\( \lambda \)-COEFFICIENT OF ORLICZ SEQUENCE SPACES

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Let \( X \) be a Banach space, and \( S(X) \) and \( B(X) \) denote the unit sphere and unit ball of \( X \), respectively. For each \( x \in B(X) \), write
\[
\lambda(x) = \sup \{ \lambda \in [0, 1] : x = \lambda e + (1 - \lambda)y, \ y \in B(X), \ e \in \text{Ext} B(X) \}.
\]
If \( \lambda(x) > 0 \) for all \( x \in B(X) \), then \( X \) is said to have the \( \lambda \)-property. Moreover, if \( \inf \{ \lambda(x) : x \in S(X) \} > 0 \), then \( X \) is said to have the uniform \( \lambda \)-property.

If \( X \) has the \( \lambda \)-property, then \( B(X) = \text{co}(\text{Ext} B(X)) \) and each element \( x \in B(X) \) can be expressed as \( x = \sum_{i=1}^{\infty} \lambda_i e_i \), where \( e_i \in \text{Ext} B(X) \) and \( \lambda_i > 0 \), \( \sum_{i=1}^{\infty} \lambda_i = 1 \). Moreover, if \( X \) has the uniform \( \lambda \)-property, then the series \( x = \sum_{i=1}^{\infty} \lambda_i e_i \) converges uniformly for all \( x \in B(X) \).

Define
\[
\lambda(X) = \inf \{ \lambda(x) : x \in S(X) \}.
\]
Obviously, \( \lambda(X) \) expresses the degree of \( \lambda \)-property; we call it the \( \lambda \)-coefficient of \( X \).

The \( \lambda \)-property of Orlicz spaces has been thoroughly discussed in the literature, and it is well known that the Orlicz function space \( L_M \) endowed with the Luxemburg norm has the uniform \( \lambda \)-property iff \( M \) is strictly convex on \([0, \infty)\) (for short, we write \( M \in \text{SC} \)). Indeed, if \( M \not\in \text{SC} \), then \( \lambda(L_M) = 0 \), and if \( M \in \text{SC} \), then \( \lambda(L_M) = 1 \). In this paper, we discuss Orlicz sequence spaces endowed with the Luxemburg norm, and get an interesting result that \( \lambda(l_M) \) may take every value in the harmonic number sequence \( \{1/n\}_{n=1}^{\infty} \) and 0. Hence, we can easily deduce a sufficient and necessary condition for \( l_M \) to have the uniform \( \lambda \)-property.

Let \( M : (-\infty, \infty) \to (0, \infty) \) be convex, even, continuous and \( M(u) = 0 \iff u = 0 \). For a given sequence \( x = (x_n)_{n=1}^{\infty} \), define \( g_M(x) = \sum_{n=1}^{\infty} M(x_n) \), \( l_M = \{ x = (x_n)_{n=1}^{\infty} : \exists \lambda > 0, \ g_M(\lambda x) < \infty \} \), and \( \|x\| = \inf \{ \lambda > 0 : g_M(x/\lambda) \leq 1 \} \) for \( x \in l_M \). Then \( (l_M, \| \cdot \|) \) is a Banach space. \( \text{Ext} B(l_M) \)

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Since affine interval of $M(u)$, i.e. an interval $[a, b]$ such that $M(u)$ is affine on $[a, b]$ and is not affine on $[a - \varepsilon, b]$ and $[a, b + \varepsilon]$ for any $\varepsilon > 0$. $S_M$ is the set of strictly convex points of $M(u)$ (i.e. $u \in S_M$ iff for any $\varepsilon > 0$, $M(u) < (M(u - \varepsilon) + M(u + \varepsilon))/2$).

**Lemma 1.** If $x, y, z \in B(X)$ and $x = \alpha y + (1 - \alpha)z$ for some $\alpha \in [0, 1]$, then $\lambda(x) \geq \alpha \lambda(y)$.

