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THE RIEMANN THEOREM AND DIVERGENT PERMUTATIONS

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In this paper the fundamental algebraic propeties of convergent and divergent permutations of $\mathbb{N}$ are presented. A permutation p of $\mathbb{N}$ is said to be divergent if at least one conditionally convergent series $\sum a_n$ of real terms is rearranged by p to a divergent series $\sum a_{p(n)}$. All other permutations of $\mathbb{N}$ are called convergent. Some generalizations of the Riemann theorem about the set of limit points of the partial sums of rearrangements of a given conditionally convergent series are also studied.

1. Basic notation and terminology. The sets of reals, positive integers, even and odd positive integers will be denoted by $\mathbb{R}, \mathbb{N}, 2\mathbb{N}$ and $2\mathbb{N}-1$, respectively.

A finite nonempty subset I of $\mathbb{N}$ is said to be an *interval*, or equivalently, an *interval of* $\mathbb{N}$ if I is the range of some increasing sequence of consecutive elements of $\mathbb{N}$.

We will use the symbols $(a,b), [a,b]$ and $[a,b)$ with $a,b \in \mathbb{N}$ to denote intervals of $\mathbb{R}$ as well as the respective intervals of $\mathbb{N}$, depending on the context. Similarly, the symbol $\sum a_n$ will denote either an infinite series of the form $\{\sum_{n=1}^{m} a_n : m \in \mathbb{N}\}$ or its sum whenever this series is convergent. The terms of all series discussed in the paper are reals. We will denote by $\{a_n\}$, also according to the context, either an infinite sequence with domain $\mathbb{N}$ or the range of this sequence.

For abbreviation, we write $A < B$ for two nonempty subsets A and B of $\mathbb{N}$ when $a < b$ for any $a \in A$ and $b \in B$.

We call a sequence $\{A_n\}$ of nonempty subsets of $\mathbb{N}$ *increasing* if $A_n < A_{n+1}$ for every $n \in \mathbb{N}$.

We say that a nonempty subset A of $\mathbb{N}$ is a union of k *mutually separated intervals* (MSI for abbreviation) if there exist k intervals $I_1, \ldots, I_k \subset \mathbb{N}$ which form a partition of A and $\operatorname{dist}(I_i, I_j) \geq 2$ for any distinct $i, j \leq k$.

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A permutation p of $\mathbb{N}$ is said to be *convergent* if for every conditionally convergent series $\sum a_n$ the p-rearranged series $\sum a_{p(n)}$ is also convergent. A permutation of $\mathbb{N}$ which is not convergent is said to be *divergent*.

A permutation p of $\mathbb{N}$ is called *sum-preserving* if for every conditionally convergent series $\sum a_n$, the convergence of $\sum a_{p(n)}$ implies that $\sum a_n = \sum a_{p(n)}$.

Proofs of all theorems presented in this paper will be published separately ([22]).

1. Algebraic properties of convergent and divergent permutations. The following two equivalent characterizations of convergent and divergent permutations are well known (see [1], [5–10], [13–17], [19–20], [23]).

LEMMA 1.1. *A permutation p of $\mathbb{N}$ is convergent iff there exists a positive integer k such that the set $p(I)$ is a union of at most k* MSI *for every interval $I \subset \mathbb{N}$ (the minimal positive integer k with this property will be denoted by $k(p)$).*

LEMMA 1.2. *A permutation p of $\mathbb{N}$ is divergent iff for every positive integer n there exists an interval $I \subset \mathbb{N}$ such that $p(I)$ is a union of at least n* MSI.

It can be easily deduced from the first characterization that the composition of two convergent permutations, p and q, is also convergent and $k(pq) \le k(p)k(q)$. We note that there exist convergent permutations such that their inverses are divergent. In other words, the set $\mathfrak{C}$ of all convergent permutations is only a semigroup under the composition of permutations.

EXAMPLE 1.1. Let us define two different convergent permutations p and q satisfying the following conditions:

(I) p^{-1} and q^{-1} are divergent;
(II) $p([1,n]) = [1,n]$ for infinitely many positive integers n;
(III) $q(I)$ is a union of at least two MSI for every interval $I \subset \mathbb{N}$ having at least two members.

The permutation p is defined as follows:

$$p(2i + 2^n - 2) = i + 2^n - 2 \quad \text{and} \quad p(2i - 1 + 2^n - 2) = i + (3 \cdot 2^{n-1}) - 2$$

for every $i = 1, \ldots, 2^{n-1}$ and $n \in \mathbb{N}$. Then $p(I)$ is a union of at most three MSI for every interval $I \subset \mathbb{N}$. In other words, p is a convergent permutation. Moreover, we have

$$p([1, 2^{n+1} - 2]) = [1, 2^{n+1} - 2]$$

for every $n \in \mathbb{N}$, which implies (II). Since the set

$$p^{-1}([2^n - 1, 3 \cdot 2^{n-1} - 2])$$

is a union of 2^{n-1} MSI for every $n \in \mathbb{N}$ we see that p^{-1} is a divergent permutation.

