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THE UNCONDITIONAL POINTWISE CONVERGENCE OF ORTHOGONAL SERIES IN L_2 OVER A VON NEUMANN ALGEBRA

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1. Introduction. The paper is devoted to some problems concerning a convergence of pointwise type in the L_2-space over a von Neumann algebra M with a faithful normal state Φ [3]. Here $L_2 = L_2(M, \Phi)$ is the completion of M under the norm $x \to \|x\|^2 = \Phi(x^*x)^{1/2}$.

Intuitively, "reducing" L_2 to L_∞, we can say that $|f|$ is smaller than ε on a subset $Z \in \mathcal{F}$, for $f \in L_2$ over a classical measure space $(X, \mathcal{F}, \mu)$, if $g_n \to f$ in L_2 and $\|g_n \mathbf{1}_Z\|_\infty < \varepsilon$ for some $g_n \in L_\infty(X, \mathcal{F}, \mu)$, $n = 1, 2, \ldots$

This leads us to the following concept. Roughly speaking, $\xi \in L_2(M, \Phi)$ has "modulus" less than ε on the subspace indicated by a projection $p \in M$ if, for some $x_n \in M$, $\|x_n - \xi\| \to 0$ and $\|x_n p\|_\infty < \varepsilon$, $n = 1, 2, \ldots$, where $\|x\|_\infty$ denotes the (operator) norm of $x \in M$.

We say that $\xi_n \to 0$ "almost surely" if, for any $\varepsilon > 0$, there exists a projection $p \in M$ such that ξ_n has "modulus" less than ε on the (subspace indicated by the) projection p for n large enough and, moreover, $\Phi(\mathbf{1} - p) < \varepsilon$.

Precise definitions are given in the next section.

It is worth noting that several limit theorems can be proved in the von Neumann algebras context by using the above concept of convergence [5; 6; 10; 11].

In comparison with other concepts of "almost sure" convergence for (unbounded observables forming) $L_2(M, \Phi)$ [4; 7; 9], our proposal seems to be intuitively clear. It gives a fairly natural extension of the almost uniform convergence in an algebra [2; 8; 12; 14; 15; 16; see also 9].

The main method used in the paper is the maximal inequality for a von Neumann algebra (Theorem 2.2) which can be proved quite elementarily and is very useful in the study of convergences on subspaces (cf. [7], compare also the maximal ergodic theorem of M. S. Goldstein [4]).

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As an illustration, we prove the noncommutative extension of the rather advanced theorem of Tandori [17; 1] which gives the weakest condition implying the unconditional convergence of orthogonal series (Theorem 2.2). Some discussion of the noncommutative Cauchy condition will also be necessary (cf. Section 3).

2. Notation and main results. Throughout the paper, M is a σ-finite von Neumann algebra with a faithful normal state Φ [3] (acting in a Hilbert space $\mathcal{K}$), with the Hilbert space $H = L_2(M,\Phi)$ being the completion of the algebra M under the norm $\|x\| = \Phi(x^*x)^{1/2}$, given by the scalar product $(x,y) = \Phi(y^*x)$. For $x \in M$, we put $|x|^2 = x^*x$. $\operatorname{Proj} M$ will denote the lattice of all self-adjoint projections in M. We write $p^\perp = \mathbf{1} - p$ for $p \in \operatorname{Proj} M$. The operator norm in M will be denoted by $\| \, \|_\infty$.

For $\xi \in H$ and $p \in \operatorname{Proj} M$, we set

$$S_{\xi,p} = \Big\{ (x_k) \subset M : \sum_{k=1}^{\infty} x_k = \xi \text{ in } H \text{ and } \sum_{k=1}^{\infty} x_k p \text{ converges in norm in } M \Big\}$$

and

$$\|\xi\|_p = \inf \Big\{ \Big\| \sum_{k=1}^{\infty} x_k p \Big\|_\infty : (x_k) \in S_{\xi,p} \Big\}$$

(with the usual convention $\inf \emptyset = +\infty$).

We adopt the following definition.

2.1. Definition. A sequence $(\xi_n) \subset H$ is said to be *almost surely* (a.s.) *convergent* to $\xi \in H$ if, for each $\varepsilon > 0$, there exists a projection p in M such that $\Phi(p^\perp) < \varepsilon$ and $\|\xi_n - \xi\|_p \to 0$ as $n \to \infty$.

