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λ-COEFFICIENT OF ORLICZ SEQUENCE SPACES

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Let X be a Banach space, and $S(X)$ and $B(X)$ denote the unit sphere and unit ball of X, respectively. For each $x \in B(X)$, write

$$\lambda(x) = \sup\{\lambda \in [0,1] : x = \lambda e + (1-\lambda)y,\ y \in B(X),\ e \in \operatorname{Ext} B(X)\}.$$

If $\lambda(x) > 0$ for all $x \in B(X)$, then X is said to have the λ-*property*. Moreover, if $\inf\{\lambda(x) : x \in S(X)\} > 0$, then X is said to have the *uniform* λ-*property*.

If X has the λ-property, then $B(X) = \mathrm{co}(\operatorname{Ext} B(X))$ and each element $x \in B(X)$ can be expressed as $x = \sum_{i=1}^{\infty} \lambda_i e_i$, where $e_i \in \operatorname{Ext} B(X)$ and $\lambda_i > 0$, $\sum_{i=1}^{\infty} \lambda_i = 1$. Moreover, if X has the uniform λ-property, then the series $x = \sum_{i=1}^{\infty} \lambda_i e_i$ converges uniformly for all $x \in B(X)$.

Define

$$\lambda(X) = \inf\{\lambda(x) : x \in S(X)\}.$$

Obviously, $\lambda(X)$ expresses the degree of λ-property; we call it the λ-*coefficient* of X.

The λ-property of Orlicz spaces has been thoroughly discussed in the literature, and it is well known that the Orlicz function space L_M endowed with the Luxemburg norm has the uniform λ-property iff $M(u)$ is strictly convex on $[0,\infty)$ (for short, we write $M \in \mathrm{SC}$). Indeed, if $M \notin \mathrm{SC}$, then $\lambda(L_M) = 0$, and if $M \in \mathrm{SC}$, then $\lambda(L_M) = 1$. In this paper, we discuss Orlicz sequence spaces endowed with the Luxemburg norm, and get an interesting result that $\lambda(l_M)$ may take every value in the harmonic number sequence $\{1/n\}_{n=1}^{\infty}$ and 0. Hence, we can easily deduce a sufficient and necessary condition for l_M to have the uniform λ-property.

Let $M : (-\infty,\infty) \to (0,\infty)$ be convex, even, continuous and $M(u) = 0 \Leftrightarrow u = 0$. For a given sequence $x = (x_n)_{n=1}^{\infty}$, define $\varrho_M(x) = \sum_{n=1}^{\infty} M(x_n)$, $l_M = \{x = (x_n)_{n=1}^{\infty} : \exists \lambda > 0,\ \varrho_M(\lambda x) < \infty\}$, and $\|x\| = \inf\{\lambda > 0 : \varrho_M(x/\lambda) \le 1\}$ for $x \in l_M$. Then $(l_M, \|\cdot\|)$ is a Banach space. $\operatorname{Ext} B(l_M)$

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denotes the set of all extreme points of $B(l_M)$. SAI represents a structural affine interval of $M(u)$, i.e. an interval $[a,b]$ such that $M(u)$ is affine on $[a,b]$ and is not affine on $[a-\varepsilon,b]$ and $[a,b+\varepsilon]$ for any $\varepsilon>0$. S_M is the set of strictly convex points of $M(u)$ (i.e. $u\in S_M$ iff for any $\varepsilon>0$, $M(u)<(M(u-\varepsilon)+M(u+\varepsilon))/2$).

LEMMA 1. *If* $x,y,z\in B(X)$ *and* $x=\alpha y+(1-\alpha)z$ *for some* $\alpha\in[0,1]$, *then* $\lambda(x)\ge\alpha\lambda(y)$.

Proof. See [2].

LEMMA 2. *Let* $x\in S(l_M)$. *Then* $x\in \operatorname{Ext}B(l_M)$ *iff* (i) $\varrho_M(x)=1$ *and* (ii) $\mu\{i:x(i)\notin S_M\}\le 1$.