Now put $I_n = [2^n - 1, 2^{n+1} - 2]$ for every $n \in \mathbb{N}$ and let us define the permutation q by the following relations:

$$q(2\mathbb{N} - 1) = \bigcup_{n\in\mathbb{N}} I_{2n-1} \quad \text{and} \quad q(2\mathbb{N}) = \bigcup_{n\in\mathbb{N}} I_{2n}, \tag{1}$$

$$q^{-1}(I_n) < q^{-1}(I_{n+2}), \qquad n \in \mathbb{N}, \tag{2}$$

and

(3) the restriction of q^{-1} to I_n is decreasing for every $n \in \mathbb{N}$.

Then it can be deduced that $q(I)$ is a union of at most five MSI for every interval $I \subset \mathbb{N}$ and that the condition (III) is satisfied. Moreover, it follows that $q^{-1}(I_n)$ is a union of 2^n MSI for every $n \in \mathbb{N}$, which means that q^{-1} is a divergent permutation.

There are also permutations p of $\mathbb{N}$ such that both p and p^{-1} are divergent.

Example 1.2. As in the previous example we are going to define two divergent permutations p and q with some extreme combinatorial properties:

(I) p^{-1} and q^{-1} are divergent;

(II) $p([1, n]) = [1, n]$ for infinitely many positive integers n;

(III) $k(q, I) \to \infty$ as $\operatorname{card} I \to \infty$, where $I \subset \mathbb{N}$ is an interval and the positive integer $k(q, I)$ is defined in such a way that $q(I)$ is a union of $k(q, I)$ MSI.

The permutation p is defined by

$$p(2i + 2^m - 2) = i + 2^m - 2, \quad p(2i - 1 + 2^m - 2) = i + 3 \cdot 2^{m-1} - 2,$$
$$p(i + 2^n - 2) = 2i + 2^n - 2 \quad \text{and} \quad p(i + 3 \cdot 2^{n-1} - 2) = 2i - 1 + 2^n - 2$$

for every $m \in 2\mathbb{N} - 1$ and $n \in 2\mathbb{N}$. Then the following three statements can be readily verified:

(1) $p([1, 2^{n+1} - 2]) = [1, 2^{n+1} - 2]$ for every $n \in \mathbb{N}$,

(2) $p([2^m - 1, 3 \cdot 2^{m-1} - 2])$ is a union of 2^{m-1} MSI for every $m \in 2\mathbb{N} - 1$,

(3) $p^{-1}([2^n - 1, 3 \cdot 2^{n-1} - 2])$ is a union of 2^{n-1} MSI for every $n \in 2\mathbb{N}$.

That p and p^{-1} are divergent follows immediately from (2) and (3), respectively. On the other hand, (1) implies (II).

Now let $\{x_n\}$ and $\{y_n\}$ be two increasing sequences of odd and even positive integers, respectively, and let

$$\lim_{n\to\infty} (x_{n+1} - x_n) = \lim_{n\to\infty} (y_{n+1} - y_n) = \infty.$$

The permutation q is defined to be the increasing mapping of the sets $2\mathbb{N}-1, \{y_n\}$ and $2\mathbb{N}\backslash\{y_n\}$ onto $\{x_n\},\ 2\mathbb{N}$ and $(2\mathbb{N}-1)\backslash\{x_n\}$, respectively. The proof of (III) is left to the reader.

Both examples presented above suggest the following two different partitions of the family $\mathfrak{P}$ of all permutations of $\mathbb{N}$:

$$\mathfrak{P}=\mathfrak{C}\cup\mathfrak{D} \quad \text{and} \quad \mathfrak{P}=\mathfrak{C}\mathfrak{C}\cup\mathfrak{C}\mathfrak{D}\cup\mathfrak{D}\mathfrak{C}\cup\mathfrak{D}\mathfrak{D},$$

where $\mathfrak{D}\subset\mathfrak{P}$ denotes the family of all divergent permutations while the sets AB with $A,B\in\{\mathfrak{C},\mathfrak{D}\}$ are defined as follows:

$$p\in AB \Leftrightarrow p\in A \text{ and } p^{-1}\in B,$$

for every permutation p of $\mathbb{N}$.

The following interesting relations between the subsets of $\mathfrak{P}$ described above are presented in [22]. Below the symbol $\circ$ stands for the composition of nonempty subsets of $\mathfrak{P}$ defined by

$$S\circ T=\{\sigma\tau : \sigma\in S,\ \tau\in T \text{ and } \sigma\tau(n):=\sigma(\tau(n)) \text{ for every } n\in\mathbb{N}\}.$$

We have

$$\mathfrak{C}\mathfrak{C}\circ\mathfrak{A}=\mathfrak{A}\circ\mathfrak{C}\mathfrak{C}=\mathfrak{A}$$

for every $\mathfrak{A}\in\{\mathfrak{C},\mathfrak{D},\mathfrak{C}\mathfrak{C},\mathfrak{C}\mathfrak{D},\mathfrak{D}\mathfrak{C},\mathfrak{D}\mathfrak{D}\}$, and thus $\mathfrak{C}\mathfrak{C}$ is the unit for the composition $\circ$. Next we have

$$\mathfrak{C}\mathfrak{D}\circ\mathfrak{C}\mathfrak{D}=\mathfrak{C}\mathfrak{D} \quad \text{and} \quad \mathfrak{D}\mathfrak{C}\circ\mathfrak{D}\mathfrak{C}=\mathfrak{D}\mathfrak{C},$$

