C r^{-sp}.$$

From (3.22) and (3.23), (3.20) follows.

For  $m > 1$ , the conclusion can be proved by induction. The proof is complete.

**4. Pluriharmonic conjugates.** It is well known that if  $f \in H(\mathcal{D})$  then both  $u = \operatorname{Re} f$  and  $v = \operatorname{Im} f$  are pluriharmonic in  $\mathcal{D}$  and  $v$  is the pluriharmonic conjugate of  $u$ . Conversely, if  $\mathcal{D}$  is simply connected [4, p. 311], then every pluriharmonic function on  $\mathcal{D}$  is the real part of a holomorphic function [14, p. 44]. In [12], Stoll proved the following:

**THEOREM A.** *Let  $f = u + iv$  be holomorphic in  $B$ , the unit ball of  $\mathbb{C}^n$ ,*



with  $f(0)$  real. Then for  $1 \leq p \leq \infty$ ,  $0 < q < \infty$ , and  $s > -1$ ,

$$\int_0^1 (1-r)^s M_p^q(v, r) dr \leq C \int_0^1 (1-r)^s M_p^q(u, r) dr.$$

In [8], Shi generalized this theorem to bounded symmetric domains of  $\mathbb{C}^n$ . Restricting himself to the unit ball of  $\mathbb{C}^n$ , Shi proved in [9] that Theorem A is still valid for  $0 < p < 1$ . In both [9, 10] Shi mentioned the problem whether Theorem A holds on arbitrary bounded symmetric domains for all possible  $p \in (0, \infty]$ . We have solved this problem affirmatively in [5]. Theorem 3 shows that Theorem A can be generalized in another direction; that is,  $B$  can be replaced by any bounded domain  $\mathcal{D}$  with  $C^\infty$  boundary for all  $p \in (0, \infty]$ .

**Proof of Theorem 3.** First, we prove that for some  $\varepsilon > 0$ ,

$$(4.1) \quad \int_0^\varepsilon r^{s+q} M_p^q(\nabla f, r) dr \leq C \int_0^\varepsilon r^s M_p^q(u, r) dr.$$

The proof of this inequality uses the same approach as in the proof of Theorem 1. Four cases will be considered.

**Case 1:**  $0 < p = q < \infty$ . Similarly to Lemma 8, we know from formula (35) of [9] that

$$(4.2) \quad |\nabla f(z)|^p \leq C \delta(z)^{-(2n+p)} \int_{B(z, \delta(z)/2)} |u(w)|^p dm(w).$$

Since  $r/C \leq \delta(z) \leq Cr$  for  $r$  small enough and  $z \in \partial\mathcal{D}_r$ , we have positive constants  $c_1, c_2$  (we may assume  $c_1 < 1 < c_2$ ) independent of  $r$  so that

$$\bigcup_{z \in \partial\mathcal{D}_r} B(z, \delta(z)/2) \subset \{w : -c_2 r \leq \lambda(w) \leq -c_1 r\}.$$

Therefore

$$\begin{aligned} (4.3) \quad & \int_{\partial\mathcal{D}_r} |\nabla f(z)|^p d\sigma_r(z) \\ & \leq Cr^{-(2n+p)} \int_{\partial\mathcal{D}_r} d\sigma_r(z) \int_{B(z, \delta(z)/2)} |u(w)|^p dm(w) \\ & \leq Cr^{-(2n+p)} \int_{\partial\mathcal{D}_r} d\sigma_r(z) \int_{-c_2 r \leq \lambda(z) \leq -c_1 r} |u(w)|^p \chi_{B(z, \delta(z)/2)}(w) dm(w) \\ & = Cr^{-(2n+p)} \int_{-c_2 r \leq \lambda(z) \leq -c_1 r} |u(w)|^p dm(w) \int_{\partial\mathcal{D}_r} \chi_{B(z, \delta(z)/2)}(w) d\sigma_r(z) \end{aligned}$$

$$\leq Cr^{-(2n+p)+(2n-1)} \int_{c_1 r}^{c_2 r} d\varrho \int_{\partial\mathcal{D}_\varrho} |u(w)|^p d\sigma_\varrho(w).$$

