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COMMUTATIVITY THEOREMS FOR NORMED *-ALGEBRAS

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1. Introduction. In ring theory much attention has been devoted to showing that certain rings must be commutative as a consequence of conditions which are seemingly too weak to imply commutativity. This work was initiated largely by Jacobson, Kaplansky and especially Herstein (see [1, Chapt. 3]) and has continued up to the present time.

In [7], [8] and [9] we pursued the same aim for Banach algebras. Here an important tool was the Baire category theorem. In this note we consider these questions for normed *-algebras which are not necessarily complete so that the Baire theorem is not available. Suppose that A is a normed *-algebra with identity. It is shown that A is commutative if for each $x \in A$ there is a positive integer $n(x)$ so that $x^{n(x)}$ is a normal element of A. An example shows this result can fail if A does not have an identity. Among our results we show that if A is a semi-prime algebra and there is a fixed positive integer n where x^n is normal modulo the center of A for each $x \in A$ then A is commutative.

Unrelated commutativity theorems for rings with involution are discussed in [4, Chapt. 3].

2. Notation. Throughout, let A be a normed *-algebra over the complex field with involution $x \to x^*$, with center Z and where H is its set of self-adjoint elements. Basic definitions on *-algebras are given in [6, Chapt. 4]. Throughout, E denotes a closed linear subspace of A. For us $E = (0)$ and $E = Z$ are the subspaces of most concern. We say that $x \in A$ is *normal modulo* E if $xx^* - x^*x \in E$.

We shall use several times the following readily established fact. Let $p(t) = \sum_{r=0}^{n} b_r t^r$ be a polynomial in the real variable t with coefficients in A. If $p(t) \in E$ for all t in an infinite subset of the reals then every $b_r \in E$.

We set $[x, y] = xy - yx$ and $x \cdot y = xy + yx$. We say that A is a *semi-prime algebra* if it has no non-zero nilpotent ideals.

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3. On commutativity

THEOREM 3.1. *Suppose that A has an identity e and that, for each $x \in A$, there is a positive integer $n(x)$ such that $x^{n(x)}$ is normal modulo E. Then $[x, y] \in E$ for all $x, y \in A$.*

Proof. It is known [7, Th. 3.4] that if A is complete it is sufficient to have $x^{n(x)}$ normal modulo E on a non-void open subset of A.

Let $x = h + ik$ where h and k lie in H. For each real t there is a positive integer $n(t)$ so that

$$[(e + tx)^{n(t)},\ (e + tx^*)^{n(t)}] \in E.$$

For each positive integer m let S_m be the set of real t where $n(t) = m$. At least one S_m must be infinite. Say S_r is infinite. Then

$$[(e + tx)^r, (e + tx^*)^r] \in E$$

for infinitely many values of t and hence for all real t. The coefficient of t^2 in this polynomial lies in E so that $xx^* - x^*x = 2i[k, h] \in E$. As $[h, k] \in E$ for every $h, k \in H$ we also have $[x, y] \in E$ for all $x, y \in A$.

COROLLARY 3.1. *If, in Theorem 3.1, $E = (0)$ then A is commutative. If, in Theorem 3.1, $E = Z$ and A is a semi-prime algebra then A is commutative.*

Proof. Suppose that $E = Z$ and A is semi-prime. Then by Theorem 3.1 we see that for all $x, y \in A$ we have $[x, [x, y]] = 0$. It follows from a sublemma of Herstein [2, p. 5] that each $x \in A$ lies in Z.

Theorem 3.1 and Corollary 3.1 can fail if A does not have an identity. For we can have a normed *-algebra which is not commutative but where $x^3 \equiv 0$ as in the following example.

Let A be the three-dimensional complex algebra with basis $\{a, b, c\}$ and multiplication given by

$$(\lambda_1 a + \mu_1 b + \nu_1 c)(\lambda_2 a + \mu_2 b + \nu_2 c) = (\lambda_1\mu_2 - \lambda_2\mu_1)c$$

where the λ_k, μ_k and ν_k are complex scalars. The multiplication is associative as the product of any three elements is zero. As a norm for A we may take

$$\|\lambda a + \mu b + \nu c\| = (|\lambda|^2 + |\mu|^2 + |\nu|^2)^{1/2}$$

and as the involution

$$(\lambda a + \mu b + \nu c)^* = \overline{\lambda} a + \overline{\mu} b - \overline{\nu} c.$$

However, in the positive direction we have the following result.

COROLLARY 3.2. *Suppose that the intersection of the two-sided modular closed *-ideals of A is (0). Suppose that for each $x \in A$ there is a posi-*

tive integer $n(x)$ where $x^{n(x)}$ is normal modulo E. If $E = (0)$ then A is commutative. If $E = Z$ and A is semi-prime then A is commutative.

