COLLOQUIUM MATHEMATICUM

VOL. LXIX 1995 FASC. 2

THE CLOSURE OF THE INVERTIBLES IN A VON NEUMANN ALGEBRA

BY

LAURA BURLANDO (GENOVA) AND ROBIN HARTE (BELFAST)

In this paper we consider a subset $\widehat{A}$ of a Banach algebra A (containing all elements of A which have a generalized inverse) and characterize membership in the closure of the invertibles for the elements of $\widehat{A}$. Thus our result yields a characterization of the closure of the invertible group for all those Banach algebras A which satisfy $\widehat{A} = A$. In particular, we prove that $\widehat{A} = A$ when A is a von Neumann algebra. We also derive from our characterization new proofs of previously known results, namely Feldman and Kadison's characterization of the closure of the invertibles in a von Neumann algebra and a more recent characterization of the closure of the invertibles in the bounded linear operators on a Hilbert space.

0. Suppose A is a ring, with identity 1 and invertible group A^{-1}: we shall write ([11], Definition 7.3.1; [10])

$$\overline{A} = \{a \in A : a \in aAa\} \tag{0.1}$$

for the *regular* or "relatively Fredholm" elements, those which have *generalized inverses* in A, and

$$A^{-1}A^{\bullet} = A^{\bullet}A^{-1} = \{a \in A : a \in aA^{-1}a\} \tag{0.2}$$

for the *decomposably regular* or "relatively Weyl" elements, with invertible generalized inverses. As our notation anticipates, these are just the products of invertibles and *idempotents*

$$A^{\bullet} = \{a \in A : a^2 = a\}. \tag{0.3}$$

For example, if $A = B(X)$ (i.e., the bounded linear operators on X) for a Hilbert space X then ([14], Theorem 3.8.2)

$$\overline{A} = \{a \in A : a(X) = \operatorname{cl} a(X)\} \tag{0.4}$$

1991 *Mathematics Subject Classification*: 46H05, 47D25.

(where "cl" denotes norm closure), with ([14], Theorem 3.8.6)

$$A^{-1}A^{\bullet} = \{a \in \overline{A} : \mathrm{nul}(a) = \mathrm{nul}(a^*)\}, \tag{0.5}$$

where $\mathrm{nul}(a) = \dim a^{-1}(0)$ is the Hilbert space dimension of the null space of a.

When A is a Hausdorff topological ring, with jointly continuous multiplication and continuous inversion, then the closure of the invertibles is a semigroup

$$\mathrm{cl}(A^{-1})\,\mathrm{cl}(A^{-1}) \subseteq \mathrm{cl}(A^{-1}), \tag{0.6}$$

and for a topological algebra over the rationals there is inclusion

$$A^{-1}A^{\bullet} \subseteq \mathrm{cl}(A^{-1}). \tag{0.7}$$

When in particular A is a real or a complex Banach algebra then ([10]; [11], Theorem 7.3.4)

$$\overline{A} \cap \mathrm{cl}(A^{-1}) = A^{-1}A^{\bullet}. \tag{0.8}$$

In this note we see how (0.8) together with (0.6) and (0.7) gives a new kind of characterization of $\mathrm{cl}(A^{-1})$ when A is the bounded linear operators on a complex Hilbert space, or more generally a von Neumann algebra; as a result we are able to offer possibly a clearer view of the existing characterization of Feldman/Kadison [8] and Olsen [17].

Throughout this paper, by a Banach algebra we shall mean a Banach algebra with identity.

1. THEOREM. *If A is a real or complex Banach algebra, and if $a \in A$ and (p_n) is a sequence of idempotents of A satisfying*

$$\|a - ap_n\| \to 0 \quad \textit{as} \quad n \to \infty \quad \textit{and} \quad \{ap_n : n \in \mathbb{N}\} \subseteq \overline{A} \tag{1.1}$$

then

$$a \in \mathrm{cl}(A^{-1}) \Leftrightarrow \{ap_n : n \in \mathbb{N}\} \subseteq A^{-1}A^{\bullet}. \tag{1.2}$$

Proof. Backward implication in (1.2) is (0.7), together with the idempotent property of the closure operation; forward implication is the semigroup property (0.6) with (0.8). ∎

If A is a Banach algebra, we define

$$\widehat{A} = \{a \in A : \text{ there exists a sequence } (p_n) \text{ of idempotents of } A \text{ satisfying (1.1)}\}. \tag{1.3}$$

Then Theorem 1 provides a characterization of $\widehat{A} \cap \mathrm{cl}(A^{-1})$. For the Banach algebras A satisfying $\widehat{A} = A$, Theorem 1 actually gives a characterization of $\mathrm{cl}(A^{-1})$. We shall show how the equality $\widehat{A} = A$ is satisfied when $A = B(X)$ for a Hilbert space X, or, more generally, when A is a von Neumann algebra. Henceforth, all Hilbert spaces we consider are assumed to be complex.