Then for some  $\varepsilon > 0$ ,

$$\begin{aligned} (4.4) \quad \int_0^{\varepsilon/c_2} r^{s+p} M_p^p(\nabla f, r) dr &\leq C \int_0^{\varepsilon/c_2} r^{s-1} dr \int_{c_1 r}^{c_2 r} d\varrho \int_{\partial\mathcal{D}_\varrho} |u(w)|^p d\sigma_\varrho(w) \\ &\leq C \int_0^\varepsilon d\varrho \int_{\partial\mathcal{D}_\varrho} |u(w)|^p d\sigma_\varrho(w) \int_{\varrho/c_2}^{\varrho/c_1} r^{s-1} dr \\ &= C \int_0^\varepsilon \varrho^s M_p^p(u, \varrho) d\varrho. \end{aligned}$$

By the plurisubharmonicity of  $|\nabla f|^p$ ,  $M_p(\nabla f, r) \leq CM_p(\nabla f, \varrho)$  whenever  $0 < \varrho < r \leq \varepsilon$ . Then

$$(4.5) \quad \int_{\varepsilon/c_2}^\varepsilon r^{s+q} M_p^p(\nabla f, r) dr \leq C \int_0^{\varepsilon/c_2} r^{s+q} M_p^p(\nabla f, r) dr.$$

Combine (4.4) and (4.5) to get (4.1).

Case 2:  $0 < q < p < \infty$ . As in Case 1,

$$|\nabla f(z)| \leq C\delta(z)^{-(2n+q)/q} \left\{ \int_{B(z, \delta(z)/2)} |u(w)|^q dm(w) \right\}^{1/q}.$$

Then

$$\begin{aligned} &\left\{ \int_{\partial\mathcal{D}_r} |\nabla f(z)|^p d\sigma_r(z) \right\}^{q/p} \\ &\leq Cr^{-(2n+q)} \left\{ \int_{\partial\mathcal{D}_r} d\sigma_r(z) \left[ \int_{B(z, \delta(z)/2)} |u(w)|^q dm(w) \right]^{p/q} \right\}^{q/p} \\ &\leq Cr^{-(2n+q)} \left\{ \int_{\partial\mathcal{D}_r} d\sigma_r(z) \left[ \int_{c_1 r}^{c_2 r} d\varrho \int_{\partial\mathcal{D}_\varrho} |u(w)|^q \chi_{B(z, \delta(z)/2)}(w) d\sigma_\varrho(w) \right]^{p/q} \right\}^{q/p}. \end{aligned}$$

Minkowski's inequality implies the above is not more than

$$Cr^{-(2n+q)} \int_{c_1 r}^{c_2 r} d\varrho \left\{ \int_{\partial\mathcal{D}_r} d\sigma_r(z) \left[ \int_{\partial\mathcal{D}_\varrho} |u(w)|^q \chi_{B(z, \delta(z)/2)}(w) d\sigma_\varrho(w) \right]^{p/q} \right\}^{q/p}.$$

Using the Hölder inequality in the inner integral with exponents  $p/q$  and  $p/(p-q)$ , we get

$$\begin{aligned}
M_p^q(\nabla f, r) &\leq Cr^{-(2n+q)} \int_{c_1 r}^{c_2 r} d\varrho \left\{ \int_{\partial \mathcal{D}_r} d\sigma_r(z) \left[ \int_{\partial \mathcal{D}_\varrho} (|u(w)|^q \chi_{B(z, \delta(z)/2)}(w)) \right. \right. \\
&\quad \left. \left. \times (\chi_{B(z, \delta(z)/2)}(w)) d\sigma_\varrho(w) \right]^{p/q} \right\}^{q/p} \\
&\leq Cr^{-(2n+q)} \int_{c_1 r}^{c_2 r} d\varrho \left\{ \int_{\partial \mathcal{D}_r} d\sigma_r(z) \right. \\
&\quad \times \left[ \int_{\partial \mathcal{D}_\varrho} |u(w)|^p \chi_{B(z, \delta(z)/2)}(w) d\sigma_\varrho(w) \right] \\
&\quad \times \left[ \int_{\partial \mathcal{D}_\varrho} \chi_{B(z, \delta(z)/2)}(w) d\sigma_\varrho(w) \right]^{(p-q)/q} \left. \right\}^{q/p} \\
&\leq Cr^{-(2n+q)+(2n-1)(p-q)/p} \int_{c_1 r}^{c_2 r} d\varrho \left\{ \int_{\partial \mathcal{D}_r} d\sigma_r(z) \right. \\
&\quad \times \left. \int_{\partial \mathcal{D}_\varrho} |u(w)|^p \chi_{B(z, \delta(z)/2)}(w) d\sigma_\varrho(w) \right\}^{q/p} \\
&= Cr^{-(2n+q)+(2n-1)(p-q)/p} \int_{c_1 r}^{c_2 r} d\varrho \left\{ \int_{\partial \mathcal{D}_\varrho} |u(w)|^p d\sigma_\varrho(w) \right. \\
&\quad \times \left. \int_{\partial \mathcal{D}_r} \chi_{B(z, \delta(z)/2)}(w) d\sigma_r(z) \right\}^{q/p} \\
&\leq Cr^{-(q+1)} \int_{c_1 r}^{c_2 r} M_p^q(u, \varrho) d\varrho.
\end{aligned}$$