Proof. Let K be a two-sided modular closed *-ideal of A. Then A/K is a normed *-algebra with identity. Suppose $E = (0)$. Then $[x, y] \in K$ by Theorem 3.1 for every $x, y \in A$, so A is commutative. In the case $E = Z$ we see that $[[x, y], z] \in K$ for all $x, y, z \in A$. Then an application of Herstein's sublemma [2, p. 5] shows that A is commutative if A is semi-prime.

In particular let A be a B^*-algebra which is strongly semi-simple [6, p. 59], that is, the intersection of its maximal modular two-sided ideals is (0). For each maximal modular two-sided ideal M, M is automatically closed and $M = M^*$ by [6, Th. 4.9.2]. Also A is semi-simple so that if for each $x \in A$ there is $n(x)$ with $x^{n(x)}$ normal modulo Z we have A commutative.

LEMMA 3.1. *Let h and k be in H and n be a positive integer. Suppose that $(h+itk)^n$ is normal modulo E for each real t where $E = (0)$ or $E = Z$. Then*

$$\Big[h^n, \sum_{j=0}^{n-1} h^j k h^{n-1-j}\Big] \in E, \tag{1}$$

$$[h^n, [h^n, k]] = 0. \tag{2}$$

Proof. If $n = 1$ then clearly $hk - kh \in E$ so that (1) and (2) are valid. Suppose that $n > 1$. We write $(h+k)^n = \sum_{r=0}^{n} V_r$ where V_r is the sum of the terms of the expansion of $(h+k)^n$ for which the sum of the exponents of the k^j factors is r. In particular $V_0 = h^n$ and $V_1 = \sum_{j=0}^{n-1} h^j k h^{n-1-j}$.

Let $\sum'$ ($\sum''$) denote the summation from $j = 0$ to $j = n$ for the odd (even) values of j. For t real we have

$$(h+itk)^n = A(t) + B(t), \qquad (h-itk)^n = A(t) - B(t)$$

where

$$A(t) = {\sum}'' i^j V_j t^j \quad \text{and} \quad B(t) = {\sum}' i^j V_j t^j.$$

As $(h+itk)^n$ is normal modulo E we see that

$$[A(t) + B(t), A(t) - B(t)] \in E$$

so that $[A(t), B(t)] \in E$.

Notice that $A(t)$ consists of h^n plus a polynomial in t with t^2 as a factor. For $t \neq 0$, $-it^{-1}B(t)$ consists of V_1 plus a polynomial in t with t^2 as a factor. Letting $t \to 0$ we see that $[h^n, V_1] \in E$, which is (1).

One checks that $hV_1 - V_1 h = [h^n, k]$ so that our task for (2) is to see that $[h^n, hV_1 - V_1 h] = 0$. For $E = (0)$ this follows from $[h^n, V_1] = (0)$. Suppose $E = Z$ so we have $[h^n, V_1] = z \in Z$. Then

$$[h^n, hV_1 - V_1 h] = h[h^n, V_1] - [h^n, V_1]h = 0.$$

LEMMA 3.2. (a) *If* $[h^n, y] \in E$ *for every* $h \in H$ *then* $[x^n, y] \in E$ *for every* $x \in A$.

(b) *If for each* $h \in H$ *there is an integer* $m(h)$ *where* $[h^{m(h)}, y] \in E$ *then for each* $x \in A$ *there is an integer* $n(x)$ *so that* $[x^{n(x)}, y] \in E$.

Proof. (a) Let $x = h + ik$, $h, k \in H$. As in the proof of Lemma 3.1 we write $(h+k)^n = \sum V_r$. Now

$$[(h+tk)^n, y] = \sum [V_r, y] t^r$$

lies in E for all real t so that each $[V_r, y] \in E$. Inasmuch as $x^n = \sum i^r V_r$ we have $[x^n, y] \in E$.

(b) Let $x = h + ik$ as above. For each real t there is a positive integer $m(t)$ so that

$$[y, (h+tk)^{m(t)}] \in E.$$

Arguing as in the proof of Theorem 3.1 we see that there is a positive integer n so that

$$[(h+tk)^n, y] \in E$$

for infinitely many t and hence for all real t. As in (a) we see that $[x^n, y] \in E$.

