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GAME-THEORETIC ALGORITHMS FOR FAIR AND STRONGLY FAIR CAKE DIVISION WITH ENTITLEMENTS

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0. Introduction. The cake division problem is the problem of dividing a cake among a certain number of individuals in such a way that each individual is "satisfied", where we assume that each such individual has some means of comparing the relative worth of different pieces of cake, but where we do not necessarily assume that different individuals agree with each other in these evaluations. Of course, it is not yet clear what we mean by "satisfied".

To make this precise, let us assume that there are n individuals, and n associated finitely additive, non-atomic measures $m_1, \ldots, m_n$, each defined on some algebra of subsets of our cake C, such that for each such m_i, $m_i(C) = 1$. Various notions of what it means for each of the n individuals to be satisfied with a partition $\langle P_1, \ldots, P_n \rangle$ of C have been considered. See, for example, Brams and Taylor [2], or Barbanel [1]. We shall consider, and then generalize, two of these notions: The partition $\langle P_1, \ldots, P_n \rangle$ of C is *fair* with respect to the sequence of measures $\langle m_1, \ldots, m_n \rangle$ if and only if for each $i = 1, \ldots, n$, $m_i(P_i) \geq 1/n$, and is *strongly fair* with respect to the sequence of measures $\langle m_1, \ldots, m_n \rangle$ if and only if for each $i = 1, \ldots, n$, $m_i(P_i) > 1/n$. Banach and Knaster (see Steinhaus [6] or [7]) presented an algorithm for obtaining a partition of a cake C among n individuals that is fair, and Woodall [8] presented an algorithm for obtaining a partition of a cake C among n individuals that is strongly fair, assuming that the measures are not identical and that we are given a witness of the disagreement of some two of the measures. We note that it is clearly impossible to obtain a strongly fair partition of C if all of the involved measures are identical.

In this paper, we study a generalization of these notions. Let $\langle \alpha_1, \ldots, \alpha_n \rangle$ be a sequence of positive numbers satisfying $\alpha_1 + \ldots + \alpha_n = 1$. We shall call such a sequence an *entitlement sequence*. Then, we say that the partition $\langle P_1, \ldots, P_n \rangle$ of C is *fair* with respect to the sequence of measures $\langle m_1, \ldots, m_n \rangle$ and the entitlement sequence $\langle \alpha_1, \ldots, \alpha_n \rangle$ if and only if for

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each $i = 1, \ldots, n$, $m_i(P_i) \geq \alpha_i$, and is *strongly fair* with respect to the sequence of measures $\langle m_1, \ldots, m_n \rangle$ and the entitlement sequence $\langle \alpha_1, \ldots, \alpha_n \rangle$ if and only if for each $i = 1, \ldots, n$, $m_i(P_i) > \alpha_i$. We note that these notions reduce to fairness and strong fairness respectively if $\alpha_1 = \ldots = \alpha_n$.

Consider the question of whether there exists an algorithm for obtaining a partition of C among n individuals that is fair, or is strongly fair, with respect to a given sequence of measures and entitlement sequence. As Dubins and Spanier [4] point out, the Banach and Knaster result discussed above generalizes easily to yield an algorithm for fairness if all of the elements of the entitlement sequence are rational. Similarly, Woodall's result generalizes easily to yield an algorithm for strong fairness if all of the elements of the entitlement sequence are rational. We shall show, for both fairness and strong fairness, that this rationality assumption is not necessary. In particular, we shall show that there exists an algorithm for obtaining a partition of C among n individuals that is fair with respect to a given sequence of measures and entitlement sequence, and that if the measures are not identical and we are given a witness of the disagreement of some two measures, then we can find a partition that is strongly fair with respect to the given sequence of measures and entitlement sequence. Our proof of the existence of a fair partition is completely different from the fairness algorithm of Banach and Knaster, but our proof of the existence of a strongly fair partition is very much inspired by the strong fairness algorithm of Woodall. The algorithms we shall consider will be "game-theoretic", in a sense that we shall discuss below.