From this we obtain, as in Case 1,

$$\int_0^{\varepsilon/c_2} r^{s+q} M_p^q(\nabla f, r) dr \leq C \int_0^\varepsilon \varrho^s M_p^q(u, \varrho) d\varrho,$$

and then (4.1) follows.

Case 3:  $0 < p < q < \infty$ . By (4.3) and the Hölder inequality with exponents  $q/p$  and  $q/(q-p)$ ,

$$M_p^q(\nabla f, r) \leq C \left\{ r^{-(p+1)} \int_{c_1 r}^{c_2 r} d\varrho \int_{\partial \mathcal{D}_\varrho} |u(w)|^p d\sigma_\varrho(w) \right\}^{q/p}$$

$$\begin{aligned}
&= Cr^{-(q+q/p)} \left\{ \int_{c_1 r}^{c_2 r} d\varrho \int_{\partial \mathcal{D}_\varrho} |u(w)|^p d\sigma_\varrho(w) \right\}^{q/p} \\
&\leq Cr^{-(q+q/p)} \left\{ \int_{c_1 r}^{c_2 r} d\varrho \left[ \int_{\partial \mathcal{D}_\varrho} |u(w)|^p d\sigma_\varrho(w) \right]^{q/p} \right\} \\
&\quad \times \left\{ \int_{c_1 r}^{c_2 r} d\varrho \right\}^{\frac{q-p}{q}, \frac{q}{p}} \\
&\leq Cr^{-(q+1)} \int_{c_1 r}^{c_2 r} M_p^q(u, \varrho) d\varrho.
\end{aligned}$$

Case 4:  $p = \infty$ ,  $0 < q < \infty$ . By (4.2) and the plurisubharmonicity of  $|u(z)|$ ,

$$\begin{aligned}
|\nabla f(z)| &\leq C\delta(z)^{-(2n+1)} \int_{B(z, \delta(z)/2)} |u(w)| dm(w) \\
&\leq C\delta(z)^{-1} \max_{w \in B(z, \delta(z)/2)} |u(w)| \\
&\leq Cr^{-1} \max_{-c_2 r \leq \lambda(w) \leq -c_1 r} |u(w)| \quad (\text{whenever } z \in \partial \mathcal{D}_r) \\
&\leq Cr^{-1} M_\infty(u, c_1 r).
\end{aligned}$$

This implies

$$\begin{aligned}
\int_0^\varepsilon r^{s+q} M_\infty^q(\nabla f, r) dr &\leq C \int_0^\varepsilon r^s M_\infty^q(u, c_1 r) dr \\
&\leq C \int_0^{c_1 \varepsilon} \varrho^s M_\infty^q(u, \varrho) d\varrho \\
&\leq C \int_0^\varepsilon \varrho^s M_\infty^q(u, \varrho) d\varrho.
\end{aligned}$$

The proof of (4.1) is complete.

Furthermore, by Theorem 2,

$$(4.6) \quad \int_0^\varepsilon r^s M_p^q(f, r) dr \leq C \left\{ |f(z_0)|^q + \int_0^\varepsilon r^{s+q} M_p^q(\nabla f, r) dr \right\}.$$

We can also fix  $\varepsilon$  so small that  $z_0 \in \mathcal{D}_\varepsilon$ . Then

$$(4.7) \quad |f(z_0)|^q = |u(z_0)|^q \leq M_\infty^q(u, \varepsilon) \leq C \int_0^\varepsilon r^s M_p^q(u, r) dr.$$

Therefore from (4.1), (4.6), and (4.7) we get

$$\int_0^\varepsilon r^s M_p^q(v, r) dr \leq C \int_0^\varepsilon r^s M_p^q(f, r) dr \leq C \int_0^\varepsilon r^s M_p^q(u, r) dr.$$

The theorem is proved.