It is convenient to make some preliminary calculations to expedite the proof of Theorem 3.2. We are concerned with a sum $S = \sum_{j=0}^{n-1} w^j y w^{n-1-j}$. We note first that

$$w^j y w^{n-1-j} + w^{n-1-j} y w^j = w^j (y \cdot w^{n-1-2j}) w^j.$$

Suppose first that n is odd. Then $n - 1 - j = j$ just when $j = (n-1)/2$. Thus if $s = (n-1)/2$ we have

$$S = \sum_{j=0}^{s-1} w^j (y \cdot w^{n-1-2j}) w^j + w^s y w^s. \tag{3}$$

Now suppose n is even and $r = [(n-1)/2]$ is the largest integer in $(n-1)/2$. Here j is never equal to $n - 1 - j$ so that

$$S = \sum_{j=0}^{r} w^j (y \cdot w^{n-1-2j}) w^j. \tag{4}$$

THEOREM 3.2. *Suppose that, for a positive integer* n, x^n *is normal modulo* E *for all* $x \in A$ *where* $E = (0)$ *or* $E = Z$. *Then* $[x^{6n^2}, y] \in E$ *for all* $x, y \in A$.

Proof. Let h and k be in H. By Lemma 3.1, (1) and (2) hold. Also (2) can be rewritten as

$$k \cdot h^{2n} = 2h^n k h^n. \tag{5}$$

In (1) we replace h by h^{2n} to obtain

$$\Big[h^{2n^2}, \sum_{j=0}^{n-1} h^{2nj} k h^{2n(n-1-j)}\Big] \in E \tag{6}$$

for all $h, k \in H$. We examine the summation in (6), which we denote by S. For typographical convenience we set $p(j) = n - 1 - 2j$. Suppose first that n is even. Then, by (4),

$$S = \sum_{j=0}^{r} h^{2nj}\{k \cdot h^{2np(j)}\}h^{2nj}.$$

By (5), $k \cdot h^{2np(j)} = 2h^{np(j)}kh^{np(j)}$. But also $h^{np(j)}h^{2nj} = h^{n(n-1)}$. Thus

$$S = 2(r+1)h^{n(n-1)}kh^{n(n-1)}.$$

In case n is odd we get an extra term so that, by (3),

$$S = 2sh^{n(n-1)}kh^{n(n-1)} + h^{2ns}kh^{2ns}.$$

However, $s = (n-1)/2$ so that, for n odd, we see that

$$S = (2s+1)h^{n(n-1)}kh^{n(n-1)}.$$

Therefore, from (6), we see that

$$[h^{2n^2}, h^{n(n-1)}kh^{n(n-1)}] \in E$$

for all $h, k \in H$. Now $v = h^{n(n+1)}kh^{n(n+1)}$ lies in H. Consequently,

$$[h^{2n^2}, h^{2n^2}kh^{2n^2}] \in E \tag{7}$$

for all $h, k \in H$. From (7) and (5) we have

$$[h^{2n^2}, k \cdot h^{4n^2}] \in E \quad \text{or} \quad [h^{6n^2}, k] - [h^{2n^2}, h^{2n^2}kh^{2n^2}] \in E$$

for all $h, k \in H$. Therefore, by (7), $[h^{6n^2}, k] \in E$. As every $y \in A$ can be written in the form $u + iv$ where $u, v \in H$ we see that $[h^{6n^2}, y] \in E$ for all $h \in H$, $y \in A$. An appeal to Lemma 3.2 completes the proof.

COROLLARY 3.3. *Let A be semi-prime. Suppose that for some integer n, x^n is normal modulo Z for each $x \in A$. Then A is commutative.*

Proof. By Theorem 3.2,

$$[[x^m, y], z] = 0 \tag{8}$$

for all $x, y, z \in A$ where $m = 6n^2$. By the theorem in [5], A is commutative if we can show that, for each prime p, the ring of two-by-two matrices over the integers modulo p fails to satisfy (8). It is readily seen that (8) does not hold with the choices

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

As in [3] by the *hypercenter* $T(A)$ of A we mean the set of $a \in A$ where for each $x \in A$ there is a positive integer $n = n(x, a)$ such that $ax^n = x^n a$. As noted in [3], $T(A)$ is a subalgebra of A containing Z.

THEOREM 3.3. *For each $x \in A$ let $W(x)$ denote the smallest *-subalgebra of A containing x. Suppose that for each $x \in A$ there is a positive integer $m(x)$ so that $y^{m(x)}$ is normal for all $y \in W(x)$. Then A coincides with its hypercenter.*

Proof. Let v be a fixed element of H. For $u \in H$ let $x = u + iv$ and note that u and v lie in $W(x)$. By Theorem 3.2 there is a positive integer $m = m(x)$ so that y^{6m^2} lies in the center of $W(x)$. Hence

$$[u^{6m^2}, v] = 0.$$

By Lemma 3.2 we see that for each $w \in A$ there is a positive integer $n(w)$ so that $[w^{n(w)}, v] = 0$. Hence $v \in T(A)$. As $T(A)$ is a subalgebra of A, we have $A = T(A)$.

COROLLARY 3.4. *Under the hypotheses of Theorem* 3.3, *A is commutative if A is a semi-prime algebra.*

Proof. If A is a semi-prime algebra then $T(A) = Z$ by [3, Th. 2].

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