Proof. See [5].

LEMMA 3. $\lambda(l_M)=\inf\{\lambda(x):\varrho_M(x)=1\}$.

Proof. Define $\lambda'=\inf\{\lambda(x):\varrho_M(x)=1\}$. Obviously, $\lambda(l_M)\le\lambda'$. For any $x\in S(l_M)$ with $\varrho_M(x)<1$ and $0<\varepsilon<1$, since $\varrho_M(x/(1-\varepsilon))=\infty$, there exists n such that

$$\sum_{j\le n}M\left(\frac{x(j)}{1-\varepsilon}\right)+\sum_{j>n}M(x(j))\ge 1.$$

Select $0<\varepsilon'<\varepsilon$ satisfying

$$\sum_{j\le n}M\left(\frac{x(j)}{1-\varepsilon'}\right)+\sum_{j>n}M(x(j))=1.$$

Take

$$y=\left(\frac{x(1)}{1-\varepsilon'},\frac{x(2)}{1-\varepsilon'},\ldots,\frac{x(n)}{1-\varepsilon'},x(n+1),x(n+2),\ldots\right).$$

Then $\varrho_M(y)=1$. Set $z=(0,\ldots,0,x(n+1),x(n+2),\ldots)$. Clearly, $z\in B(L_M)$ and $x=(1-\varepsilon')y+\varepsilon' z$. By Lemma 1, $\lambda(x)\ge(1-\varepsilon')\lambda(y)\ge(1-\varepsilon')\lambda'$. Since x, ε' are arbitrary, we have $\lambda(l_M)\ge\lambda'$.

Now define

$$d_M=\sup\{d\ge 0: M(u)\text{ is strictly convex on }[0,d]\}.$$

The main result of this paper is the following:

THEOREM. *Let* l_M *be an Orlicz sequence space.*

(i) *If* $d_M\ge M^{-1}(1/2)$, *then* $\lambda(l_M)=1$.

(ii) *If* $M^{-1}(1/(n+1))\le d_M<M^{-1}(1/n)$, *then* $\lambda(l_M)=1/n$ $(n=2,3,\ldots)$.

(iii) *If* $d_M=0$, *then* $\lambda(l_M)=0$.

Proof. (i) For any $x\in l_M$ with $\varrho_M(x)=1$, since $d_M\ge M^{-1}(1/2)$ and $M^{-1}(1/2)\in S_M$, we see that $\{j:x(j)\notin S_M\}$ contains at most one element. Hence by Lemma 2, $x \in \operatorname{Ext} B(l_M)$. According to Lemma 3, $\lambda(l_M) = \inf\{\lambda(x) : \varrho_M(x) = 1\} = 1$.

(ii) First we show

$$\lambda(l_M) \le 1/n.$$

Since $M^{-1}(1/(n+1)) \le d_M < M^{-1}(1/n)$, there exists a SAI $[a, b]$ of $M(u)$ such that $d_M \le a < M^{-1}(1/n)$. Choose $a < c < b$ satisfying $[a, c] \subset [d_M, M^{-1}(1/n))$. Since $M(u)$ is strictly convex on $[0, M^{-1}(1/(n+1)))$, we can construct a sequence

$$x = \Big(\overbrace{\Big(1 - \frac{1}{n}\Big)a + \frac{1}{n}c, \dots, \Big(1 - \frac{1}{n}\Big)a + \frac{1}{n}c}^{n}, x(n+1), \dots \Big)$$

with $\varrho_M(x) = 1$ and $x(j) \in S_M$ $(j > n)$.