**Remark.** In the proof of Theorems 1 and 2, we have actually proved that for  $0 < p \leq \infty$ ,  $s > 0$ ,  $m \in \mathbb{N}$ , and  $f \in H(\mathcal{D})$ ,

$$M_p(f, r) = O(r^s) \quad \text{iff} \quad \sum_{|\alpha|=m} M_p(D^\alpha f, r) = O(r^{s+m}).$$

Then, the proof of Theorem 3 tells us that for  $f = u + iv$ ,

$$M_p(u, r) = O(r^s) \quad \text{iff} \quad M_p(v, r) = O(r^s).$$

**5. Applications.** The first application is to get a generalization of the following theorem.

**THEOREM B.** *Let  $m$  be a positive integer and  $f \in H(B)$ . Then  $f \in L^p(dm)$  for  $0 < p < \infty$  iff the functions  $(1 - |z|^2)^m (D^\alpha f)(z)$  with  $|\alpha| = m$  are in  $L^p(dm)$ .*

For  $1 \leq p < \infty$ , Theorem B was first proved by Zhu (see [15]). In [9], Shi gave a proof for all  $p \in (0, \infty)$ . To state our result precisely, we need some notation. Recall that  $\mathcal{D}$  is a bounded domain with  $C^\infty$  boundary, and  $\lambda(z)$  is its defining function. Let  $z_0 \in \mathcal{D}$  be fixed, and  $0 < p < \infty$ . Set

$$(T_\alpha f)(z) = (-\lambda(z))^{|\alpha|} (D^\alpha f)(z),$$

$$\|f\|_{m,p} = \sum_{|\alpha| < m} |(D^\alpha f)(z_0)| + \sum_{|\alpha|=m} \|T_\alpha f\|_p,$$

where  $\|T_\alpha f\|_p = \{\int_{\mathcal{D}} |T_\alpha f|^p(z) dm(z)\}^{1/p}$ .

**THEOREM 5.** *Let  $z_0 \in \mathcal{D}$  be fixed,  $0 < p < \infty$ , and  $m \in \mathbb{N}$ . Then for  $f \in H(\mathcal{D})$ , we have  $f \in L^p(dm)$  iff all functions  $T_\alpha f$  with  $|\alpha| = m$  are in  $L^p(dm)$ . Furthermore,  $\|f\|_p$  and  $\|f\|_{p,m}$  are equivalent.*

The expression “ $A$  and  $B$  are equivalent” (denoted by  $A \sim B$ ) means  $A/C \leq B \leq CB$  for some positive constant  $C$ .

Before proving the theorem, we first prove the following lemma:

**LEMMA 12.** *Let  $0 < p < \infty$  and  $s > -1$ . Then for  $f \in H(\mathcal{D})$ ,  $f \in L^p(|\lambda(z)|^s dm)$  iff for some  $\varepsilon > 0$ ,*

$$\int_0^\varepsilon r^s M_p^p(f, r) dr < \infty.$$

Moreover,  $\{\int_{\mathcal{D}} |f(z)|^p |\lambda(z)|^s dm(z)\}^{1/p}$  and  $\{\int_0^\varepsilon r^s M_p^p(f, r) dr\}^{1/p}$  are equivalent.

**Proof.** Let  $\varepsilon > 0$  be fixed and small enough. Using polar coordinates (as in [7, p. 160]) we obtain

$$(5.1) \quad \int_{\mathcal{D} \setminus \mathcal{D}_\varepsilon} |f(z)|^p |\lambda(z)|^s dm(z) = \int_0^\varepsilon \int_{\partial \mathcal{D}_r} |f(z)|^p |\lambda(z)|^s w(r, z) d\sigma_r(z) dr \\ \sim \int_0^\varepsilon \int_{\partial \mathcal{D}_r} |f(z)|^p |\lambda(z)|^s d\sigma_r(z) dr \\ = \int_0^\varepsilon r^s M_p^p(f, r) dr,$$

where  $w(r, z)$  is a  $C^1$  function and  $1/C \leq w(r, z) \leq C$ . This implies

$$(5.2) \quad \int_{\mathcal{D}} |f(z)|^p |\lambda(z)|^s dm(z) \geq C \int_0^\varepsilon r^s M_p^p(f, r) dr.$$

On the other hand,

$$\sup_{z \in \mathcal{D}_\varepsilon} |f(z)|^p \leq C M_p^p(f, \varepsilon/2) \leq C \int_{\varepsilon/4}^{\varepsilon/2} r^s M_p^p(f, r) dr \leq C \int_0^\varepsilon r^s M_p^p(f, r) dr.$$