**Proof.** See [2].

**Lemma 2.** Let $x \in S(l_M)$. Then $x \in \text{Ext } B(l_M)$ iff (i) $g_M(x) = 1$ and (ii) $\mu \{i : x(i) \notin S_M\} \leq 1$.

**Proof.** See [5].

**Lemma 3.** $\lambda(l_M) = \inf\{\lambda(x) : g_M(x) = 1\}$.

**Proof.** Define $\lambda' = \inf\{\lambda(x) : g_M(x) = 1\}$. Obviously, $\lambda(l_M) \leq \lambda'$. For any $x \in S(l_M)$ with $g_M(x) < 1$ and $0 < \varepsilon < 1$, since $g_M(x/(1 - \varepsilon)) = \infty$, there exists $n$ such that

$$\sum_{j \leq n} M\left(\frac{x(j)}{1 - \varepsilon}\right) + \sum_{j > n} M(x(j)) \geq 1.$$ 

Select $0 < \varepsilon^* < \varepsilon$ satisfying

$$\sum_{j \leq n} M\left(\frac{x(j)}{1 - \varepsilon^*}\right) + \sum_{j > n} M(x(j)) = 1.$$ 

Take

$$y = \left(\frac{x(1)}{1 - \varepsilon'}, \frac{x(2)}{1 - \varepsilon'}, \ldots, \frac{x(n)}{1 - \varepsilon'}, x(n + 1), x(n + 2), \ldots\right).$$

Then $g_M(y) = 1$. Set $z = (0, \ldots, 0, x(n + 1), x(n + 2), \ldots)$. Clearly, $z \in B(L_M)$ and $x = (1 - \varepsilon')y + \varepsilon'z$. By Lemma 1, $\lambda(x) \geq (1 - \varepsilon')\lambda(y) \geq (1 - \varepsilon')\lambda'$. Since $x, \varepsilon'$ are arbitrary, we have $\lambda(l_M) \geq \lambda'$.

Now define

$$d_M = \sup\{d \geq 0 : M(u) \text{ is strictly convex on } [0, d]\}.$$ 

The main result of this paper is the following:

**Theorem.** Let $l_M$ be an Orlicz sequence space.

(i) If $d_M \geq M^{-1}(1/2)$, then $\lambda(l_M) = 1$.
(ii) If $M^{-1}(1/(n + 1)) \leq d_M < M^{-1}(1/n)$, then $\lambda(l_M) = 1/n$ ($n = 2, 3, \ldots$).
(iii) If $d_M = 0$, then $\lambda(l_M) = 0$.

**Proof.** (i) For any $x \in l_M$ with $g_M(x) = 1$, since $d_M \geq M^{-1}(1/2)$ and $M^{-1}(1/2) \in S_M$, we see that $\{j : x(j) \notin S_M\}$ contains at most one
element. Hence by Lemma 2, \( x \in \text{Ext} B(l_M) \). According to Lemma 3, 
\( \lambda(l_M) = \inf \{ \lambda(x) : g_M(x) = 1 \} = 1 \).

(ii) First we show 
\( \lambda(l_M) \leq 1/n. \)

Since \( M^{-1}(1/(n + 1)) \leq d_M < M^{-1}(1/n) \), there exists a SAI \([a, b]\) of \( M(u) \) such that \( d_M \leq a < M^{-1}(1/n) \). Choose \( a < c < b \) satisfying \([a, c] \subset [d_M, M^{-1}(1/n)]\). Since \( M(u) \) is strictly convex on \([0, M^{-1}(1/(n + 1))]\), we can construct a sequence 
\[
x = \left( \frac{1}{n} - \frac{1}{n} \right) a + \frac{1}{n} c, \ldots, \left( \frac{1}{n} - \frac{1}{n} \right) a + \frac{1}{n} c, x(n + 1), \ldots
\]

with \( g_M(x) = 1 \) and \( x(j) \in S_M (j > n) \).