Let $x = \lambda e + (1-\lambda)y$, where $e \in \operatorname{Ext} B(l_M)$ and $y \in B(l_M)$. Since

$$\begin{aligned} 1 = \varrho_M(x) &= \sum_j M(\lambda e(j) + (1-\lambda)y(j)) \\ &\le \lambda \sum_j M(e(j)) + (1-\lambda) \sum_j M(y(j)) \le \lambda + (1-\lambda) = 1, \end{aligned}$$

we get $M(\lambda e(j) + (1-\lambda)y(j)) = \lambda M(e(j)) + (1-\lambda)M(y(j))$ for any j, which shows that either $x(j)$, $e(j)$ and $y(j)$ are in the same SAI of M, or $x(j) = y(j) = e(j)$. Using $x(j) \in S_M$ for any $j > n$, we have $x(j) = y(j) = e(j)$ for any $j > n$. Thus

$$\begin{aligned} \sum_{j=1}^{n} M(e(j)) &= 1 - \sum_{j>n} M(e(j)) = 1 - \sum_{j>n} M(x(j)) = \sum_{j=1}^{n} M(x(j)) \\ &= \sum_{j=1}^{n} M\Big(\Big(1 - \frac{1}{n}\Big)a + \frac{1}{n}c\Big) = (n-1)M(a) + M(c). \end{aligned}$$

Since $e \in \operatorname{Ext} B(l_M)$, all elements of $\{e(j) : 1 \le j \le n\}$ except possibly one are equal to a or b. By the above equality, there exists no j satisfying $e(j) = b$ $(1 \le j \le n)$. So $\{j : e(j) = a\}$ contains $n-1$ elements, and there exists only one index j_0 $(1 \le j_0 \le n)$ such that $e(j_0) = c$. Therefore

$$\begin{aligned} \Big(1 - \frac{1}{n}\Big)a + \frac{1}{n}c = x(j_0) &= \lambda e(j_0) + (1-\lambda)y(j_0) \\ &= \lambda c + (1-\lambda)y(j_0) \ge (1-\lambda)a + \lambda c. \end{aligned}$$

This implies $\lambda \le 1/n$ and we have $\lambda(x) \le 1/n$ as the decomposition $x = \lambda e + (1-\lambda)y$ is arbitrary. So we get $\lambda(l_M) \le 1/n$.

From the above proof, we can deduce (iii).

Now we prove

$$\lambda(l_M) \geq 1/n.$$

For any $x \in S(l_M) \setminus \operatorname{Ext} B(l_M)$, by Lemma 3, assume $\varrho_M(x) = 1$. Without loss of generality, we may assume $x(j) \geq 0$ for any j. This part of the proof will be split into two steps. Let $\{[a_k, b_k]\}_{k=1}^{\infty}$ be all the SAI of M.

Step I: We show that $\lambda(x) \geq \min\{\sigma, 1-\sigma\}$. For each $\lambda \in [0,1]$, define

$$x_\lambda(j) = \begin{cases} b_k, & b_k > x(j) > \lambda a_k + (1-\lambda)b_k, \\ a_k, & \lambda a_k + (1-\lambda)b_k \geq x(j) > a_k, \\ x(j), & \text{otherwise.} \end{cases}$$

Then the function $f(\lambda) = \varrho_M(x_\lambda)$ is nondecreasing. As $\{j : x(j) \notin S_M\}$ contains at least two elements, $\varrho_M(x_0) < \varrho_M(x) = 1$ and $\varrho_M(x_1) > \varrho_M(x) = 1$.