Then

$$(5.3) \quad \int_{\mathcal{D}} |f(z)|^p |\delta(z)|^s dm(z) \\ = \left\{ \int_{\mathcal{D} \setminus \mathcal{D}_\varepsilon} + \int_{\mathcal{D}_\varepsilon} \right\} |f(z)|^p |\delta(z)|^s dm(z) \\ \leq C \left\{ \int_{\mathcal{D} \setminus \mathcal{D}_\varepsilon} |f(z)|^p |\delta(z)|^s dm(z) + \sup_{z \in \mathcal{D}_\varepsilon} |f(z)|^p \right\} \\ \leq C \int_0^\varepsilon r^s M_p^p(f, r) dr.$$

From (5.2) and (5.3) the lemma follows.

**Proof of Theorem 5.** We first assume  $f \in L^p(dm)$ . Take  $p = q$  in Theorem 1, and use Lemma 12, to obtain  $\varepsilon > 0$  so that

$$\int_{\mathcal{D}} |\delta(z)|^{p|\alpha|} |(D^\alpha f)(z)|^p dm(z) \leq C \int_0^\varepsilon r^{p|\alpha|} M_p^p(D^\alpha f, r) dr$$

$$\leq C \int_0^\varepsilon M_p^p(f, r) dr \leq C \int_{\mathcal{D}} |f(z)|^p dm(z).$$

This proves that  $T_\alpha f \in L^p(dm)$  for any multi-index  $\alpha$  and  $\|T_\alpha f\|_p \leq C\|f\|_p$ . Therefore

$$\sum_{|\alpha|=m} \|T_\alpha f\|_p \leq C\|f\|_p,$$

and by the plurisubharmonicity of  $|D^\alpha f|$ ,

$$\sum_{|\alpha|<m} |(D^\alpha f)(z_0)| \leq C \sum_{|\alpha|<m} \|T_\alpha f\|_p \leq C\|f\|_p.$$

That is,

$$\|f\|_{p,m} \leq C\|f\|_p.$$

Next, suppose  $T_\alpha f \in L^p(dm)$  for all  $\alpha$  with  $|\alpha| = m$ . Take  $p = q$  in Theorem 2 and use Lemma 12 again to obtain

$$\begin{aligned} \|f\|_p^p &\leq C \int_0^\varepsilon M_p^p(f, r) dr \\ &\leq C \left\{ \sum_{|\alpha|<m} |(D^\alpha f)(z_0)|^p + \sum_{|\alpha|=m} \int_0^\varepsilon r^{p|\alpha|} M_p^p(D^\alpha f, r) dr \right\} \\ &\leq C \left\{ \sum_{|\alpha|<m} |(D^\alpha f)(z_0)|^p + \sum_{|\alpha|=m} \int_{\mathcal{D}} |\delta(z)|^{pm} |(D^\alpha f)(z)|^p dm(z) \right\} \\ &= C\|f\|_{p,m}^p. \end{aligned}$$

Theorem 5 is proved.

**Remark.** Theorem 5 can of course be further generalized to the case of weighted Bergman spaces. More precisely, one can prove the following theorem.

**THEOREM 6.** *Let  $z_0 \in \mathcal{D}$  be fixed,  $0 < p < \infty$ ,  $s > -1$ , and  $m \in \mathbb{N}$ . Then for  $f \in H(\mathcal{D})$ , we have  $f \in L^p(|\lambda(z)|^s dm)$  iff all functions  $T_\alpha f$  with  $|\alpha| = m$  are in  $L^p(|\lambda(z)|^s dm)$ . Furthermore,*

$$\begin{aligned} &\left\{ \int_{\mathcal{D}} |f(z)|^p |\lambda(z)|^s dm(z) \right\}^{1/p} \\ &\sim \sum_{|\alpha|<m} |(D^\alpha f)(z_0)| + \sum_{|\alpha|=m} \left\{ \int_{\mathcal{D}} |(D^\alpha f)(z)|^p |\lambda(z)|^{s+mp} dm(z) \right\}^{1/p}. \end{aligned}$$

The proof of Theorem 6 will be omitted here because it runs along the same lines as that of Theorem 5.

Our second application is to Bloch functions. For this purpose we restrict  $\mathcal{D}$  to be a bounded strongly pseudoconvex domain with  $C^\infty$  boundary (for the definition of strong pseudoconvexity we refer to [6]). A function  $f \in H(\mathcal{D})$  is called a *Bloch function* (denoted by  $f \in \mathcal{B}(\mathcal{D})$ ) if

$$\|f\|_{\mathcal{B}(\mathcal{D})} := \sup_{z \in \mathcal{D}} |\nabla f(z)| \cdot |\lambda(z)| < \infty.$$

An equivalent definition of Bloch functions can be found in [7].