Let \( x = \lambda e + (1 - \lambda)y \), where \( e \in \text{Ext} B(l_M) \) and \( y \in B(l_M) \). Since 
\[
1 = g_M(x) = \sum_j M(\lambda e(j) + (1 - \lambda)y(j)) \leq \lambda \sum_j M(e(j)) + (1 - \lambda) \sum_j M(y(j)) \leq \lambda + (1 - \lambda) = 1,
\]
we get \( M(\lambda e(j) + (1 - \lambda)y(j)) = \lambda M(e(j)) + (1 - \lambda)M(y(j)) \) for any \( j \), which shows that either \( x(j) = e(j) \) and \( y(j) \) are in the same SAI of \( M \), or \( x(j) = y(j) = e(j) \). Using \( x(j) \in S_M \) for any \( j > n \), we have \( x(j) = y(j) = e(j) \) for any \( j > n \). Thus 
\[
\sum_{j=1}^n M(e(j)) = 1 - \sum_{j>n} M(e(j)) = 1 - \sum_{j>n} M(x(j)) = \sum_{j=1}^n M(x(j)) \\
= \sum_{j=1}^n M \left( \frac{1}{n} - \frac{1}{n} \right) a + \frac{1}{n} c = (n - 1)M(a) + M(c).
\]

Since \( e \in \text{Ext} B(l_M) \), all elements of \( \{e(j) : 1 \leq j \leq n\} \) except possibly one are equal to \( a \) or \( b \). By the above equality, there exists no \( j \) satisfying \( e(j) = b \) \((1 \leq j \leq n)\). So \( \{j : e(j) = a\} \) contains \( n - 1 \) elements, and there exists only one index \( j_0 \) \((1 \leq j_0 \leq n)\) such that \( e(j_0) = c \). Therefore 
\[
\left( \frac{1}{n} - \frac{1}{n} \right) a + \frac{1}{n} c = x(j_0) = \lambda e(j_0) + (1 - \lambda)y(j_0) \\
= \lambda c + (1 - \lambda)y(j_0) \geq (1 - \lambda)a + \lambda c.
\]

This implies \( \lambda \leq 1/n \) and we have \( \lambda(x) \leq 1/n \) as the decomposition \( x = \lambda e + (1 - \lambda)y \) is arbitrary. So we get \( \lambda(l_M) \leq 1/n \).

From the above proof, we can deduce (iii).
Now we prove 
\[ \lambda(l_M) \geq 1/n. \]

For any \( x \in S(l_M) \setminus \text{Ext } B(l_M) \), by Lemma 3, assume \( g_M(x) = 1 \). Without loss of generality, we may assume \( x(j) \geq 0 \) for any \( j \). This part of the proof will be split into two steps. Let \( \{[a_k, b_k]\}_{k=1}^{\infty} \) be all the SAI of \( M \).

Step I: We show that \( \lambda(x) \geq \min\{\sigma, 1 - \sigma\} \). For each \( \lambda \in [0, 1] \), define
\[
x_\lambda(j) = \begin{cases} 
  b_k, & b_k > x(j) > \lambda a_k + (1 - \lambda)b_k, \\
  a_k, & \lambda a_k + (1 - \lambda)b_k \geq x(j) > a_k, \\
  x(j), & \text{otherwise}.
\end{cases}
\]
Then the function \( f(\lambda) = g_M(x_\lambda) \) is nondecreasing. As \( \{j : x(j) \notin S_M\} \) contains at least two elements, \( g_M(x_0) < g_M(x) = 1 \) and \( g_M(x_1) > g_M(x) = 1 \).

Define
\[
\sigma = \sup\{\lambda : g_M(x_\lambda) \leq 1\}.
\]
As \( d_M \geq M^{-1}(1/(n+1)) \), \( \{j : x(j) \notin S_M\} \) is a finite set. Clearly, \( 0 < \sigma < 1 \).