Define

$$\sigma = \sup\{\lambda : \varrho_M(x_\lambda) \leq 1\}.$$

As $d_M \geq M^{-1}(1/(n+1))$, $\{j : x(j) \notin S_M\}$ is a finite set. Clearly, $0 < \sigma < 1$. Write

$$N_k = \{j : x(j) = \sigma a_k + (1-\sigma)b_k\}.$$

If $\varrho_M(x_\sigma) = 1$, then set $e = x_\sigma$. If $\varrho_M(x_\sigma) < 1$, then $\bigcup_k N_k \neq \emptyset$. Thus there exist $E_k \subset N_k$ $(k \geq 1)$ such that $\varrho_M(u_\sigma) \leq 1$, where

$$u_\sigma(j) = \begin{cases} b_k, & b_k > x(j) > \sigma a_k + (1-\sigma)b_k \text{ or } j \in E_k, \\ a_k, & \sigma a_k + (1-\sigma)b_k > x(j) > a_k \text{ or } j \in N_k \setminus E_k, \\ x(j), & \text{otherwise,} \end{cases}$$

and for any $j \in N_k \setminus E_k$, if we set $u_\sigma(j) = b_k$, then $\varrho_M(u_\sigma) > 1$.

If $\varrho_M(u_\sigma) = 1$, set $e = u_\sigma$. If $\varrho_M(u_\sigma) < 1$, we can take an index k' such that $N_{k'} \setminus E_{k'} \neq \emptyset$. Select $\alpha \in (a_{k'}, b_{k'})$ and $j' \in N_{k'} \setminus E_{k'}$ satisfying $\varrho_M(e) = 1$, where

$$e(j) = \begin{cases} \alpha, & j = j', \\ u_\sigma(j), & j \neq j'. \end{cases}$$

By Lemma 2, $e \in \operatorname{Ext} B(l_M)$.

If $\sigma \geq 1/2$, take z with $x = (1-\sigma)e + \sigma z$, and if $\sigma < 1/2$, take z with $x = \sigma e + (1-\sigma)z$. In both these cases, we can prove $\varrho_M(z) = 1$. We only discuss the case $\sigma \geq 1/2$ (the case $\sigma < 1/2$ is similar).

If $x(j) = e(j)$, then $z(j) = x(j) = e(j)$.

If $x(j) \le \sigma a_k + (1-\sigma)b_k$ and $e(j) = a_k$, then

$$\begin{aligned} a_k < x(j) \le z(j) &= \frac{1}{\sigma}(x(j) - (1-\sigma)e(j)) \\ &\le \frac{1}{\sigma}(\sigma a_k + (1-\sigma)b_k - (1-\sigma)a_k) = a_k + \left(\frac{1}{\sigma} - 1\right)(b_k - a_k) \\ &\le a_k + (b_k - a_k) = b_k. \end{aligned}$$

If $x(j) \ge \sigma a_k + (1-\sigma)b_k$ and $e(j) = b_k$, then

$$b_k > x(j) \ge z(j) \ge \frac{1}{\sigma}(\sigma a_k + (1-\sigma)b_k - (1-\sigma)b_k) = a_k.$$

If $x(j) = \sigma a_k + (1-\sigma)b_k < \alpha = e(j)$, then

$$\begin{aligned} b_k > x(j) \ge z(j) &= \frac{1}{\sigma}(\sigma a_k + (1-\sigma)b_k - (1-\sigma)\alpha) \\ &\ge \frac{1}{\sigma}(\sigma a_k + (1-\sigma)b_k - (1-\sigma)b_k) = a_k. \end{aligned}$$

If $x(j) = \sigma a_k + (1-\sigma)b_k \ge \alpha = e(j)$, then

$$\begin{aligned} a_k < x(j) \le z(j) &= \frac{1}{\sigma}(\sigma a_k + (1-\sigma)b_k - (1-\sigma)\alpha) \\ &\le \frac{1}{\sigma}(\sigma a_k + (1-\sigma)b_k - (1-\sigma)a_k) = a_k + \left(\frac{1}{\sigma} - 1\right)(b_k - a_k) \le b_k. \end{aligned}$$

Thus either $x(j) = e(j) = z(j)$, or $x(j)$, $e(j)$ and $z(j)$ are in the same SAI of M. Hence

$$\begin{aligned} 1 = \varrho_M(x) &= \varrho_M((1-\sigma)e + \sigma z) = (1-\sigma)\varrho_M(e) + \sigma\varrho_M(z) \\ &= 1 - \sigma + \sigma\varrho_M(z). \end{aligned}$$

This shows that $\varrho_M(z) = 1$, and thus $\lambda(x) \ge 1 - \sigma$. Similarly, if $\sigma < 1/2$, we can get $\lambda(x) \ge \sigma$. Consequently, $\lambda(x) \ge \min\{\sigma, 1-\sigma\}$.