in other words, both families $\mathfrak{C}\mathfrak{D}$ and $\mathfrak{D}\mathfrak{C}$ are closed under $\circ$. On the contrary, the family $\mathfrak{D}\mathfrak{D}$ is very large with respect to the composition $\circ$ since

$$\mathfrak{D}\mathfrak{D}\circ\mathfrak{D}\mathfrak{D}=\mathfrak{P}.$$

Moreover, we have

$$\mathfrak{D}\mathfrak{C}\circ\mathfrak{D}\mathfrak{D}=\mathfrak{D}\mathfrak{D}\circ\mathfrak{D}\mathfrak{C}=\mathfrak{D} \quad \text{and} \quad \mathfrak{C}\mathfrak{D}\circ\mathfrak{D}\mathfrak{D}=\mathfrak{D}\mathfrak{D}\circ\mathfrak{C}\mathfrak{D}=\mathfrak{C}\mathfrak{D}\cup\mathfrak{D}\mathfrak{D},$$

$$\bigcup_{n\in\mathbb{N}}\mathfrak{C}_n=\bigcup_{n\in\mathbb{N}}\mathfrak{D}_n=\mathfrak{G},$$

$$\mathfrak{C}_n\subseteq\mathfrak{C}_{n+1},\quad \mathfrak{D}_n\subseteq\mathfrak{D}_{n+1} \quad \text{and} \quad \mathfrak{C}_n\cup\mathfrak{D}_n\subseteq\mathfrak{C}_{n+1}\cap\mathfrak{D}_{n+1}$$

for every $n\in\mathbb{N}$, where $\mathfrak{G}$ denotes the group generated by $\mathfrak{C}$,

$$\mathfrak{C}_1:=\mathfrak{C}\mathfrak{D},\ \mathfrak{D}_1:=\mathfrak{D}\mathfrak{C},\ \mathfrak{C}_{n+1}:=\mathfrak{C}\mathfrak{D}\circ\mathfrak{D}_n \quad \text{and} \quad \mathfrak{D}_{n+1}:=\mathfrak{D}\mathfrak{C}\circ\mathfrak{C}_n,\ n\in\mathbb{N}.$$

It is interesting that both families $\mathfrak{C}_2$ and $\mathfrak{D}_2$ are large, as can be seen from the following relations:

$$\mathfrak{P}\backslash\mathfrak{D}\mathfrak{D}\subseteq\mathfrak{C}_2\cap\mathfrak{D}_2 \quad \text{and} \quad \mathfrak{D}\mathfrak{D}\cap\mathfrak{C}_2\cap\mathfrak{D}_2\neq\emptyset.$$

At the same time we have

$$\mathfrak{C}_2\cup\mathfrak{D}_2\neq\mathfrak{P} \quad \text{and} \quad \mathfrak{C}_2\backslash\mathfrak{D}_2\neq\emptyset.$$

We note that $\mathfrak{G}$ is a proper subset of $\mathfrak{P}$, as shown in [13].

2. A strengthening of the Riemann theorem. The following beautiful theorem was proved by Riemann.

If $\sum a_n$ is a conditionally convergent series then for every closed interval I of $\mathbb{R}^\star$ (:= 2-point compactification of $\mathbb{R}$) there exists a permutation p of $\mathbb{N}$ such that $\sigma a_{p(n)} = I$.

Here and subsequently, σb_n stands for the set of limit points of the partial sums of the series $\sum b_n$. Notice that the permutation p in the Riemann theorem is divergent whenever $I \neq \{\sum a_n\}$.

In this section it will be shown that a permutation p satisfying the assertion of the Riemann theorem can always be found in the family $\mathfrak{DD}$, but not always in $\mathfrak{D}(k) := \{p \in \mathfrak{D}$: there exists an increasing sequence $\{r_n\}$ of positive integers such that $p^{-1}([1, r_n])$ is a union of at most k MSI for every $n \in \mathbb{N}\}$, for any $k \in \mathbb{N}$.

We note that

$$\mathfrak{D}(k) \cap \mathfrak{DD} \neq \emptyset, \quad \mathfrak{D}(k) \cap \mathfrak{DC} \neq \emptyset,$$
$$\mathfrak{D}(k) \cap \mathfrak{DC} \subset \mathfrak{D}(k+1) \cap \mathfrak{DC}$$

for every $k \in \mathbb{N}$, and

$$\bigcup_{k\in\mathbb{N}} \mathfrak{D}(k) \cap \mathfrak{DC} = \mathfrak{DC}.$$

Our considerations start with the theorem describing, for each conditionally convergent series $\sum a_n$, two classes $\mathfrak{J}_3$ and $\mathfrak{J}_5$ of closed intervals of $\mathbb{R}^\star$ such that for every $I \in \mathfrak{J}_3$ and $J \in \mathfrak{J}_5$ there exist permutations $p \in \mathfrak{D}(3) \cap \mathfrak{DC}$ and $q \in \mathfrak{D}(5) \cap \mathfrak{DC}$ such that $\sigma a_{p(n)} = I$ and $\sigma a_{q(n)} = J$.