The following theorem gives a kind of maximal inequality and is crucial in our considerations.

2.2. Theorem. *Let* $0 < \varepsilon < 1/16$, $D_n \in M^+$ *for* $n = 1, 2, \ldots$ *and*

$$\sum_{k=1}^{\infty} \Phi(D_k) < \varepsilon. \tag{1}$$

Then there exists $p \in \operatorname{Proj} M$ *such that*

$$\Phi(p^\perp) < \varepsilon^{1/4}, \tag{2}$$

$$\Big\| p\Big(\sum_{k=1}^{n} D_k\Big)p \Big\|_\infty < 4\varepsilon^{1/2}, \qquad n = 1, 2, \ldots \tag{3}$$

As a main example of consequences of this maximal inequality we prove the following extension to the noncommutative context of the classical Tandori theorem on the unconditional almost sure convergence of orthogonal series.

2.3. THEOREM. *Let* $(\xi_n)_{n=1}^\infty$ *be a sequence of pairwise orthogonal elements in* H *and*

$$\sum_{k=0}^{\infty}\Big(\sum_{n\in I_k}\|\xi_n\|^2\log^2(n+1)\Big)^{1/2}<\infty, \tag{4}$$

where $I_k=\{2^{2^k}+1,\ldots,2^{2^{k+1}}\}$. *Then*, *for each permutation* π *of the set* $\mathbb{N}$ *of positive integers*, *the series* $\sum_{k=0}^\infty \xi_{\pi(k)}$ *is a.s. convergent.*

The theorem below, an analogue of Orlicz's theorem, can be deduced from the previous one.

2.4. THEOREM. *Let* $(\xi_n)_{n=1}^\infty$ *be an orthogonal sequence in* H. *Let* (w_n) *be a nondecreasing sequence of positive numbers such that*

$$\sum_{m=1}^{\infty} 1/w_{\nu_m}<\infty$$

for some increasing sequence (ν_m) *of positive integers satisfying the inequalities*

$$\log\nu_{m+1}\le c\log\nu_m \qquad (c>1,\ m=1,2,\ldots).$$

If

$$\sum_{n=1}^{\infty}\|\xi_n\|^2 w_n\log^2(n+1)<\infty,$$

then, *for each permutation* π *of* $\mathbb{N}$, *the series* $\sum_{k=1}^\infty \xi_{\pi(k)}$ *is a.s. convergent.*

3. Noncommutative Cauchy condition. We start with some properties of a.s. convergence, interesting in their own right.

3.1. LEMMA. *Let* $(\eta_n)\subset H$ *and* $p\in\operatorname{Proj}M$. *If* $\sum_{n=1}^\infty \eta_n$ *is convergent in* H, *then*

$$\Big\|\sum_{n=1}^{\infty}\eta_n\Big\|_p\le\sum_{n=1}^{\infty}\|\eta_n\|_p. \tag{5}$$

Proof. Without any loss of generality we may assume that

$$\sum_{n=1}^{\infty}\|\eta_n\|_p<\infty. \tag{6}$$

Let $\varepsilon>0$ and $\varepsilon_n=\varepsilon 2^{-n}$, $\varepsilon_{n,s}=\varepsilon 2^{-n-s}$ $(n=1,2,\ldots)$. Then there exist $y_{n,k}\in M$ $(n,k=1,2,\ldots)$ such that

$$\Big\|\eta_n-\sum_{k=1}^{s}y_{n,k}\Big\|<\varepsilon_{n,s}, \tag{7}$$

$$\|y_{n,s+1}p\|_\infty<\varepsilon_{n,s}, \tag{8}$$

$$\Big\| \sum_{k=1}^{s} y_{n,k} p \Big\|_\infty < \|\eta_n\|_p + \varepsilon_n, \qquad n, s = 1, 2, \ldots \tag{9}$$

