COLLOQUIUM MATHEMATICUM

VOL. LXIX 1995 FASC. 2

A NOTE ON STRICTLY POSITIVE RADON MEASURES

BY

GRZEGORZ PLEBANEK (WROCŁAW)

Recently van Casteren [1] presented the following characterization of topological spaces admitting a strictly positive Radon measure.

THEOREM 1. *The following are equivalent for a topological space X*:

(i) *there exists a strictly positive Radon measure on X*;

(ii) *for every open subset U of X one can choose a compact set $K(U) \subseteq U$ such that for every sequence $(U_n)_{n\in\omega}$ with $\mathrm{int}(\bigcap_{n\in\omega} U_n) \neq \emptyset$ there is a set $A \subseteq \omega$ of non-zero density such that $\bigcap_{n\in A} K(U_n) \neq \emptyset$.*

Here and below every topological space is assumed to be Hausdorff. By a *Radon measure* on a space X we mean a finite compact-regular measure defined on the Borel σ-algebra of X. A measure is said to be *strictly positive* if it is non-zero on every non-empty open set. Thus a Radon measure μ on a space X is strictly positive if and only the support of μ is the whole space. A subset A of the set ω of natural numbers is said to be of *non-zero density* if

$$\limsup_{n\to\infty} \frac{|A \cap n|}{n} > 0,$$

where $n = \{0, 1, \ldots, n-1\}$.

We find the condition (ii) of Theorem 1 to be a concise topological characterization of spaces admitting a strictly positive measure. However, the proof given in [1] is rather complicated. In this note we present a shorter argument based on some classical results.

We shall first recall the notion of intersection numbers introduced by Kelley [4].

Given a finite sequence $(P_1, \ldots, P_n)$ of sets, $\mathrm{cal}(P_1, \ldots, P_n)$ is the maximum of k such that there are $1 \le m_1 < \ldots < m_k \le n$ with $\bigcap_{i=1}^k P_{m_i} \neq \emptyset$. Note that

$$\mathrm{cal}(P_1, \ldots, P_n) = \Big\| \sum_{i=1}^n \chi_{P_i} \Big\|,$$

1991 *Mathematics Subject Classification*: Primary 28C15.

Supported in part by KBN grant 2 P 301 043 07.

where $\|\cdot\|$ denotes the supremum norm and χ_P is the characteristic function of a set P. The *intersection number* $\kappa(\mathcal{P})$ of an arbitrary family $\mathcal{P}$ is defined as

$$\kappa(\mathcal{P}) = \inf\{\mathrm{cal}(P_1, \ldots, P_n)/n : P_i \in \mathcal{P},\ n \geq 1\}.$$

THEOREM 2 (Kelley [4], see also [2] and [10]). *Given a family $\mathcal{P}$ of subsets of a set X and $r > 0$, the following are equivalent*:

(a) *there is a probability quasi-measure μ with $\mu(P) \geq r$ for all $P \in \mathcal{P}$*;
(b) $\kappa(\mathcal{P}) \geq r$.

By a *quasi-measure* we mean a finitely additive and non-negative set function defined on an algebra of sets. The condition ensuring the existence of a measure as in (a) of Theorem 2 is much more complicated (see [9]). However, in special cases σ-additivity is for free—the following Theorem 3 seems to be well known (see the remark at the end of this paper).

THEOREM 3. *If $\mathcal{P}$ is a family of compact subsets of a space X then there exists a probability Radon measure μ such that $\mu(K) \geq \kappa(\mathcal{P})$ for $K \in \mathcal{P}$.*

Theorem 2 made it possible to give a combinatorial characterization of measurable Boolean algebras and was subsequently used to describe compact spaces having a strictly positive Radon measure; this is exposed in Chapter 6 of [2]. As we noted in [8], Theorem 3 yields the following:

THEOREM 4. *A topological space X has a strictly positive Radon measure if and only if there exists a family $\mathcal{P}$ of non-empty compact subsets of X such that $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{P}_n$, where $\kappa(\mathcal{P}_n) > 0$, and $\mathcal{P}$ is a π-base for X* (*that is*, *every non-empty open set contains an element of $\mathcal{P}$*).

The proof of Theorem 1 we shall present below shows that characterizations of spaces having a strictly positive Radon measure contained in Theorem 1 and Theorem 4 are in fact closely related. Our proof is based on two lemmata we shall now prove.

LEMMA 1. *If μ is a finite measure and P_n's are measurable sets with $\mu(P_n) \geq r$, where $r > 0$, then $\bigcap_{n\in A} P_n \neq \emptyset$ for some set $A \subseteq \omega$ of non-zero density.*

Proof. Let $g_n = (1/n)\sum_{i=1}^{n} \chi_{P_i}$ and $g = \limsup_{n\to\infty} g_n$. Since $\int g_n\, d\mu \geq r$, it follows from Fatou's lemma that $\int g\, d\mu \geq r$. Hence $g(x_0) > 0$ for some x_0, and, consequently, $A = \{n \in \omega : x_0 \in P_n\}$ has non-zero density.