Step II: We prove $\lambda(x) \ge 1/n$. If $\sigma \ge 1/2$, then by Step I, $\lambda(x) \ge 1-\sigma$. If $1 - \sigma \ge 1/n$, then the proof is complete. Conversely, if $1 - \sigma < 1/n$, then rearrange $x(j)$ by putting $x(j)$ at the beginning if $x(j) \notin S_M$. Assume $x(j) \notin S_M$ $(j = 1, \ldots, m)$, i.e. for $1 \le j \le m$, $x(j) = (1-\lambda_j)a_j + \lambda_j b_j$, where $0 < \lambda_j < 1$ and $[a_j, b_j]$ is a SAI of M.

Now $x \notin \operatorname{Ext} B(l_M)$ implies $m \ge 2$. Notice that $d_M \ge M^{-1}(1/(n+1))$. We deduce that

$$\begin{aligned} 1 = \varrho_M(x) &\ge \sum_{j=1}^{m} M(x(j)) > \sum_{j=1}^{m} M(d_M) \\ &\ge \sum_{j=1}^{m} M\left(M^{-1}\left(\frac{1}{n+1}\right)\right) = \frac{m}{n+1}. \end{aligned}$$

So $m \le n$. Define

$$J = \{1 \le j \le m : \lambda_j \le 1/n,\ \lambda_j \text{ is the coefficient of } x(j) = (1-\lambda_j)a_j + \lambda_j b_j\}.$$

Then $J \ne \emptyset$. Otherwise, if $\lambda_j > 1/n$ for any $1 \le j \le m$, then $x_{1-1/n}(j) = b_j$ $(1 \le j \le m)$. Hence $\varrho_M(x_{1-1/n}) > 1$. But $\varrho_M(x_\sigma) \le 1$, and we obtain $\sigma < 1 - 1/n$, which contradicts $1 - \sigma < 1/n$.

By rearranging again, assume $J = \{1, \dots, r\}$ $(r \le m)$ with

$$\lambda_r(M(b_r) - M(a_r)) = \max_{i \le r} \lambda_i(M(b_i) - M(a_i)).$$

For each $\delta \in [0, 1]$, consider

$$y_\delta(j) = \begin{cases} a_j, & j < r, \\ (1-\delta)a_j + \delta b_j, & j = r, \\ b_j, & r < j \le m, \\ x(j), & j > m. \end{cases}$$

Clearly the function $f(\delta) = \varrho_M(y_\delta)$ is nondecreasing, and $\varrho_M(y_0) = \varrho_M(x_{1-1/n}) \le \varrho_M(x_\sigma) \le 1$. Notice that $r\lambda_r \le m/n \le 1$, and therefore, $y_{r\lambda_r}$ has a meaning. We have

$$\begin{aligned} \varrho_M(y_{r\lambda_r}) - 1 = {} & \sum_{j<r} M(a_j) + M((1 - r\lambda_r)a_r + r\lambda_r b_r) \\ & + \sum_{j=r+1}^{m} M(b_j) + \sum_{j>m} M(x(j)) \\ & - \sum_{j=1}^{r}((1-\lambda_j)M(a_j) + \lambda_j M(b_j)) \\ & - \sum_{j=r+1}^{m} M(x(j)) - \sum_{j>m} M(x(j)) \\ \ge {} & -\sum_{j=1}^{r} \lambda_j(M(b_j) - M(a_j)) + r\lambda_r(M(b_r) - M(a_r)) \\ \ge {} & -r\lambda_r(M(b_r) - M(a_r)) + r\lambda_r(M(b_r) - M(a_r)) = 0. \end{aligned}$$

Hence there exists $\delta \in [0, r\lambda_r]$ such that $\varrho_M(y_\delta) = 1$.