Put $\xi = \sum_{k=1}^\infty \eta_k$ and $x_n = \sum_{k,l=1}^n y_{k,l}$ for $n = 1, 2, \ldots$ Then $x_n \in H$. First, we remark that $\|\xi - x_n\| \to 0$. In fact, by (7), we have

$$\begin{aligned} \|\xi - x_n\| &\le \sum_{k=1}^n \Big\| \eta_k - \sum_{l=1}^n y_{k,l} \Big\| + \Big\| \sum_{k=n+1}^\infty \eta_k \Big\| \\ &\le \sum_{k=1}^n \varepsilon_{k,n} + \varrho_n < \varepsilon_n + \varrho_n, \end{aligned}$$

where $\varrho_n = \| \sum_{k=n+1}^\infty \eta_k \| \to 0$ as $n \to \infty$. Now, we notice that $(x_n p)_{n=1}^\infty$ is convergent in M. Indeed, by (8) and (9), we have

$$\begin{aligned} \|(x_{n+1} - x_n)p\|_\infty &\le \sum_{k=1}^n \|y_{k,n+1} p\|_\infty + \Big\| \sum_{l=1}^{n+1} y_{n+1,l} p \Big\|_\infty \\ &< \sum_{k=1}^n \varepsilon_{k,n} + \|\eta_{n+1}\|_p + \varepsilon_n < 2\varepsilon_n + \|\eta_{n+1}\|_p, \end{aligned}$$

and (6) yields the Cauchy condition for $(x_n p)_{n=1}^\infty$. Finally, by (9), we have

$$\|x_n p\|_\infty \le \sum_{k=1}^n \Big\| \sum_{l=1}^n y_{k,l} p \Big\|_\infty \le \sum_{k=1}^n (\|\eta_k\|_p + \varepsilon_k) < \sum_{k=1}^n \|\eta_k\|_p + \varepsilon.$$

Hence we get (5). ∎

The next theorem gives a kind of noncommutative Cauchy condition for a.s. convergence.

3.2. PROPOSITION. *Let* $(\sigma_n) \subset H$ *and* $\|\sigma_n - \sigma\| \to 0$ *as* $n \to \infty$, *where* $\sigma \in H$. *If*, *for each* $\varepsilon > 0$, *there exists some* $p \in \operatorname{Proj} M$ *with* $\Phi(p^\perp) < \varepsilon$ *such that* $\|\sigma_n - \sigma_m\|_p \to 0$ *as* $n, m \to \infty$, *then* $\sigma_n \to \sigma$ *a.s.*

Proof. By the assumption, for $\varepsilon > 0$, there are $p \in \operatorname{Proj} M$ with $\Phi(p^\perp) < \varepsilon$ and a sequence of indices $m_0 < m_1 < \ldots$ such that

$$\|\sigma_n - \sigma_m\|_p < \varepsilon 2^{-k} \qquad \text{for } n, m \ge m_k. \tag{10}$$

Let $n \ge m_0$. Fix k such that $n < m_k$. Putting $\eta_0 = \sigma_{m_k} - \sigma_n$, $\eta_1 = \sigma_{m_{k+1}} - \sigma_{m_k}, \ldots, \eta_j = \sigma_{m_{k+j}} - \sigma_{m_{k+j-1}}, \ldots$, we obtain $\sigma - \sigma_n = \sum_{j=0}^\infty \eta_j$, where the series is convergent in H. Moreover, by (10), we have $\|\eta_0\|_p < \varepsilon$, $\|\eta_1\|_p < \varepsilon 2^{-k}, \ldots, \|\eta_j\|_p < \varepsilon 2^{-k-j+1}, \ldots$ Thus $\sum_{j=0}^\infty \|\eta_j\|_p < 2\varepsilon$ and, by Lemma 3.1, we get $\|\sigma - \sigma_n\|_p < 2\varepsilon$ for $n > m_0$. This completes the proof. ∎

4. Auxiliary consequences of the maximal inequality. For the sake of completeness we reproduce the proof given in [7, 3.10].

Proof of Theorem 2.2. For brevity, we define $B_n = \sum_{k=1}^{n} D_k$, $n = 1, 2, \ldots$ Put

$$p_n = e_{B_n}([0, \varepsilon^{1/2}]), \qquad n = 1, 2, \ldots, \tag{11}$$

where $B_n = \int_0^\infty \lambda\, e_{B_n}(d\lambda)$ is the spectral representation. The sequence $(p_n)_{n=1}^\infty$ of projections is conditionally weakly operator compact. Let a be a limit point, i.e.

$$a = \text{w.o.-}\lim_{k\to\infty} p_{n(k)} \tag{12}$$

for some subsequence $(n(k))$. Obviously, $a \in M$, $0 \le a \le \mathbf{1}$. Put

$$p = e_a([1 - \varepsilon^{1/4}, 1]), \tag{13}$$

where $a = \int_0^1 \lambda e_a(d\lambda)$.