**THEOREM 7.** *Let  $\mathcal{D}$  be a bounded strongly pseudoconvex domain with  $C^\infty$  boundary, and  $m \in \mathbb{N}$ . Then for  $f \in H(\mathcal{D})$ , we have  $f \in \mathcal{B}(\mathcal{D})$  iff all functions  $T_\alpha f$  with  $|\alpha| = m$  are in  $L^\infty(dm)$ . Moreover, for fixed  $z_0 \in \mathcal{D}$ ,*

$$(5.4) \quad \|f\|_{\mathcal{B}(\mathcal{D})} \sim \sum_{1 \leq |\alpha| < m} |(D^\alpha f)(z_0)| + \sum_{|\alpha|=m} \|T_\alpha f\|_\infty.$$

**Proof.** To prove (5.4), it suffices to prove

$$\| |(\nabla f)(z)| \lambda(z) \|_\infty \sim \sum_{1 \leq |\alpha| < m} |(D^\alpha f)(z_0)| + \sum_{|\alpha|=m} \|T_\alpha f\|_\infty,$$

or equivalently,

$$(5.5) \quad \sum_{|\alpha|=1} \|T_\alpha f\|_\infty \sim \sum_{1 \leq |\alpha| < m} |(D^\alpha f)(z_0)| + \sum_{|\alpha|=m} \|T_\alpha f\|_\infty.$$

Let  $\varepsilon > 0$  be small enough. Then

$$\begin{aligned} & \sup_{0 < r \leq \varepsilon} r^{|\alpha|} M_\infty(D^\alpha f, r) \\ &= \sup_{z \in \mathcal{D} \setminus \mathcal{D}_\varepsilon} |\lambda(z)|^{|\alpha|} |(D^\alpha f)(z)| \leq \sup_{z \in \mathcal{D}} |\lambda(z)|^{|\alpha|} |(D^\alpha f)(z)| = \|T_\alpha f\|_\infty \\ &= \sup_{z \in \mathcal{D} \setminus \mathcal{D}_\varepsilon} |\lambda(z)|^{|\alpha|} |(D^\alpha f)(z)| + \sup_{z \in \mathcal{D}_\varepsilon} |\lambda(z)|^{|\alpha|} |(D^\alpha f)(z)| \\ &\leq \sup_{0 < r \leq \varepsilon} r^{|\alpha|} M_\infty(D^\alpha f, r) + C M_\infty(D^\alpha f, \varepsilon) \\ &\leq C \sup_{0 < r \leq \varepsilon} r^{|\alpha|} M_\infty(D^\alpha f, r). \end{aligned}$$

Hence

$$(5.6) \quad \|T_\alpha f\|_\infty \sim \sup_{0 < r \leq \varepsilon} r^{|\alpha|} M_\infty(D^\alpha f, r).$$

For  $p = \infty$  and  $s = 1$ , applying Theorems 1 and 4 to  $\partial f / \partial z_j$  ( $j = 1, \dots, n$ ), we get



$$\begin{aligned} & \sup_{0 < r \leq \varepsilon} r M_\infty \left( \frac{\partial f}{\partial z_j}, r \right) \\ & \sim \sum_{|\alpha| < m-1} \left| \left( D^\alpha \left( \frac{\partial f}{\partial z_j} \right) \right) (z_0) \right| + \sum_{|\alpha|=m-1} \sup_{0 < r \leq \varepsilon} r^{1+|\alpha|} M_\infty \left( D^\alpha \left( \frac{\partial f}{\partial z_j} \right), r \right). \end{aligned}$$

Summing over  $j$  from 1 to  $n$  gives

$$(5.7) \quad \begin{aligned} & \sum_{|\alpha|=1} \sup_{0 < r \leq \varepsilon} r M_\infty (D^\alpha f, r) \\ & \sim \sum_{1 \leq |\alpha| < m} |(D^\alpha f)(z_0)| + \sum_{|\alpha|=m} \sup_{1 < r \leq \varepsilon} r^m M_\infty (D^\alpha f, r). \end{aligned}$$

Now (5.6) and (5.7) imply (5.5). The proof is complete.

Theorem 7 is a generalization of Stroethoff's results in [13].

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