When $A = B(X)$ for a Hilbert space X then every operator $a \in A$ has sequences (p_n) of projections satisfying (1.1): this uses the *polar decomposition* $a = \mathrm{sgn}(a)|a|$ of $a \in A$ and the *spectral theorem* for the positive operator $|a| \in A$. We recall that the "modulus" of $a \in A$ is given by the square root ([11], Theorem 9.9.5; [14]; [16], Theorem 2.2.1):

$$|a| = (a^*a)^{1/2} \in B(X), \tag{1.4}$$

computed using the continuous function calculus ([16], Theorem 2.1.13) derived from the commutative Gelfand–Naimark theorem, and then the "sign" or "argument" of a is the partial isometry defined ([11], (10.8.1.5); [16], Theorem 2.3.4; [20]) by setting

$$\mathrm{sgn}(y+z) = \lim_n ax_n \quad \text{if } y = \lim_n |a|x_n \in \mathrm{cl}|a|(X) \text{ and } z \in |a|(X)^{\perp}. \tag{1.5}$$

This is well defined and bounded because

$$\||a|x\| = \|ax\| \quad (x \in X). \tag{1.6}$$

The operator $u = \mathrm{sgn}(a)$ now satisfies

$$u = uu^*u; \quad u^{-1}(0) = |a|^{-1}(0) = a^{-1}(0); \quad a = u|a|, \tag{1.7}$$

and there is implication

$$ba = ab \text{ and } ba^* = a^*b \Rightarrow \mathrm{sgn}(a)b = b\,\mathrm{sgn}(a) \quad (\textit{as } b|a| = |a|b). \tag{1.8}$$

The polar decomposition of the adjoint is given ([20], Exercise 7.26) by

$$\mathrm{sgn}(a^*) = \mathrm{sgn}(a)^* \quad \text{and} \quad |a^*| = \mathrm{sgn}(a)|a|\,\mathrm{sgn}(a)^*, \tag{1.9}$$

and finally if f is a polynomial then

$$\mathrm{sgn}(a)f(a^*a) = f(aa^*)\,\mathrm{sgn}(a). \tag{1.10}$$

Since the positive operator $|a| = (a^*a)^{1/2}$ is selfadjoint the *spectral theorem* is valid: the continuous function calculus extends ([16], Theorems 2.5.4 and 2.5.5) to a norm-decreasing *-homomorphism

$$f \to f(|a|) : D \to A \tag{1.11}$$

into A from the C^*-algebra D of bounded Borel measurable complex-valued functions on the spectrum $\sigma(|a|) \subseteq \mathbb{C}$ of $|a| \in A$. When $K \subseteq \mathbb{C}$ is a Borel set we shall write

$$e_a K = \chi_{K\cap\sigma(|a|)}(|a|), \tag{1.12}$$

where χ_S denotes the *characteristic function* of $S \subseteq \mathbb{C}$.

2. Theorem. *If $a \in A = B(X)$ for a Hilbert space X and $0 < \varepsilon \le \|a\|$ then $p_\varepsilon = e_a[\varepsilon, \|a\|]$ satisfies*

$$p_\varepsilon^2 = p_\varepsilon, \quad \|a - ap_\varepsilon\| \le \varepsilon \quad \textit{and} \quad ap_\varepsilon \in \overleftarrow{A}. \tag{2.1}$$

Proof. The first part follows from the idempotent property of a characteristic function $\chi_K = \chi_K^2 \in D$ and the second from (1.6) and the inequality $|(1-\chi_K)z| < \varepsilon$ on $\sigma(|a|)$, which holds for $K = [\varepsilon, \|a\|] \cap \sigma(|a|)$. For the last part observe ([20], pp. 196–197) that

$$\|ax\| \geq \varepsilon \|x\| \quad \text{if } x = p_\varepsilon x, \tag{2.2}$$

so that $ap_\varepsilon(X)$ is closed, and appeal to (0.4): explicitly $ap_\varepsilon = ap_\varepsilon bap_\varepsilon$ with $b \in A$ defined by setting

$$b(ap_\varepsilon x + z) = p_\varepsilon x \quad \text{if } x \in X \text{ and } z \in ap_\varepsilon(X)^\perp. \ \blacksquare \tag{2.3}$$