THEOREM 2.1. *Let $\sum a_n$ be a conditionally convergent series and let I be a closed interval of $\mathbb{R}^\star$. Then there exists a permutation $p \in \mathfrak{DD}$ such that $\sigma a_{p(n)} = I$. Additionally, if $\sum a_n \in I$ or I has the form $[\alpha, +\infty]$ or $[-\infty, \beta]$ with $\alpha, \beta \in \mathbb{R}^\star$, $\alpha < +\infty$ and $\beta > -\infty$, then there exists a permutation $q \in \mathfrak{DC}$ with $k(q^{-1}) \leq 5$ such that $\sigma a_{q(n)} = I$. In the case when $\sum a_n \in I$ it may be assumed that $k(q^{-1}) \leq 3$.*

The reason that for each $k \in \mathbb{N}$ and for any conditionally convergent series $\sum a_n$ we cannot find a permutation $p \in \mathfrak{D}(k)$ such that $\sigma a_{p(n)} = I$ for some compact intervals $I \subset \mathbb{R}$, is given in the following easy lemma.

LEMMA 2.2. *Let $p \in \mathfrak{D}(k)$ for some $k \in \mathbb{N}$. Then for every conditionally convergent series $\sum a_n$ the following implication holds: if $\sigma a_{p(n)} = [\alpha, \beta] \subset \mathbb{R}$ then*

$$(\star) \qquad k(\alpha - \beta) + \beta \leq \sum a_n \leq k(\beta - \alpha) + \alpha.$$

COROLLARY 2.2.1. *Each $p \in \bigcup_{k\in\mathbb{N}} \mathfrak{D}(k)$ is a sum-preserving permutation.*

Corollary 2.2.2. *Let $p \in \mathfrak{P}$ and let $\sum a_n$ be a conditionally convergent series. If the p-rearranged series $\sum a_{p(n)}$ is convergent and $\sum a_n \neq \sum a_{p(n)}$ then $p \in \mathfrak{D}\mathfrak{D}$ and $p \notin \bigcup_{k\in\mathbb{N}} \mathfrak{D}(k)$.*

Corollary 2.2.3. *Let $p \in \mathfrak{D}\mathfrak{C}$ and let $\sum a_n$ be a conditionally convergent series. If $\sigma a_{p(n)} = [\alpha, \beta] \subset \mathbb{R}$ then*

$$c(p)(\alpha - \beta) + \beta \leq \sum a_n \leq c(p)(\beta - \alpha) + \alpha,$$

where $c(p) := \min\{n \in \mathbb{N}$: there exists an infinite subset B of $\mathbb{N}$ such that $p^{-1}([1, b])$ is a union of n MSI for every $b \in B\}$. Note that $2c(p) - 1 \leq k(p^{-1})$.

Corollary 2.2.4. *For every conditionally convergent serie $\sum a_n$ and for every $k \in \mathbb{N}$ there exist closed intervals $I \subset \mathbb{R}$ such that $\sigma a_{p(n)} \neq I$ for each $p \in \mathfrak{D}(k)$.*

In the following example we show that both inequalities ($\star$) from the assertion of Lemma 2.2 are sharp. Only the case $k = 2$ will be discussed in the example below.

Example 2.1. Let $s, t \in \mathbb{N}$. Let $\{I_n\}$ and $\{J_n\}$ be increasing sequences of intervals which form two different partitions of $\mathbb{N}$ defined by the relations

$$\operatorname{card} I_n = 2(s+t)t^{n-1}, \qquad \operatorname{card} J_2 = s,$$
$$\operatorname{card} J_{2n+2} = t^{n-1} + st^n \quad \text{and} \quad \operatorname{card} J_{2n-1} = (s+t)t^{n-1}$$

for every $n \in \mathbb{N}$. The permutation p is defined to be the increasing mapping of the sets $\bigcup_{n\in\mathbb{N}} J_{2n-1}$ and $\bigcup_{n\in\mathbb{N}} J_{2n}$ onto $2\mathbb{N}$ and $2\mathbb{N} - 1$, respectively. It is then easily seen that $p \in \mathfrak{D}\mathfrak{C}$ and

$$p^{-1}([1, \max I_n]) = [1, \max J_{2n}] \cup [\min J_{2n+2}, t^n + t^{n-1} - 2 + \min J_{2n+2}] \tag{1}$$

for every n.

Consider the series $\sum a_n$ whose terms are defined as follows:

$$a_j = -(s+t)^{-1}t^{-n+1} \quad \text{and} \quad a_{j+1} = (s+t)^{-1}t^{-n+1}$$

for every index j of the form $j = 2i - 2 + \min I_n$ with $i = 1, \ldots, (s+t)t^{n-1}$ and $n \in \mathbb{N}$. We see that

$$\sum a_n = 0 \quad \text{and} \quad \sigma a_{p(n)} = \left[1 - \frac{s-1}{s+t}, 2 - \frac{s-1}{s+t}\right].$$

Using Lemma 2.2 we get

$$-\frac{s-1}{s+t} \leq \sum a_n \leq 3 - \frac{s-1}{s+t} \tag{2}$$

by (1). Additionally, if we set $b_n = -a_n$ for every $n \in \mathbb{N}$ then

$$\sum b_n = 0, \quad \sigma b_{p(n)} = \left[-2 + \frac{s-1}{s+t}, -1 + \frac{s-1}{s+t}\right]$$

and consequently we get

$$-3 + \frac{s-1}{s+t} \le \sum b_n \le \frac{s-1}{s+t}. \tag{3}$$

Immediately from (2) and (3) it may be concluded, for $k = 2$, that neither of the two k's occurring in the inequality $(\star)$ of Lemma 2.2 can be replaced by a constant of the form $\gamma_k k$ with $\gamma_k \in (0, 1)$.