Write
\[
N_k = \{j : x(j) = \sigma a_k + (1 - \sigma)b_k\}.
\]
If \( g_M(x_\sigma) = 1 \), then set \( e = x_\sigma \). If \( g_M(x_\sigma) < 1 \), then \( \bigcup_k N_k \neq \emptyset \). Thus there exist \( E_k \subset N_k \) (\( k \geq 1 \)) such that \( g_M(u_\sigma) \leq 1 \), where
\[
u_\sigma(j) = \begin{cases} 
  b_k, & b_k > x(j) > \sigma a_k + (1 - \sigma)b_k \text{ or } j \in E_k, \\
  a_k, & \sigma a_k + (1 - \sigma)b_k > x(j) > a_k \text{ or } j \in N_k \setminus E_k, \\
  x(j), & \text{otherwise},
\end{cases}
\]
and for any \( j \in N_k \setminus E_k \), if we set \( u_\sigma(j) = b_k \), then \( g_M(u_\sigma) > 1 \).

If \( g_M(u_\sigma) = 1 \), set \( e = u_\sigma \). If \( g_M(u_\sigma) < 1 \), we can take an index \( k' \) such that \( N_{k'} \setminus E_{k'} \neq \emptyset \). Select \( \alpha \in (a_{k'}, b_{k'}) \) and \( j' \in N_{k'} \setminus E_{k'} \) satisfying \( g_M(e) = 1 \), where
\[
e(j) = \begin{cases} 
  \alpha, & j = j', \\
  u_\sigma(j), & j \neq j'.
\end{cases}
\]
By Lemma 2, \( e \in \text{Ext } B(l_M) \).

If \( \sigma \geq 1/2 \), take \( z \) with \( x = (1 - \sigma)e + \sigma z \), and if \( \sigma < 1/2 \), take \( z \) with \( x = \sigma e + (1 - \sigma)z \). In both these cases, we can prove \( g_M(z) = 1 \). We only discuss the case \( \sigma \geq 1/2 \) (the case \( \sigma < 1/2 \) is similar).

If \( x(j) = e(j) \), then \( z(j) = x(j) = e(j) \).
If \( x(j) \leq \sigma a_k + (1 - \sigma) b_k \) and \( e(j) = a_k \), then
\[
a_k < x(j) \leq z(j) = \frac{1}{\sigma}(x(j) - (1 - \sigma)e(j))
\]
\[
\leq \frac{1}{\sigma} (\sigma a_k + (1 - \sigma) b_k - (1 - \sigma) a_k) = a_k + \left( \frac{1}{\sigma} - 1 \right) (b_k - a_k)
\]
\[
\leq a_k + (b_k - a_k) = b_k.
\]
If \( x(j) \geq \sigma a_k + (1 - \sigma) b_k \) and \( e(j) = b_k \), then
\[
b_k > x(j) \geq z(j) \geq \frac{1}{\sigma} (\sigma a_k + (1 - \sigma) b_k - (1 - \sigma) a_k) = a_k.
\]
If \( x(j) = \sigma a_k + (1 - \sigma) b_k < \alpha = e(j) \), then
\[
b_k > x(j) \geq z(j) = \frac{1}{\sigma} (\sigma a_k + (1 - \sigma) b_k - (1 - \sigma) \alpha)
\]
\[
\geq \frac{1}{\sigma} (\sigma a_k + (1 - \sigma) b_k - (1 - \sigma) b_k) = a_k.
\]
If \( x(j) = \sigma a_k + (1 - \sigma) b_k \geq \alpha = e(j) \), then
\[
a_k < x(j) \leq z(j) = \frac{1}{\sigma} (\sigma a_k + (1 - \sigma) b_k - (1 - \sigma) \alpha)
\]
\[
\leq \frac{1}{\sigma} (\sigma a_k + (1 - \sigma) b_k - (1 - \sigma) a_k) = a_k + \left( \frac{1}{\sigma} - 1 \right) (b_k - a_k) \leq b_k.
\]

Thus either \( x(j) = e(j) = z(j) \), or \( x(j), e(j) \) and \( z(j) \) are in the same SAI of \( M \). Hence
\[
1 = \varrho_M(x) = \varrho_M((1 - \sigma)e + \sigma z) = (1 - \sigma) \varrho_M(e) + \sigma \varrho_M(z) = 1 - \sigma + \sigma \varrho_M(z).
\]
This shows that \( \varrho_M(z) = 1 \), and thus \( \lambda(x) \geq 1 - \sigma \). Similarly, if \( \sigma < 1/2 \), we can get \( \lambda(x) \geq \sigma \). Consequently, \( \lambda(x) \geq \min\{\sigma, 1 - \sigma\} \).