Theorem 2 tells us that $\widehat{A} = A$ when $A = B(X)$ for a Hilbert space X, and then Theorem 1 tells us that the closure of the invertibles in A is characterised by (1.2). We can further recognise the right hand side of (1.2) as the condition written down by Bouldin [2], in terms of the "essential nullity" and "essential defect":

$$\operatorname{ess\,nul}(a) = \inf_{0<\varepsilon\leq\|a\|} \operatorname{nul} e_a[\varepsilon, \|a\|] \tag{2.4}$$

and

$$\operatorname{ess\,def}(a) = \operatorname{ess\,nul}(a^*). \tag{2.5}$$

The first of these coincides ((2.1), (2.2) and [7], Lemma 1.2) with the "approximate nullity" of Edgar, Ernest and Lee ([7], Definition 1.3), which is a refinement of the concept of Kato ([15], IV, (5.9)). By the well ordering property of the cardinal numbers it follows that there is $\varepsilon_a > 0$ for which

$$0 < \varepsilon \leq \varepsilon_a \Rightarrow \operatorname{ess\,nul}(a) = \operatorname{nul} e_a[\varepsilon, \|a\|]. \tag{2.6}$$

We are ready to recover, in Theorem 3 below, the characterization of the closure of $B^{-1}(X)$ as obtained by Bouldin ([2], Theorem 3), and independently by Burlando ([4], Theorem 1.10). Our proof will be shorter, as Theorem 1 enables us to deal with regular elements and then appeal to (0.8) and (0.5).

3. Theorem. *If $a \in A = B(X)$ for a Hilbert space X then*

$$a \in \operatorname{cl}(A^{-1}) \Leftrightarrow \operatorname{ess\,nul}(a) = \operatorname{ess\,def}(a). \tag{3.1}$$

Proof. By Theorem 1 and (2.5) it is sufficient to show, with $p_\varepsilon = e_a[\varepsilon, \|a\|]$ and $q_\varepsilon = e_{a^*}[\varepsilon, \|a\|]$,

$$ap_\varepsilon \in A^{-1}A^\bullet \Leftrightarrow \operatorname{nul} p_\varepsilon = \operatorname{nul} q_\varepsilon. \tag{3.2}$$

To establish (3.2) note that by (2.2),

$$(ap_\varepsilon)^{-1}(0) = p_\varepsilon^{-1}(0) \quad \text{and} \quad (a^*q_\varepsilon)^{-1}(0) = q_\varepsilon^{-1}(0), \tag{3.3}$$

while, using (1.10),

$$a^* q_\varepsilon = \operatorname{sgn}(a^*)|a^*|q_\varepsilon = \operatorname{sgn}(a^*)q_\varepsilon|a^*| = p_\varepsilon \operatorname{sgn}(a)^*|a^*| = p_\varepsilon a^* = (ap_\varepsilon)^*$$

and hence

$$(ap_\varepsilon)^{*-1}(0) = q_\varepsilon^{-1}(0). \tag{3.4}$$

Now (3.3) and (3.4), together with (0.5), give (3.2). ■

When the space X is separable then (3.1) can be rewritten in an entertaining fashion (which could also be derived from [1], Theorem 3):

4. THEOREM. *If $A = B(X)$ for a separable Hilbert space X then*

$$\operatorname{cl}(A^{-1}) = \overline{A} \cup (A \setminus \overline{A}), \tag{4.1}$$

where

$$\overline{A} = \{a \in A : \operatorname{nul}(a) = \operatorname{def}(a)\ (= \operatorname{nul}(a^*))\}. \tag{4.2}$$

Proof. With no separability assumption, inclusion one way is clear from (0.8) together with (0.5):

$$\operatorname{cl}(A^{-1}) \subseteq (\overline{A} \cap \operatorname{cl}(A^{-1})) \cup (A \setminus \overline{A}) = A^{-1}A^\bullet \cup (A \setminus \overline{A}) \subseteq \overline{A} \cup (A \setminus \overline{A}). \tag{4.3}$$