Remark 2.1. There is an interesting connection between the family of sets $\mathfrak{D}(k)$, $k \in \mathbb{N}$, introduced in this section and the permutations of $\mathbb{N}$ which rearrange some conditionally convergent series into a series diverging to infinity. It was proved in [21] that for a given divergent permutation p there is a conditionally convergent series $\sum a_n$ such that $\sum a_{p(n)} = +\infty$ iff $L_n \to \infty$ as $n \to \infty$. Here L_n denotes the number of mutually separated intervals which form a partition of $p([1, n])$ for every $n \in \mathbb{N}$. It is easy to verify that $L_n \to \infty$ as $n \to \infty$ iff p^{-1} belongs to the complement of $\bigcup_{k\in\mathbb{N}} \mathfrak{D}(k)$.

3. The Riemann theorem with fixed permutations and variable series. The following theorem was shown independently in [7] and [21] (in [7] the authors do not discuss unbounded intervals).

THEOREM 3.1. *For each divergent permutation p, for any closed interval $I \subset \mathbb{R}^\star, I \ne \{-\infty\}, \{+\infty\}$, and for any $\alpha \in I \cap \mathbb{R}$ there exists a conditionally convergent series $\sum a_n$ such that*

$$\sum a_n = \alpha \quad \textit{and} \quad \sigma a_{p(n)} = I.$$

Notice that the condition $\sum a_n \in I$ above is essential due to the fact that $\mathfrak{D}(1) \ne \emptyset$.

To the author's knowledge, the next result seems to be new. It is an important supplement to Theorem 3.1.

THEOREM 3.2. *Let $p \in \mathfrak{D}$. Suppose that there exists a conditionally convergent series $\sum a_n$ such that the p-rearranged series $\sum a_{p(n)}$ is also convergent and that $\sum a_n \ne \sum a_{p(n)}$. Then for every $\alpha \in \mathbb{R}$ and every nonempty closed interval $I \subset \mathbb{R}^\star$ there exists a conditionally convergent series $\sum b_n$ such that*

$$\sum b_n = \alpha \quad \textit{and} \quad \sigma b_{p(n)} = I.$$

COROLLARY 3.2.1. *Each divergent permutation falls into one of the two classes: the sum-preserving permutations and the permutations satisfying* *the conclusion of Theorem* 3.2. *Additionally*, *if* $\mathfrak{S}$ *denotes the family of divergent sum-preserving permutations*, *then* $\mathfrak{S} \circ \mathfrak{S} = \mathfrak{D}(1) \circ \mathfrak{D}(1) = \mathfrak{P}$ (see [19]), *and* $(\mathfrak{D} \backslash \mathfrak{S}) \circ (\mathfrak{D} \backslash \mathfrak{S}) = \mathfrak{P}$.

Remark 3.1. For every positive integer $k \geq 2$ there exists a sum-preserving permutation $p \in \mathfrak{D}(k) \backslash \bigcup_{i=1}^{k-1} \mathfrak{D}(k)$ such that $p^{-1} \notin \bigcup_{i \in \mathbb{N}} \mathfrak{D}(i)$. Furthermore, there exists a sum-preserving permutation p such that neither p nor p^{-1} belongs to $\bigcup_{i \in \mathbb{N}} \mathfrak{D}(i)$ (see [7]).

Our next aim is to extend Theorem 3.1 to the cases where a divergent permutation p is replaced by an arbitrary set of divergent permutations.

THEOREM 3.3. (i) *For any nonempty countable family* $\mathfrak{F}$ *of divergent permutations and for any compact interval* $I = [a, b] \subset \mathbb{R}$ *there exists a conditionally convergent series* $\sum a_n$ *such that*

$$\sum a_n = \frac{1}{2}(a + b) \quad \textit{and} \quad \sigma a_{f(n)} = I$$

for every permutation $f \in \mathfrak{F}$.

(ii) *Let* $p \in \mathfrak{D}$ *and let* $\mathfrak{F}$ *be a nonempty countable subset of* $\mathfrak{D}$. *Then for any two* $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, *there exists a conditionally convergent series* $\sum b_n$ *such that*

$$\sum b_n = \beta, \qquad \sigma b_{p(n)} = [\alpha, +\infty]$$

and

$$[\alpha, 2\beta - \alpha] \subseteq \sigma b_{f(n)} \qquad \textit{for every } f \in \mathfrak{F}.$$

Moreover, *there exists a conditionally convergent series* $\sum c_n$ *such that* $\sum c_n = 0$ *and* $\sigma c_{f(n)} = \mathbb{R}$ *for every* $f \in \mathfrak{F}$ (see [21]).