LEMMA 2. *For a family $\mathcal{P}$ of sets the following are equivalent*:

(a) $\kappa(\mathcal{P}) > 0$;

(b) *for every sequence $(P_n)_{n\in\omega}$ from $\mathcal{P}$ there exists a set $A \subseteq \omega$ of non-zero density such that the family $\{P_n : n \in A\}$ has the finite intersection property.*

P r o o f. (a)⇒(b). Put $r = \kappa(\mathcal{P})$ and let $\mathcal{A}$ be the algebra generated by $\mathcal{P}$. By the Stone representation theorem there is an isomorphism $\widehat{\ }$ between $\mathcal{A}$ and the algebra of open and closed subsets of a certain compact space S. Clearly $\kappa(\{\widehat{P} : P \in \mathcal{P}\}) = r$, so by Theorem 3 there is a Radon measure μ on S with $\mu(\widehat{P}) \geq r$ for $P \in \mathcal{P}$. Now to check (b) it suffices to apply Lemma 1, and notice that $\bigcap_{n\in A} \widehat{P}_n \neq \emptyset$ means that the family $\{P_n : n \in A\}$ has the finite intersection property.

(b)⇒(a). Suppose that $\kappa(\mathcal{P}) = 0$. This means that for every k there are $n_k \in \omega$ and $P_1^k, \ldots, P_{n_k}^k \in \mathcal{P}$ such that $\|\sum_{i=1}^{n_k} \chi_{P_i^k}\| \leq n_k 2^{-k}$. Put $m_k = \sum_{i=1}^{k} n_i$, and let $(Q_n)_n$ be an enumeration of the sequence

$$P_1^1, \ldots, P_{n_1}^1, P_1^2, \ldots, P_{n_2}^2, \ldots$$

We have $\|\sum_{i=1}^{m_k} \chi_{Q_i}\| \leq n_1 2^{-1} + \ldots + n_k 2^{-k}$, which implies easily that $(1/m_k)\|\sum_{i=1}^{m_k} \chi_{Q_i}\|$ tends to 0. Note that this would give $(1/n)\|\sum_{i=1}^{n} \chi_{Q_i}\| \to 0$ if we knew that the sequence m_{k+1}/m_k were bounded from above.

Consider now a sequence $R_1, R_2, \ldots$ in which the segment $P_1^k, \ldots, P_{n_k}^k$ appears $r_k = [n_{k+1}/n_k] + 1$ times. We just apply the above remarks to R_j's and the subsequence of natural numbers that can be written as $r_1 n_1 + \ldots + r_i n_i + j n_{i+1}$, where $j \leq r_{i+1}$.

It follows that $(1/n)\|\sum_{i=1}^{n} \chi_{R_i}\| \to 0$, which means that $A \subseteq \omega$ has zero density whenever $\bigcap_{n\in A} R_n \neq \emptyset$, a contradiction.

P r o o f o f T h e o r e m 1. If μ is a strictly positive Radon measure on X we can find for every open $V \subseteq X$ a compact $K(V) \subseteq V$ with $\mu(K(V)) \geq (1/2)\mu(V)$. Now (ii) follows immediately from Lemma 1.

To check that (ii) is sufficient for the existence of a strictly positive Radon measure on X, consider the family $\mathcal{S}$ of all (closed) subsets of X which are supporting some Radon measure. Note that if $S_0, S_1, \ldots \in \mathcal{S}$ then $S = \overline{\bigcup_{n\in\omega} S_n}$ is again in $\mathcal{S}$. Indeed, if S_n is the support of a probability Radon measure μ_n then S is the support of $\sum_{n\in\omega} 2^{-n}\mu_n$. We are to check that $X \in \mathcal{S}$.

Let $\mathcal{C}$ be the family of sets $K(V)$, where $X \setminus S \subseteq V$ for some $S \in \mathcal{S}$. Then $\kappa(\mathcal{C}) = 0$; indeed, otherwise by Theorem 3 there is a Radon measure μ which is positive on elements from $\mathcal{C}$. In particular, $\mu(K(X \setminus S)) > 0$, where S is the support of μ, a contradiction with $K(X \setminus S) \subseteq X \setminus S$.

It follows that there is $S_0 \in \mathcal{S}$ such that $\kappa(\{K(V) : V \supseteq X \setminus S_0\}) = 0$ (since the intersection number is attained on some countable subfamily; this is a consequence of the fact that $\mathcal{S}$ is countably upward directed). Now (ii) and Lemma 2 imply that $X \setminus S_0 = \emptyset$ and we are done.

Let us note that subsets of ω of non-zero density play a crucial role in (b) of Lemma 2. Consider, for instance, the following example of a family $\mathcal{C}$ with $\kappa(\mathcal{C}) = 0$, having the property that every sequence from $\mathcal{C}$ has a subsequence with non-empty intersection.