By (1) and (11), we obtain

$$\Phi(p_n^\perp) = \Phi(e_{B_n}((\varepsilon^{1/2}, \infty))) \le \varepsilon^{-1/2}\Phi(B_n) < \varepsilon^{1/2}.$$

Consequently, $\Phi(a) = \lim_{k\to\infty} \Phi(p_{n(k)}) \ge 1 - \varepsilon^{1/2}$ and, finally,

$$\Phi(\mathbf{1} - a) \le \varepsilon^{1/2}. \tag{14}$$

On the other hand, by (13), we can write $p = e_{\mathbf{1}-a}([0, \varepsilon^{1/4}])$. Then, by (14), we obtain

$$\Phi(p^\perp) = \Phi(e_{\mathbf{1}-a}((\varepsilon^{1/4}, 1])) \le e^{-1/4}\Phi(1 - a) \le \varepsilon^{1/4},$$

which proves (2).

To show (3), we estimate $(B_n\xi, \xi)$ for all $\xi \in p\mathcal{K}$ with $\|\xi\| = 1$.

Obviously, the subspace $p\mathcal{K}$ is invariant for a. Moreover, by (13), the spectrum of the operator $a_p = a|_{p\mathcal{K}}$ is contained in the interval $[1 - \varepsilon^{1/4}, 1]$. Thus, a_p is invertible, a_p^{-1} is defined on $p\mathcal{K}$ and $\|a_p^{-1}\|_\infty \le (1 - \varepsilon^{1/4})^{-1}$. Fix $\xi \in p\mathcal{K}$, $\|\xi\| = 1$ and put $\zeta = a_p^{-1}\xi$. Then $\zeta \in p\mathcal{K}$ and

$$\|\zeta\| \le (1 - \varepsilon^{1/4})^{-1}. \tag{15}$$

Define $\eta_k = p_{n(k)}\zeta - \xi$. By (12), η_k converges weakly to 0 as $k \to \infty$. Therefore, by the positivity of B_n,

$$\liminf_{k\to\infty}((B_n\eta_k, \eta_k) + (B_n\eta_k, \xi) + (B_n\xi, \eta_k)) \ge 0.$$

Hence, by (11) and (15), we get

$$\begin{aligned}(B_n\xi,\xi) &\le \liminf_{k\to\infty}(B_n(\eta_k+\xi),\eta_k+\xi) = \liminf_{k\to\infty}(B_n p_{n(k)}\zeta, p_{n(k)}\zeta)\\ &\le \liminf_{k\to\infty}\|p_{n(k)}B_n p_{n(k)}\|_\infty\,\|\zeta\|^2\\ &\le \liminf_{k\to\infty}\|p_{n(k)}B_{n(k)} p_{n(k)}\|_\infty\,\|\zeta\|^2\\ &\le \varepsilon^{1/2}(1-\varepsilon^{1/4})^{-2} < 4\varepsilon^{1/2},\end{aligned}$$

which gives (3). The proof is complete. ∎

4.1. Lemma. *Let* $0<\varepsilon<1/16$, $D_n\in M^+$, $\zeta_n\in H$ *for* $n=1,2,\ldots$ *and*

$$\sum_{k=1}^{\infty}\Phi(D_k)<\varepsilon, \tag{16}$$

$$\sum_{k=1}^{\infty}\|\zeta_k\|^{1/2}<\varepsilon. \tag{17}$$

Then there exists $p\in\operatorname{Proj}M$ *such that*

$$\Phi(p^\perp)<2\varepsilon^{1/4}, \tag{18}$$

$$\Big\|p\Big(\sum_{k=1}^{n}D_k\Big)p\Big\|_\infty<9\varepsilon^{1/2},\qquad n=1,2,\ldots, \tag{19}$$

$$\sum_{k=1}^{\infty}\|\zeta_k\|_p<8\varepsilon^{5/4}. \tag{20}$$

Moreover, *if condition* (17) *is replaced by*

$$\sum_{k=1}^{\infty}\|\zeta_k\|^2<\varepsilon, \tag{21}$$

then (20) *can be replaced by*

$$\|\zeta_n\|_p<8\varepsilon^{1/4},\qquad n=1,2,\ldots \tag{22}$$

P r o o f. Choose $z_{k,l}\in M$ such that $\zeta_k=\sum_{l=1}^{\infty}z_{k,l}$ in H, and