By Lemma 1, $y_\delta \in \operatorname{Ext} B(l_M)$. Suppose that z satisfies $x = (1/n)y_\delta + (1 - 1/n)z$. To prove $\lambda(x) \ge 1/n$, it suffices to verify $z \in B(l_M)$. As in Step I, we need to show that either $z(j) = y_\delta(j) = x(j)$, or $z(j)$, $y_\delta(j)$ and $x(j)$ are in the same SAI of M.

If $j > m$, then $z(j) = y_\delta(j) = x(j)$.

If $j < r$, notice that $\lambda_j \le 1/n$ and $n \ge 2$; then

$$\begin{aligned} a_j < x(j) \le z(j) &= \frac{1}{1-1/n}\left(x(j) - \frac{1}{n}y_\delta(j)\right) \\ &= \frac{1}{1-1/n}\left((1-\lambda_j)a_j + \lambda_j b_j - \frac{1}{n}a_j\right) \\ &\le \frac{1}{1-1/n}\left(\left(1-\frac{1}{n}\right)a_j + \frac{1}{n}(b_j - a_j)\right) \\ &= a_j + \frac{1/n}{1-1/n}(b_j - a_j) \le a_j + (b_j - a_j) = b_j. \end{aligned}$$

If $r < j \le m$, notice that $\lambda_j > 1/n$; then

$$\begin{aligned} b_j > x(j) \ge z(j) &= \frac{1}{1-1/n}\left((1-\lambda_j)a_j + \lambda_j b_j - \frac{1}{n}b_j\right) \\ &\ge \frac{1}{1-1/n}\left(\left(1-\frac{1}{n}\right)a_j + \frac{1}{n}b_j - \frac{1}{n}b_j\right) = a_j. \end{aligned}$$

If $j = r$ and $(1-\delta)a_r + \delta b_r \le (1-\lambda_r)a_r + \lambda_r b_r = x(r)$, then

$$\begin{aligned} a_r &< x(r) \le z(r) \\ &= \frac{1}{1-1/n}\left((1-\lambda_r)a_r + \lambda_r b_r - \frac{1}{n}((1-\delta)a_r + b_r)\right) \\ &\le \frac{1}{1-1/n}\left(\left(1-\frac{1}{n}\right)a_r + \frac{1}{n}b_r - \frac{1}{n}a_r\right) \\ &= a_r + \frac{1/n}{1-1/n}(b_r - a_r) \le b_r. \end{aligned}$$

If $j = r$ and $(1-\delta)a_r + \delta b_r > (1-\lambda_r)a_r + \lambda_r b_r = x(r)$, then

$$\begin{aligned} b_r &> x(r) \ge z(r) \\ &= \frac{1}{1-1/n}\left((1-\lambda_r)a_r + \lambda_r b_r - \frac{1}{n}((1-\delta)a_r + \delta b_r)\right) \\ &= \frac{1}{1-1/n}\left(\left(1-\frac{1}{n}\right)a_r + \left(\lambda_r - \frac{\delta}{n}\right)(b_r - a_r)\right) \\ &= a_r + \frac{1}{1-1/n}\left(\lambda_r - \frac{\delta}{n}\right)(b_r - a_r). \end{aligned}$$

By $\lambda_r \ge \delta/r \ge \delta/n$, we have $b_r > z(r) \ge a_r$. Thus $\lambda(x) \ge 1/n$. Since x is arbitrary, by Lemma 3 we conclude that $\lambda(l_M) \ge 1/n$.

The theorem immediately yields

COROLLARY. *l_M has the uniform λ-property iff $d_M > 0$.*

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