Put $K = \{A \subseteq \omega : |A \cap n| \leq \sqrt{n}\}$. Identifying the power set of ω with the Cantor set 2^ω, we may treat K as a closed subset of 2^ω. Let the family $\mathcal{C}$ consist of the sets $C_n = \{A \in K : n \in A\}$. Since every $A \in K$ has zero density, using Lemma 2 we get $\kappa(\mathcal{C}) = 0$. On the other hand, for every increasing sequence $(n_k)_{k\in\omega}$, $A = \{n_{k^2} : k \in \omega\} \in K$, so $\bigcap_{k\in\omega} C_{n_{k^2}} \neq \emptyset$.

Some of the results from [1] dealing with families of functions rather than families of sets can be proved accordingly. In our opinion, it is convenient to start such considerations from the following.

Consider a lattice $\mathcal{L}$ of subsets of an abstract set X, and an algebra $\mathcal{A}$ generated by $\mathcal{L}$. Note that for every quasi-measure on $\mathcal{A}$ the associated integral is well defined for all functions that are uniform limits of $\mathcal{A}$-measurable simple functions.

Recall that a quasi-measure μ on $\mathcal{A}$ is said to be *$\mathcal{L}$-regular* if

$$\mu(A) = \sup\{\mu(L) : L \in \mathcal{L},\ L \subseteq A\}$$

for every $A \in \mathcal{A}$.

THEOREM 5. *Let G be a family of non-negative and bounded functions on X such that $\{g \geq t\} \in \mathcal{L}$ whenever $g \in G$ and $t \geq 0$. The following are equivalent for $r \geq 0$:*

(a) *there exists an $\mathcal{L}$-regular probability quasi-measure μ on $\mathcal{A}$ such that $\int g\,d\mu \geq r$ for every $g \in G$;*

(b) *$\|\sum_{i=1}^n g_i\| \geq nr$ whenever $g_1, \ldots, g_n \in G$ and $n \geq 1$.*

Theorem 5 is likely to be known but we do not know whether its proof is written down somewhere. We shall sketch a possible argument.

(a)⇒(b) follows immediately from the inequality

$$\Big\|\sum_{i=1}^n g_i\Big\| \geq \int \sum_{i=1}^n g_i\,d\mu.$$

To prove the reverse implication one can apply the Mazur–Orlicz–Kaufman interpolation theorem, stating that if p is a subadditive function on an Abelian semigroup H and q is any function on H such that $p(h_1 + \ldots + h_n) \geq q(h_1) + \ldots + q(h_n)$ whenever $h_1, \ldots, h_n \in H$, then there exists an additive function ξ such that $q(h) \leq \xi(h) \leq p(h)$ for $h \in H$ (see [3], cf. [10] and [5]).

We take H to be a semigroup of non-negative functions on X that are uniform limits of $\mathcal{A}$-measurable simple functions, put $p(h) = \|h\|$, and put $q(h) = r$ if $h \in G$, $q(h) = 0$ otherwise. Now (b) is what we need to verify the assumption of the theorem mentioned above. Thus there is an additive and non-negative function ξ on H with $\xi(h) \le \|h\|$ for all $h \in H$ and $\xi(g) \ge r$ for $g \in G$.

Consider now a quasi-measure m given by $m(A) = \xi(\chi_A)$. According to a result due to Lembcke [6], Korollar 2.12 (see also Pachl [7], Proposition 3.4), there is an $\mathcal{L}$-regular probability quasi-measure μ on $\mathcal{A}$ such that $\mu(L) \ge m(L)$ for $L \in \mathcal{L}$. We shall check that $\int g\,d\mu \ge r$ whenever $g \in G$.

For a given $g \in G$ and a natural number $n \ge 1$, we consider the function

$$g_n = \frac{1}{n}\sum_{i=1}^{n} \chi_{L_i}, \qquad \text{where } L_i = \left\{x \in X : g(x) \ge \|g\|\frac{i}{n}\right\}.$$

We have $g \ge g_n \ge g - \|g\|/n$ and $L_i \in \mathcal{L}$, so

$$\begin{aligned}\int g\,d\mu \ge \int g_n\,d\mu &= \frac{1}{n}\sum_{i=1}^{n}\mu(L_i) \ge \frac{1}{n}\sum_{i=1}^{n} m(L_i)\\ &= \frac{1}{n}\xi\Big(\sum_{i=1}^{n} L_i\Big) = \xi(g_n) = \xi(g) - \xi(g-g_n) \ge r - \frac{\|g\|}{n},\end{aligned}$$

and this shows that μ is as required.

Note that if G is a family of characteristic functions then Theorem 5 gives Kelley's result (Theorem 2), since its condition (b) means that $\kappa(\{P : \chi_P \in G\}) \ge r$. In case $\mathcal{L}$ is a lattice of compact subsets of a topological space X, every $\mathcal{L}$-regular quasi-measure is σ-additive and extends to a Radon measure. Thus, in such a setting, we may demand in condition (a) of Theorem 5 that μ is a Radon measure. In particular, Theorem 3 is a consequence of Theorem 5.

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INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND
E-mail: GRZES@MATH.UNI.WROC.PL

Reçu par la Rédaction le 4.11.1994