**Step II:** We prove \( \lambda(x) \geq 1/n \). If \( \sigma \geq 1/2 \), then by Step I, \( \lambda(x) \geq 1 - \sigma \).

If \( 1 - \sigma \geq 1/n \), then the proof is complete. Conversely, if \( 1 - \sigma < 1/n \), then rearrange \( x(j) \) by putting \( x(j) \) at the beginning if \( x(j) \not\in S_M \). Assume \( x(j) \not\in S_M \) \( (j = 1, \ldots, m) \), i.e. for \( 1 \leq j \leq m \), \( x(j) = (1 - \lambda_j) a_j + \lambda_j b_j \), where \( 0 < \lambda_j < 1 \) and \( |a_j, b_j| \) is a SAI of \( M \).

Now \( x \not\in \text{Ext} B(l_M) \) implies \( m \geq 2 \). Notice that \( d_M \geq M^{-1}(1/(n + 1)) \).

We deduce that
\[
1 = \varrho_M(x) \geq \sum_{j=1}^{m} M(x(j)) > \sum_{j=1}^{m} M(d_M)
\]
\[
\geq \sum_{j=1}^{m} M \left( M^{-1} \left( \frac{1}{n + 1} \right) \right) = \frac{m}{n + 1}.
\]
So \( m \leq n \). Define
\[
J = \{1 \leq j \leq m : \lambda_j \leq 1/n, \ \lambda_j \text{ is the coefficient of} \ x(j) = (1 - \lambda_j) a_j + \lambda_j b_j \}.
\]
Then \( J \neq \emptyset \). Otherwise, if \( \lambda_j > 1/n \) for any \( 1 \leq j \leq m \), then \( x_{1-1/n}(j) = b_j \) (\( 1 \leq j \leq m \)). Hence \( g_M(x_{1-1/n}) > 1 \). But \( g_M(x_\sigma) \leq 1 \), and we obtain \( \sigma < 1 - 1/n \), which contradicts \( 1 - \sigma < 1/n \).

By rearranging again, assume \( J = \{1, \ldots, r\} \) (\( r \leq m \)) with
\[
\lambda_r(M(b_r) - M(a_r)) = \max_{i \leq r} \lambda_i(M(b_i) - M(a_i)).
\]
For each \( \delta \in [0, 1] \), consider
\[
y_\delta(j) = \begin{cases} 
a_j, & j < r, 
(1 - \delta)a_j + \delta b_j, & j = r, 
b_j, & r < j \leq m, 
x(j), & j > m.
\end{cases}
\]
Clearly the function \( f(\delta) = \varphi_M(y_\delta) \) is nondecreasing, and \( \varphi_M(y_0) = \varphi_M(x_\sigma) \leq 1 \). Notice that \( r\lambda_r \leq m/n \leq 1 \), and therefore, \( y_{r\lambda_r} \) has a meaning. We have
\[
\varphi_M(y_{r\lambda_r}) - 1 = \sum_{j < r} M(a_j) + M((1 - r\lambda_r)a_r + r\lambda_r b_r) + \sum_{j = r+1}^m M(b_j) + \sum_{j > m} M(x(j)) - \sum_{j = 1}^r ((1 - \lambda_j)M(a_j) + \lambda_j M(b_j)) - \sum_{j = r+1}^m M(x(j)) - \sum_{j > m} M(x(j)) \geq - \sum_{j = 1}^r \lambda_j(M(b_j) - M(a_j)) + r\lambda_r(M(b_r) - M(a_r)) \geq - r\lambda_r(M(b_r) - M(a_r)) + r\lambda_r(M(b_r) - M(a_r)) = 0.
\]
Hence there exists \( \delta \in [0, r\lambda_r] \) such that \( \varphi_M(y_\delta) = 1 \).