Conversely, suppose the Hilbert space X to be separable. Then ([15], Theorem IV.5.10)

$$a \in \overline{A} \Rightarrow \operatorname{ess\,nul}(a) = \operatorname{nul}(a) \text{ and } \operatorname{ess\,def}(a) = \operatorname{def}(a) \tag{4.4}$$

and

$$a \notin \overline{A} \Rightarrow \operatorname{ess\,nul}(a) = \infty \text{ and } \operatorname{ess\,def}(a) = \infty, \tag{4.5}$$

where we write "∞" for the first infinite cardinal (alternatively, for (4.4) and (4.5), there is a direct argument using the polar decomposition). Hence

$$a \in \overline{A} \cup (A \setminus \overline{A}) \Rightarrow \operatorname{ess\,nul}(a) = \operatorname{ess\,def}(a) \Rightarrow a \in \operatorname{cl}(A^{-1}). \tag{4.6}$$ ■

Theorem 3 extends to von Neumann algebras: when X is a (possibly non-separable) Hilbert space, we shall say that a C^*-subalgebra A of $B(X)$ is a *von Neumann algebra on* X if it coincides with its *double commutant* $\operatorname{comm}^2(A)$ on X. Notice that this forces A to contain the identity operator on X. Necessary and sufficient for an abstract C^*-algebra B to be *-isomorphic to a von Neumann algebra ([19], Theorem 1.16.7; [16], Remark 4.1.2) is that B is a dual Banach space. If $a \in B(X)$ then by (1.8),

$$\{|a|, \operatorname{sgn}(a)\} \subseteq \operatorname{comm}^2(a, a^*), \tag{4.7}$$

so that the polar decomposition can be performed within a von Neumann algebra (namely, when $a \in A$ for a von Neumann algebra A on X, then $|a|$ and $\operatorname{sgn}(a)$ also belong to A, as $\operatorname{comm}^2(a, a^*) \subseteq \operatorname{comm}^2(A) = A$). If $K \subseteq \mathbb{C}$ is a Borel set then ([16], Theorem 2.5.5)

$$e_a K \in \operatorname{comm}^2(|a|) \subseteq \operatorname{comm}^2(a, a^*). \tag{4.8}$$

When A is a von Neumann algebra on X it follows that, in particular, the spectral projections $e_a[\varepsilon, \|a\|]$ and $e_{a^*}[\varepsilon, \|a\|]$ belong to A for any $\varepsilon > 0$ and for any $a \in A$.

To express conditions like (3.2) in the von Neumann algebra we need "equivalence" of projections: if $p^* = p = p^2$ and $q^* = q = q^2$ in a C^*-algebra A we shall write

$$p \sim q \Leftrightarrow \exists w \in A,\ w^*w = p \text{ and } ww^* = q. \tag{4.9}$$

When $A = B(X)$, this reduces to the condition that their ranges have equal Hilbert space dimension:

$$\dim p(X) = \dim q(X). \tag{4.10}$$

Notice that condition (4.10) is necessary for equivalence of p and q also in the case of a generic C^*-subalgebra of $B(X)$; when $A = B(X)$ then (4.10) implies $p \sim q$: if $w_0 : p(X) \to q(X)$ is an (isometric) isomorphism put $wx = w_0px$ for each $x \in X$.

If $a \in A$ for a von Neumann algebra A on X then by (4.8),

$$\{R(a), N^\perp(a)\} = \{e_{a^*}(0, \|a\|], e_a(0, \|a\|]\} \subseteq \operatorname{comm}^2(a, a^*), \tag{4.11}$$

where $R(a)$ is the orthogonal projection on the closure of the range of a and $N^\perp(a)$ the orthogonal projection with the same null space as a; therefore $R(a), N^\perp(a) \in A$. In addition, since

$$R(a) = \operatorname{sgn}(a)\operatorname{sgn}(a)^* \quad \text{and} \quad N^\perp(a) = \operatorname{sgn}(a)^*\operatorname{sgn}(a), \tag{4.12}$$

it follows that

$$R(a) \sim N^\perp(a). \tag{4.13}$$

Necessary and sufficient for $a \in A$ to be decomposably regular is that the complementary projections $R^\perp(a)$ and $N(a)$ (where $R^\perp(a) = 1 - R(a)$ and $N(a) = 1 - N^\perp(a)$) are equivalent:

5. THEOREM. *If $A \subseteq B = B(X)$ is a von Neumann algebra on a Hilbert space X then*

$$\overline{A} = A \cap \overline{B} = \{a \in A : a(X) = \operatorname{cl} a(X)\}, \tag{5.1}$$

and

$$A^{-1}A^\bullet = \{a \in \overline{A} : N(a) \sim R^\perp(a)\} \subseteq A \cap B^{-1}B^\bullet. \tag{5.2}$$