$$\|z_{k,l}\|\le 2^{-l+1}\|\zeta_k\|,\qquad k,l=1,2,\ldots \tag{23}$$

Putting

$$D_{k,0}=D_k,\quad D_{k,l}=2^l\|\zeta_k\|^{-1}|z_{k,l}|^2,\quad k,l=1,2,\ldots,$$

we obtain, by (23), $\Phi(D_{k,l})\le 2^{-l+2}\|\zeta_k\|$ for $k,l=1,2,\ldots$ Thus, by (16) and (17),

$$\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\Phi(D_{k,l}) \le \sum_{k=1}^{\infty}\Phi(D_k) + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}2^{-l+2}\|\zeta_k\| < \varepsilon + 4\sum_{k=1}^{\infty}\|\zeta_k\| < 5\varepsilon.$$

Now, by Theorem 2.2, there exists $p \in \operatorname{Proj} M$ such that $\Phi(p^{\perp}) < (5\varepsilon)^{1/4} < 2\varepsilon^{1/4}$ and

$$\Big\|p\Big(\sum_{k=1}^{n}\sum_{l=0}^{m}D_{k,l}\Big)p\Big\|_{\infty} < 4(5\varepsilon)^{1/2} < 9\varepsilon^{1/2}, \quad n = 1,2,\ldots,\ m = 0,1,2,\ldots$$

In particular, $\|p(\sum_{k=1}^{n}D_k)p\|_{\infty} < 9\varepsilon^{1/2}$ and $\|z_{k,l}p\|_{\infty}^2 \le 2^{-l}\|\zeta_k\|\,9\varepsilon^{1/2}$ for $k,l = 1,2,\ldots$ Thus

$$\|\zeta_k\|_p \le \Big\|\sum_{l=1}^{\infty}z_{k,l}\Big\|_{\infty} \le \sum_{l=1}^{\infty}2^{-l/2}\,3\varepsilon^{1/4}\|\zeta_k\|^{1/2} < 8\varepsilon^{1/4}\|\zeta_k\|^{1/2},$$

which, by (17), gives (20) easily.

The last part of the theorem can be proved in the same manner. ∎

The following two propositions are simple consequences of Lemma 4.1.

4.2. Proposition. *Let $D_n \in M^+$, $\zeta_n \in H$ for $n = 1,2,\ldots$ and*

$$\sum_{k=1}^{\infty}\Phi(D_k) < \infty, \qquad \sum_{k=1}^{\infty}\|\zeta_k\|^{1/2} < \infty.$$

Then for each $\varepsilon > 0$ there exists $p \in \operatorname{Proj} M$ with $\Phi(p^{\perp}) < \varepsilon$ such that the sequence $(\|p(\sum_{k=1}^{n}D_k)p\|_{\infty})_{n=1}^{\infty}$ is bounded and

$$\sum_{k=1}^{\infty}\|\zeta_k\|_p < \infty.$$

4.3. Proposition. *Let $D_n \in M^+$, $B_n \in M^+$, $\zeta_n \in H$ for $n = 1,2,\ldots$ and*

$$\sum_{k=1}^{\infty}\Phi(D_k) < \infty, \quad \sum_{k=1}^{\infty}\Phi(B_k) < \infty, \quad \sum_{k=1}^{\infty}\|\zeta_k\|^2 < \infty.$$

Then for each $\varepsilon > 0$ there exists $p \in \operatorname{Proj} M$ with $\Phi(p^{\perp}) < \varepsilon$ such that the sequence $(\|p(\sum_{k=1}^{n}D_k)p\|_{\infty})_{n=1}^{\infty}$ is bounded, $\|pB_np\|_{\infty} \to 0$ and $\|\zeta_n\|_p \to 0$ as $n \to \infty$.

Proof. Consider $(D_k + B_k)_{k=1}^{\infty}$ and apply Lemma 4.1. ∎

5. Other auxiliary results. The following simple lemma is in the spirit of the classical Schwarz inequality and is very useful in many estimations.