It is not surprising that none of the following two generalizations of Theorem 3.3(i) is true in full generality:

(i) one where the interval $I \subset \mathbb{R}$ is replaced by a given family $\{I_f : f \in \mathfrak{F}\}$ of finite closed intervals of $\mathbb{R}$;

(ii) one where the condition $\sum a_n = \frac{1}{2}(a + b)$ is replaced by $\sum a_n = \alpha$ with α a given member of I.

On the other hand, it is amazing that neither (i) nor (ii) holds for the family $\{D : D \subset \mathfrak{DC} \text{ and } \operatorname{card} D = 2\}$ because there exist permutations $p, q \in \mathfrak{DC}$ satisfying the following condition:

For every conditionally convergent series $\sum a_n$ there exist $\varepsilon, \delta \in [0, +\infty]$ such that

(iii) $\sigma a_{p(n)} = [-\varepsilon + \sum a_n, \delta + \sum a_n]$ and $\sigma a_{q(n)} = [-\delta + \sum a_n, \varepsilon + \sum a_n]$.

In the sequel, if $\sigma a_{p(n)} = \sigma a_{q(n)}$ for some conditionally convergent series $\sum a_n$ then

$$\sigma a_{p(n)} = \sigma a_{q(n)} = \Big[-\varepsilon + \sum a_n, \varepsilon + \sum a_n\Big]$$

for some $\varepsilon \in [0, +\infty]$.

EXAMPLE 3.1. Let us put $I_n = [2^n - 1, 2^{n+1} - 2]$,

$$U_{k,n} = [2^n - 1, 2^n - 1 + k] \quad \text{and} \quad V_{k,n} = [2^n - 1, 2^{n+1} - 2 - k)$$

for every $k = 0, 1, \ldots, 2^n - 2$ and $n \in \mathbb{N}$. Let p and q be defined by

$$p(2^n - 1 + i) = 2^n - 1 + 2i, \qquad p(3 \cdot 2^{n-1} - 1 + i) = 2^n + 2i,$$
$$q(2^n - 1 + i) = 2^{n+1} - 2 - 2i \quad \text{and} \quad q(3 \cdot 2^{n-1} - 1 + i) = 2^{n+1} - 3 - 2i$$

for every $i = 0, 1, \ldots, 2^{n-1} - 1$ and $n \in \mathbb{N}$. The verification of the following relations is then immediate:

$$\sum_{i \in U_{k,n}} a_{p(i)} = \Big(\sum_{i \in I_n} a_i\Big) - \sum_{i \in V_{k,n}} a_{q(i)} \tag{1}$$

and

$$p(I_n) = q(I_n) = I_n \Rightarrow \sum_{i \in I_n} a_{p(i)} = \sum_{i \in I_n} a_{q(i)} = \sum_{i \in I_n} a_i \tag{2}$$

for every series $\sum a_n$ and for any indices $k = 1, \ldots, 2^n - 1$ and $n \in \mathbb{N}$. Moreover, if a series $\sum a_n$ is convergent then from (1) and (2) we deduce that there exist $\varepsilon, \delta \in [0, +\infty]$ such that the relations (iii) above hold.

The generalizations of the type (i) and (ii) of Theorem 3.1 to some infinite sets of divergent permutations are rather unexpected. For example, there exists a sequence $\{p_k\}$ of divergent permutations satisfying the following two conditions:

(1) for every conditionally convergent series $\sum a_n$ the sequence $\{\sigma a_{p_k(n)} : k \in \mathbb{N}\}$ is nonincreasing, i.e. we have

$$\sum a_n \in \sigma a_{p_{k+1}(n)} \subseteq \sigma a_{p_k(n)}$$

for every $k \in \mathbb{N}$;

(2) for any two nonincreasing sequences $\{\alpha_k\}$ and $\{\beta_k\}$ of members of the interval $(0, +\infty] \subset \mathbb{R}^\star$ there exists a conditionally convergent series $\sum a_n$ which is convergent to 0 and such that

(2a) $\sigma a_{p_k(n)} = [-\alpha_k, \beta_k]$ for every $k \in \mathbb{N}$.

EXAMPLE 3.2. We set

$$p_k(2^n - 2 + i) = 2^n - 2 + 2i \quad \text{and} \quad p_k(2^n - 2 + 2^{n-k-1} + i) = 2^n + 2^{n-k} - 2i - 3$$

for every $i = 0, 1, \ldots, 2^{n-k-1} - 1$, $k = 0, 1, \ldots, n-1$, $n \in \mathbb{N}$. Moreover, we set $p_k(i) = i$ for all other $i \in \mathbb{N}$, i.e. for $i \in \mathbb{N} \setminus \bigcup_{n=k+2}^{\infty} [2^n - 2, 2^n - 2 + 2^{n-k})$.