Proof. It is clear that

$$\overline{A} \subseteq A \cap \overline{B} = \{a \in A : a(X) = \operatorname{cl} a(X)\}.$$

If $a \in A$ has closed range then so does $|a|$, and hence $|a| + N(a)$ is invertible in B; the inverse is given by

$$(|a| + N(a))^{-1} = gN^{\perp}(a) + N(a)$$
$$\text{(with } g \in B \text{ satisfying } g|a| = N^{\perp}(a) = |a|g).$$

But now $b = (|a| + N(a))^{-1}\,\mathrm{sgn}(a)^*$ satisfies $a = aba$, and thus is a generalized inverse for a in B. Since $A^{-1} = A \cap B^{-1}$ ([11], Theorem 9.9.7 and (9.9.4.4); [16], Theorem 2.1.11; [14], Theorem 3.1) and $|a| + N(a) \in A$, it follows that $b \in A$, which proves (5.1). For (5.2) suppose that $a = cp$ with $c \in A^{-1}$ and $p \in A^{\bullet}$; we claim that

$$R^{\perp}(a) = R(d) \quad \text{and} \quad N(a) = N^{\perp}(d) \quad \text{with } d = R^{\perp}(a)cN(a). \tag{5.3}$$

For example, the null space of d is the same as that of $1 - N^{\perp}(a)$, which is the same as that of $c(1 - N^{\perp}(a))$, because ([11], Theorem 10.9.1) the intersection of the range of $a = cp$, which is the null space of $R^{\perp}(a)$, and the range of $c(1-p)$, reduces to $\{0\}$. By (4.13) and (5.3) the first part of (5.2) is contained in the second; conversely, if $a \in A$ has closed range and there is $w \in A$ for which $N(a) = w^*w$ and $R^{\perp}(a) = ww^*$ then $w = ww^*w$ and

$$v = w + \mathrm{sgn}(a) \Rightarrow v^*v = 1 = vv^* \quad \text{and} \quad \mathrm{sgn}(a) = vN^{\perp}(a). \tag{5.4}$$

Again $|a| + 1 - N^{\perp}(a)$ is invertible in A, as is v; thus

$$a = v|a| = v(|a| + N(a))N^{\perp}(a) \in A^{-1}A^{\bullet}, \tag{5.5}$$

giving the equality part of (5.2). The inclusion at the end is clear. ∎

Actually, (5.1) holds in the more general case of a C^*-subalgebra A of $B(X)$: as remarked in [13], this can be deduced from the representations formulae for the Moore–Penrose inverse in $B(X)$ provided by several authors (see [9]). In the von Neumann algebra case, (5.1) is proved also in [12] (Theorems 5 and 6), by means of a different technique.

(5.1), together with (4.8) and Theorem 2, tells us that, when A is a von Neumann algebra on a Hilbert space X, then equality $\widehat{A} = A$ holds, and consequently $\mathrm{cl}(A^{-1})$ is characterized by Theorem 1.

The following extended version of Theorem 3 is due to Feldman and Kadison ([8], Theorem 1) and has also been proved by Olsen ([17], Theorem 2.2; [18], page 357); we give here a new proof of this result, by deriving it from the general characterization we have given in Theorem 1.

6. THEOREM. *If A is a von Neumann algebra on a Hilbert space X then for any $\varepsilon_0 > 0$,*

$$\mathrm{cl}(A^{-1}) = \{a \in A : 1 - e_a[\varepsilon, \|a\|] \sim 1 - e_{a^*}[\varepsilon, \|a\|] \text{ if } 0 < \varepsilon \le \varepsilon_0\}. \tag{6.1}$$

Proof. If we set $p_\varepsilon = e_a[\varepsilon, \|a\|]$ and $q_\varepsilon = e_{a^*}[\varepsilon, \|a\|]$, by (2.1) and (5.1) the ap_ε are in $\widehat{A}$. Repeating the argument for Theorem 3, it is sufficient to show that

$$ap_\varepsilon \in A^{-1}A^{\bullet} \Leftrightarrow 1 - p_\varepsilon \sim 1 - q_\varepsilon, \tag{6.2}$$

which follows from (5.2) and a closer look at (3.3) and (3.4):

$$N^{\perp}(ap_{\varepsilon}) = p_{\varepsilon} \quad \text{and} \quad R(ap_{\varepsilon}) = q_{\varepsilon}. \ \blacksquare \tag{6.3}$$

Olsen ([17], Theorem 2.2) goes further, finding a formula for the distance of an arbitrary von Neumann algebra element from the invertible group. For operators Burlando ([5] Theorem 2.9) also does, in a form not involving the spectral resolution of the modulus. Another formula for the distance to the invertible group in $B(X)$ can be found in [3], Theorem 7 (see [6] for some comments about the proof).