5.1. Lemma. *Let* $\varepsilon_k > 0$, $x_k \in M$, $E_k \in M^+$ *and* $|x_k|^2 \le \varepsilon_k E_k$ *for* $k = 1, \ldots, n$. *Then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_\infty \le \Big\|\sum_{k=1}^n E_k\Big\|_\infty^{1/2}\Big(\sum_{k=1}^n \varepsilon_k\Big)^{1/2}.$$

Proof. For $\xi \in \mathcal{K}$ with $\|\xi\| = 1$, we have

$$\begin{aligned}\Big\|\sum_{k=1}^n x_k\xi\Big\|^2 &\le \Big(\sum_{k=1}^n \varepsilon_k^{-1/2}\|x_k\xi\|\varepsilon_k^{1/2}\Big)^2\\ &\le \Big(\sum_{k=1}^n \varepsilon_k^{-1}\|x_k\xi\|^2\Big)\sum_{k=1}^n \varepsilon_k \le \Big(\sum_{k=1}^n E_k\xi, \xi\Big)\sum_{k=1}^n \varepsilon_k\\ &\le \Big\|\sum_{k=1}^n E_k\Big\|_\infty \sum_{k=1}^n \varepsilon_k. \ \blacksquare\end{aligned}$$

The following lemma is, in fact, an easy modification of the Lemma of [8] (cf. also Proposition 4.2 of [9]).

5.2. Lemma. *Let* $J \subset \mathbb{N}$ *have cardinality* $\#J = \mu$. *Let* $(y_i)_{i=1}^\infty$ *be a sequence of operators in* M *such that* $y_i = 0$ *for* $i \notin J$. *Then there exists an operator* $B \in M^+$ *such that*

$$\Big|\sum_{j=1}^n y_j\Big|^2 \le B, \quad n = 1, 2, \ldots,$$

and

$$\Phi(B) \le (1 + \log\mu)^2 \sum_{i,j\in J} |\Phi(y_i^* y_j)|.$$

The next lemma is also a slight modification of Lemma 4.2 of [6] (cf. also Lemma 5.2.2 of [11]).

5.3. Lemma. *Let* $J \subset \mathbb{N}$ *with* $\#J = \mu$. *Let* $(\eta_i)_{i=1}^\infty$ *be a sequence of pairwise orthogonal elements in* H *such that* $\eta_i = 0$ *for* $i \notin J$. *Let* (ε_i) *be a sequence of positive numbers. Then there exist operators* $B \in M^+$ *and* $y_i \in M$ $(i \in \mathbb{N})$ *with* $y_i = 0$ *when* $\eta_i = 0$, *such that*

$$\|\eta_i - y_i\| < \varepsilon_i, \quad i = 1, 2, \ldots,$$

$$\Big|\sum_{i=1}^n y_i\Big|^2 \le B, \quad n = 1, 2, \ldots,$$

$$\Phi(B) \le 2(1 + \log\mu)^2 \sum_{i=1}^\infty \|\eta_i\|^2.$$

6. Proof of Tandori's theorem

P r o o f o f T h e o r e m 2.3. First, we notice that, by assumption (4), the sequence $(\sigma_n) = (\sum_{k=1}^{n} \xi_{\pi(k)})_{n=1}^{\infty}$ is convergent in H. By Proposition 3.2, it remains to prove the Cauchy condition for a.s. convergence. We put additionally $I_0 = \{1, \ldots, 4\}$.

For brevity, we write

$$\alpha_k = \Big(\sum_{n\in I_k} \|\xi_n\|^2 \log^2(n+1)\Big)^{1/2}, \qquad k = 1, 2, \ldots, \tag{24}$$

and set

$$\eta_{k,i} = \begin{cases} \alpha_k^{-1}\xi_{\pi(i)} & \text{as } \pi(i) \in I_k, \\ 0 & \text{otherwise.} \end{cases}$$

Fix k and, taking $J = \pi^{-1}[I_k]$, apply Lemma 5.3 to the sequence $(\eta_{k,i})_{i=1}^{\infty}$. Then there exist operators $D_k \in M^+$, $y_{k,i} \in M$ $(i = 1, 2, \ldots)$ such that

$$\begin{aligned} \|\eta_{k,i} - y_{k,i}\| &< 2^{-i} \quad \text{if } \pi(i) \in I_k, \\ y_{k,i} &= 0 \quad \text{if } \pi(i) \notin I_k, \end{aligned} \tag{26}$$

$$|s_{k,l}|^2 \le D_k \quad \text{for } l = 1, 2, \ldots, \tag{27}$$

where $s_{k,l} = y_{k,1} + \ldots + y_{k,l}$ $(l = 1, 2, \ldots)$ and

$$\Phi(D_k) \le 2(2^{k+1}+1)^2 \sum_{i=1}^{\infty} \|\eta_{k,i}\|^2$$

(the cardinality $\# I_k$ is less than $2^{2^{k+1}}$).