By Lemma 1, \( y_\delta \in \text{Ext} B(l_M) \). Suppose that \( z \) satisfies \( x = (1/n)y_\delta + (1 - 1/n)z \). To prove \( \lambda(x) \geq 1/n \), it suffices to verify \( z \in B(l_M) \). As in Step 1, we need to show that either \( z(j) = y_\delta(j) = x(j) \), or \( z(j) = y_\delta(j) \) and \( x(j) \) are in the same SAI of \( M \).

If \( j > m \), then \( z(j) = y_\delta(j) = x(j) \).
If $j < r$, notice that $\lambda_j \leq 1/n$ and $n \geq 2$; then
\[
a_j < x(j) \leq z(j) = \frac{1}{1 - 1/n} \left( x(j) - \frac{1}{n} y_j(j) \right)
\]
\[
= \frac{1}{1 - 1/n} \left( 1 - \lambda_j \right) a_j + \lambda_j b_j - \frac{1}{n} a_j
\]
\[
\leq \frac{1}{1 - 1/n} \left( 1 - \frac{1}{n} \right) a_j + \frac{1}{n} (b_j - a_j)
\]
\[
= a_j + \frac{1}{1 - 1/n} (b_j - a_j) \leq a_j + (b_j - a_j) = b_j.
\]

If $r < j \leq m$, notice that $\lambda_j > 1/n$; then
\[
b_j > x(j) \geq z(j) = \frac{1}{1 - 1/n} \left( 1 - \lambda_j \right) a_j + \lambda_j b_j - \frac{1}{n} b_j
\]
\[
\geq \frac{1}{1 - 1/n} \left( 1 - \frac{1}{n} \right) a_j + \frac{1}{n} b_j - \frac{1}{n} b_j = a_j.
\]

If $j = r$ and $(1 - \delta)a_r + \delta b_r \leq (1 - \lambda_r)a_r + \lambda_r b_r = x(r)$, then
\[
a_r < x(r) \leq z(r)
\]
\[
= \frac{1}{1 - 1/n} \left( 1 - \lambda_r \right) a_r + \lambda_r b_r - \frac{1}{n} ((1 - \delta)a_r + b_r)
\]
\[
\leq \frac{1}{1 - 1/n} \left( 1 - \frac{1}{n} \right) a_r + \frac{1}{n} b_r - \frac{1}{n} a_r
\]
\[
= a_r + \frac{1}{1 - 1/n} (b_r - a_r) \leq b_r.
\]

If $j = r$ and $(1 - \delta)a_r + \delta b_r > (1 - \lambda_r)a_r + \lambda_r b_r = x(r)$, then
\[
b_r > x(r) \geq z(r)
\]
\[
= \frac{1}{1 - 1/n} \left( 1 - \lambda_r \right) a_r + \lambda_r b_r - \frac{1}{n} ((1 - \delta)a_r + b_r)
\]
\[
= \frac{1}{1 - 1/n} \left( 1 - \frac{1}{n} \right) a_r + \frac{1}{n} (1 - \lambda_r)(b_r - a_r)
\]
\[
= a_r + \frac{1}{1 - 1/n} \left( 1 - \frac{1}{n} \right) (b_r - a_r).
\]

By $\lambda_r \geq \delta/r \geq \delta/n$, we have $b_r > z(r) \geq a_r$. Thus $\lambda(x) \geq 1/n$. Since $x$ is arbitrary, by Lemma 3 we conclude that $\lambda(l_M) \geq 1/n$.

The theorem immediately yields

**Corollary.** $l_M$ has the uniform $\lambda$-property iff $d_M > 0$. 
REFERENCES