We are unable to deduce (6.2) from (3.2), as inclusion at the end of (5.2) cannot be replaced by equality: the element w of $B(X)$ which ensures equivalence in $B(X)$ of two projections $p, q \in A$ may not be in A. For example let $D = B(\ell_2)$ and look at

$$a = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \in A = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \subseteq \begin{pmatrix} D & D \\ D & D \end{pmatrix} = B = B(X), \tag{6.4}$$

where $X = \ell_2 \times \ell_2$ and $u : (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots)$ is the unilateral shift on ℓ_2. The projections

$$p = N(a) \quad \text{and} \quad q = R^{\perp}(a) \tag{6.5}$$

satisfy (4.10) but not (4.9) (in A; of course p and q satisfy (4.9) in B).

Finally, we remark that, generally speaking, the equality $\widehat{A} = A$ is not satisfied by C^*-algebras. For example, let A be the C^*-algebra of all complex-valued continuous functions on $[0, 1]$. Then $A^{\bullet} = \{0, 1\}$, so that $\widehat{A} = \overline{A} = \{0\} \cup A^{-1}$, which is strictly contained in A.

REFERENCES

[1] R. Bouldin, *The essential minimum modulus*, Indiana Univ. Math. J. 30 (1981), 513–517.

[2] —, *Closure of the invertible operators in Hilbert space*, Proc. Amer. Math. Soc. 108 (1990), 721–726.

[3] —, *Approximating Fredholm operators on a nonseparable Hilbert space*, Glasgow Math. J. 35 (1993), 167–178.

[4] L. Burlando, *On continuity of the spectral radius function in Banach algebras*, Ann. Mat. Pura Appl. (4) 156 (1990), 357–380.

[5] —, *Distance formulas on operators whose kernel has fixed Hilbert dimension*, Rend. Mat. (7) 10 (1990), 209–238.

[6] —, *Approximation by semi-Fredholm and semi-α-Fredholm operators in Hilbert spaces of arbitrary dimension*, to appear.

[7] G. Edgar, J. Ernest and S. G. Lee, *Weighing operator spectra*, Indiana Univ. Math. J. 21 (1971), 61–80.

[8] J. Feldman and R. V. Kadison, *The closure of the regular operators in a ring of operators*, Proc. Amer. Math. Soc. 5 (1954), 909–916.

[9] C. W. Groetsch, *Representations of the generalized inverse*, J. Math. Anal. Appl. 49 (1975), 154–157.

[10] R. E. Harte, *Regular boundary elements*, Proc. Amer. Math. Soc. 99 (1987), 328–330.

[11] —, *Invertibility and Singularity for Bounded Linear Operators*, Dekker, New York, 1988.

[12] R. E. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math. 103 (1992), 71–77.

[13] —, —, *Generalized inverses in C^*-algebras II*, ibid. 106 (1993), 129–138.

[14] R. E. Harte and M. Ó Searcóid, *Positive elements and the B^* condition*, Math. Z. 193 (1986), 1–9.

[15] T. Kato, *Perturbation Theory for Linear Operators*, Springer, New York, 1966.

[16] G. J. Murphy, *C^*-algebras and Operator Theory*, Academic Press, 1990.

[17] C. Olsen, *Unitary approximation*, J. Funct. Anal. 85 (1989), 392–419.

[18] G. K. Pedersen, *Unitary extension and polar decomposition in a C^*-algebra*, J. Operator Theory 17 (1987), 357–364.

[19] S. Sakai, *C^*-algebras and W^*-algebras*, Springer, New York, 1971.

[20] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer, New York, 1980.

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI GENOVA
VIA DODECANESO 35
16146 GENOVA, ITALY
E-mail: BURLANDO@DIMA.UNIGE.IT

QUEEN'S UNIVERSITY
BELFAST BT7 1NN
NORTHERN IRELAND, UK
E-mail: RHARTE@MATHS.TCD.IE

Reçu par la Rédaction le 4.7.1994