Let $\{\alpha_k : k \in \mathbb{N}_0\}$ and $\{\beta_k : k \in \mathbb{N}_0\}$ be nonincreasing sequences of elements of $(0, +\infty] \subset \mathbb{R}^\star$. First we choose an auxiliary nondecreasing sequence $\{f_n\}$ of positive integers such that $f_1 = 1, f_n \le n-1$ for every $n \in \mathbb{N}, n \ge 2$, and

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty}(n - f_n) = \infty.$$

When $\alpha_0, \beta_0 \in \mathbb{R}$ we define the terms of the desired series $\sum a_n$ in the following way. For every $n \in 2\mathbb{N}$ we set

$$a_r = \begin{cases} \beta_{n-f_n} 2^{-f_n+2} & \text{for } r = 2^n + 2^{f_n-1} - 2, 2^n + 2^{f_n-1}, \\ & \qquad 2^n + 2^{f_n-1} + 2, \dots, 2^n + 2^{f_n} - 4, \\ (\beta_{n-f_n-1} - \beta_{n-f_n}) 2^{-f_n+1} & \text{for } r = 2^n + 2^{f_n} - 2, \\ & \qquad 2^n + 2^{f_n}, \dots, 2^n + 2^{f_n+1} - 4, \\ \dots\dots\dots & \\ (\beta_1 - \beta_2) 2^{-n+3} & \text{for } r = 5 \cdot 2^{n-2} - 2, 5 \cdot 2^{n-2}, \\ & \qquad \dots, 3 \cdot 2^{n-1} - 6, 3 \cdot 2^{n-1} - 4, \\ (\beta_0 - \beta_1) 2^{-n+2} & \text{for } r = 3 \cdot 2^{n-1} - 2, 3 \cdot 2^{n-1}, \\ & \qquad \dots, 2^{n+1} - 6, 2^{n+1} - 4. \end{cases}$$

For every $n \in 2\mathbb{N} - 1$ we set

$$a_r = \begin{cases} \alpha_{n-f_n} 2^{-f_n+2} & \text{for } r = 2^n + 2^{f_n-1} - 2, 2^n + 2^{f_n-1}, \\ & \qquad 2^n + 2^{f_n-1} + 2, \dots, 2^n + 2^{f_n} - 4, \\ (\alpha_{n-f_n} - \alpha_{n-f_n-1}) 2^{-f_n+1} & \text{for } r = 2^n + 2^{f_n} - 2, \\ & \qquad 2^n + 2^{f_n}, \dots, 2^n + 2^{f_n+1} - 4, \\ \dots\dots\dots & \\ (\alpha_2 - \alpha_1) 2^{-n+3} & \text{for } r = 5 \cdot 2^{n-2} - 2, 5 \cdot 2^{n-2}, \\ & \qquad \dots, 3 \cdot 2^{n-1} - 6, 3 \cdot 2^{n-1} - 4, \\ (\alpha_1 - \alpha_0) 2^{-n+2} & \text{for } r = 3 \cdot 2^{n-1} - 2, 3 \cdot 2^{n-1}, \\ & \qquad \dots, 2^{n+1} - 6, 2^{n+1} - 4. \end{cases}$$

Furthermore, we set $a_r = 0$ for all remaining indices $r \in 2\mathbb{N}$ and $a_r = -a_{r+1}$ for every $r \in 2\mathbb{N} - 1$.

If a finite number of elements of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are equal to $+\infty$ then the definition of terms of the series $\sum a_n$ requires some modifications. For example, if $\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = \beta_2 = +\infty$ and $\alpha_2, \beta_3 \in \mathbb{R}$ then for every even positive integer n we set

$$a_r = \begin{cases} \beta_{n-f_n} 2^{-f_n+2} & \text{for } r = 2^n + 2^{f_n-1} - 2, 2^n + 2^{f_n-1}, \\ & \qquad 2^n + 2^{f_n-1} + 2, \dots, 2^n + 2^{f_n} - 4, \\ (\beta_{n-f_n-1} - \beta_{n-f_n}) 2^{-f_n+1} & \text{for } r = 2^n + 2^{f_n} - 2, \\ & \qquad 2^n + 2^{f_n}, \dots, 2^n + 2^{f_n+1} - 4, \\ \dots\dots\dots & \\ (\beta_3 - \beta_4) 2^{-n+5} & \text{for } r = 17 \cdot 2^{n-4} - 2, 17 \cdot 2^{n-4}, \\ & \qquad \dots, 9 \cdot 2^{n-3} - 6, 9 \cdot 2^{n-3} - 4, \\ n 2^{-n} & \text{for } r = 9 \cdot 2^{n-3} - 2, 9 \cdot 2^{n-3}, \\ & \qquad \dots, 2^{n+1} - 6, 2^{n+1} - 4. \end{cases}$$

Furthermore, for every odd positive integer n we set

$$a_r = \begin{cases} \alpha_{n-f_n} 2^{-f_n+2} & \text{for } r = 2^n + 2^{f_n-1} - 2, 2^n + 2^{f_n-1}, \\ & \qquad 2^n + 2^{f_n-1} + 2, \dots, 2^n + 2^{f_n} - 4, \\ (\alpha_{n-f_n} - \alpha_{n-f_n-1}) 2^{-f_n+1} & \text{for } r = 2^n + 2^{f_n} - 2, \\ & \qquad 2^n + 2^{f_n}, \dots, 2^n + 2^{f_n+1} - 4, \\ \dots\dots\dots\dots\dots & \\ (\alpha_3 - \alpha_2) 2^{-n+4} & \text{for } r = 9 \cdot 2^{n-3} - 2, 9 \cdot 2^{n-3}, \\ & \qquad \dots, 5 \cdot 2^{n-2} - 6, 5 \cdot 2^{n-2} - 4, \\ -n 2^{-n} & \text{for } r = 5 \cdot 2^{n-2} - 2, 5 \cdot 2^{n-2}, \\ & \qquad \dots, 2^{n+1} - 6, 2^{n+1} - 4. \end{cases}$$

All remaining terms are defined as in the previous case. The verification of the properties (1) and (2a) is left to the reader.