Thus, by (25), (24) and the definition of I_k, we obtain

$$\begin{aligned} \Phi(D_k) &\le 4 \sum_{n\in I_k} (1+\log n)^2 \alpha_k^{-1} \|\xi_n\|^2 \\ &\le 16\alpha_k^{-1} \sum_{n\in I_k} \log^2(n+1)\|\xi_n\|^2 = 16\alpha_k. \end{aligned}$$

Therefore, by assumption (4) of our theorem, we get

$$\sum_{k=1}^{\infty} \Phi(D_k) < \infty. \tag{28}$$

Now, put

$$\zeta_i = \alpha_k^{1/2}(\eta_{k,i} - y_{k,i}), \qquad i = 1, 2, \ldots, \tag{29}$$

where k is uniquely determined by i via $\pi(i) \in I_k$. From (26) and (4) we obtain

$$\sum_{i=1}^{\infty} \|\zeta_i\|^{1/2} < \infty. \tag{30}$$

From (28) and (30), by Proposition 4.2, for every $\varepsilon > 0$, there exists $p \in \operatorname{Proj} M$ with $\Phi(p^{\perp}) < \varepsilon$ such that

$$\Big\| p\Big(\sum_{k=1}^{n} D_k\Big)p\Big\|_{\infty} \le C, \qquad n = 1, 2, \ldots,$$

and

$$\sum_{k=1}^{\infty} \|\zeta_k\|_p < \infty. \tag{31}$$

Now, let us define two sequences of indices. Namely, for $n = 1, 2, \ldots$, denote by $k(n)$ the smallest k such that

$$\{\pi(1), \ldots, \pi(n)\} \subset I_1 \cup \ldots \cup I_k,$$

whereas $j(n)$ is the greatest j satisfying

$$I_1 \cup \ldots \cup I_j \subset \{\pi(1), \ldots, \pi(n)\}.$$

Obviously, both $(k(n))$ and $(j(n))$ are nondecreasing and tend to infinity as $n \to \infty$.

Then, for $m < n$, by (25), (27), (29), we have

$$\begin{aligned}\sigma_n - \sigma_m = \xi_{\pi(m+1)} + \ldots + \xi_{\pi(n)} &= \sum_{k=j(m)+1}^{k(n)} \sum_{i=m+1}^{n} \alpha_k^{1/2} \eta_{k,i} \\ &= \sum_{k=j(m)+1}^{k(n)} \alpha_k^{1/2}(s_{k,n} - s_{k,m}) + \sum_{i=m+1}^{n} \zeta_i.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\sigma_n - \sigma_m\|_p \le \Big\| \sum_{k=j(m)+1}^{k(n)} \alpha_k^{1/2} s_{k,n} p\Big\|_{\infty} + \Big\| \sum_{k=j(m)+1}^{k(n)} \alpha_k^{1/2} s_{k,m} p\Big\|_{\infty} \\ + \sum_{i=m+1}^{n} \|\zeta_i\|_p.\end{aligned} \tag{32}$$

By (27) and Lemma 5.1, we have

$$(33)\qquad \Big\|\sum_{k=j(m)+1}^{k(n)} \alpha_k^{1/2} s_{k,n} p\Big\|_\infty \le \Big\|p\Big(\sum_{k=j(m)+1}^{k(n)} D_k\Big)p\Big\|_\infty^{1/2}\Big(\sum_{k=j(m)+1}^{k(n)} \alpha_k\Big)^{1/2} \le C^{1/2}\Big(\sum_{k=j(m)+1}^{k(n)} \alpha_k\Big)^{1/2}.$$

Analogously,

$$(34)\qquad \Big\|\sum_{k=j(m)+1}^{k(n)} \alpha_k^{1/2} s_{k,m}\Big\|_\infty \le C^{1/2}\Big(\sum_{k=j(m)+1}^{k(n)} \alpha_k\Big)^{1/2}.$$

By (32), (33), (34), (1) and (31), the proof is complete. ■

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