To end this section we give an example of divergent permutations p and q and conditionally convergent series $\sum a_n$ and $\sum b_n$ satisfying the following conditions:

$$\sum a_{p(n)} = \sum b_{q(n)} = +\infty$$

and

$$\sum a_n = \sum b_n = \sum a_{q(n)} = \sum b_{p(n)} = 0.$$

This surprising example is a substantial supplement to the generalizations of Theorem 3.3.

EXAMPLE 3.3. Let us fix a partition $\{I_n\}$ of $\mathbb{N}$ such that

$$I_n < I_{n+1} \quad \text{and} \quad \operatorname{card} I_{2n-1} = \operatorname{card} I_{2n} = 2n^2 + 2(n-1)^2$$

for every $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ denote the increasing sequences of all members of $\bigcup_{n\in\mathbb{N}} I_{2n-1}$ and $\bigcup_{n\in\mathbb{N}} I_{2n}$, respectively.

The permutation p is defined to be the increasing mapping of

$$\bigcup_{n\in\mathbb{N}} [\min I_{2n-1}, \min I_{2n-1} + 2n^2) \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} [\min I_{2n-1} + 2n^2, \max I_{2n-1}]$$

onto $\{x_{2n}\}$ and $\{x_{2n-1}\}$, respectively. Moreover, we put $p(y_{2n-1}) = y_{2n}$ and $p(y_{2n}) = y_{2n-1}$ for every $n \in \mathbb{N}$. On the other hand, the permutation q is defined to be the increasing mapping of

$$\bigcup_{n\in\mathbb{N}} [\min I_{2n}, \min I_{2n} + 2n^2) \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} [\min I_{2n} + 2n^2, \max I_{2n}]$$

onto $\{y_{2n}\}$ and $\{y_{2n-1}\}$, respectively. Additionally, we put $q(x_{2n-1}) = x_{2n}$ and $q(x_{2n}) = x_{2n-1}$ for every $n \in \mathbb{N}$. The terms of the series $\sum a_n$ and $\sum b_n$ are defined in the following way:

$$a_k = \begin{cases} n^{-1}, & k \in I_{2n-1} \cap \{x_{2i}\},\ n \in \mathbb{N}, \\ -n^{-1}, & k \in I_{2n-1} \cap \{x_{2i-1}\},\ n \in \mathbb{N}, \\ 0, & k \in \bigcup_{n\in\mathbb{N}} I_{2n}; \end{cases}$$

and

$$b_k = \begin{cases} n^{-1}, & k \in I_{2n} \cap \{y_{2i}\},\ n \in \mathbb{N}, \\ -n^{-1}, & k \in I_{2n} \cap \{y_{2i-1}\},\ n \in \mathbb{N}, \\ 0, & k \in \bigcup_{n\in\mathbb{N}} I_{2n-1}. \end{cases}$$

Then the following relations can be easily verified:

$$\sum_{i=u_{2n+1}}^{j} a_{p(i)} = \begin{cases} 3n + \frac{1}{n+1} - [u_{2n+1} + (n+1)^2 - j - 1]_+ \cdot \frac{1}{n+1} \\ \quad +[j - u_{2n+1} - (n+1)^2 + 1]_+ \cdot \frac{1}{n+2} \\ \qquad\qquad \text{for } u_{2n+1} \le j < u_{2n+1} + 2(n+1)^2, \\ 3n + \frac{1}{n+1} + \frac{1}{n+2} + [u_{2n+2} - n^2 - 1 - j]_+ \cdot \frac{1}{n} \\ \quad -[j + 1 + n^2 - u_{2n+2}]_+ \cdot \frac{1}{n+1} \\ \qquad\qquad \text{for } u_{2n+1} + 2(n+1)^2 \le j < u_{2n+2}, \end{cases}$$

and

$$\sum_{i=u_{2n+2}}^{j} b_{q(i)} = \begin{cases} 3n + \frac{1}{n+1} - [u_{2n+2} + (n+1)^2 - j - 1]_+ \cdot \frac{1}{n+1} \\ \quad +[j - u_{2n+2} - (n+1)^2 + 1]_+ \cdot \frac{1}{n+2} \\ \qquad\qquad \text{for } u_{2n+2} \le j < u_{2n+2} + 2(n+1)^2, \\ 3n + \frac{1}{n+1} + \frac{1}{n+2} + [u_{2n+3} - n^2 - 1 - j]_+ \cdot \frac{1}{n} \\ \quad -[j + 1 + n^2 - u_{2n+3}]_+ \cdot \frac{1}{n+1} \\ \qquad\qquad \text{for } u_{2n+2} + 2(n+1)^2 \le j < u_{2n+3}, \end{cases}$$

for every $n \in \mathbb{N}$, where $u_n := \min I_n$, $n \in \mathbb{N}$ and $[m]_+ := \max\{0, m\}$ for every $m \in \mathbb{R}$.

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