It will be necessary, at various stages in our algorithms, for individuals to be able to partition C into a certain number of pieces that are equal according to that player's measure. Hence, it will be necessary that measures satisfy the *Partitioning Postulate*: A measure m, defined on some algebra of subsets of C, satisfies the Partitioning Postulate if, for any positive integer k, there exists a partition of C into k pieces, each of which is in the algebra, such that each piece is of equal size according to m.

It will also be necessary, at various stages in our algorithms, for players to be able to trim off a bit of a given subset of C to obtain a set of a given smaller size. Hence, it will be necessary that measures satisfy the *Trimming Postulate*: A measure m, defined on some algebra of subsets of C, satisfies the Trimming Postulate if, given any set A in our algebra, and given any positive real number $r < m(A)$, there exists $B \subseteq A$ in our algebra such that $m(B) = r$.

Throughout this paper, "measure" shall always mean a finitely additive, non-atomic measure, defined on some algebra of subsets of C, which gives measure 1 to C, and which satisfies the Partitioning Postulate and the Trimming Postulate. We assume that all subsets of C that are mentioned are in some common algebra on which all relevant measures are defined.

1. Game-theoretic algorithms. We begin by describing in general terms what we mean by a game-theoretic algorithm. A game-theoretic algorithm consists of three ingredients:

1. A set of *rules* for the game. The game proceeds by discrete stages. The players must follow the given rules at each stage. The rules shall contain no reference to the players' measures. The intuition is that we may view a player's measure as private information that only he or she has access to, but the rules are such that an outside observer, without access to this private information, would be able to ascertain whether the rules were being obeyed.

In general, each player has many choices to make. In other words, each player can generally obey the rules in many different ways at each stage.

2. A *strategy* for each player. Each player's strategy gives instructions as to how to obey the rules at each stage in which there are choices to be made. A player's strategy can involve that player's measure, but cannot assume knowledge of the measures of the other players or the strategies of the other players.

Our general intention is that strategies be purely deterministic (they may depend only upon the rules and upon the previous actions that the players have taken), but we will allow three types of "non-deterministic" exceptions:

(a) By the Partitioning Postulate, there exists a partition of a given piece of cake into any fixed number of pieces which are of equal size according to a given player's measure. We shall assume that each player, in following his or her strategy, can create such a partition.

(b) By the Trimming Postulate, any piece of cake has a subpiece of any given smaller size, according to any given player's measure. We shall assume that each player, in following his or her strategy, can specify such a subpiece.

(c) A particular player's strategy may involve, for example, choosing the p largest (according to that player's measure) pieces from some collection of s many pieces of cake created at a previous step in the game. Whenever a player has such a strategy, we shall assume that if there is a tie for the pth largest piece, then choices can be made arbitrarily among the pieces that are tied.

The intuition behind a player's strategy is that if all players follow the rules, then each individual player can achieve his or her goals by following his or her strategy, regardless of whether the other players follow their strategies or not.

3. A proof that the rules and strategies can be followed, and that each player's strategy is "successful" (in the sense given by the previous paragraph).

We point out that our notion of a game-theoretic algorithm is the same as what Even and Paz [5] call a “protocol”. See also Brams and Taylor [3]. We also note that Banach and Knaster’s presentation of an algorithm for fair division and Woodall’s presentation of an algorithm for strongly fair division were not explicitly game-theoretic, but it is straightforward to see that their algorithms can easily be recast as game-theoretic algorithms.

2. Fairness

THEOREM. *Let C be a non-empty set and suppose n individuals, which we shall refer to as Player 1, ..., Player n, have measures $m_1, \dots, m_n$ respectively that each uses to evaluate subsets of C. Let $\langle \alpha_1, \dots, \alpha_n \rangle$ be an entitlement sequence. Then, there is a game-theoretic algorithm for fairness with respect to these measures and this entitlement sequence. In other words, there is a game-theoretic algorithm that produces a partition $\langle P_1, \dots, P_n \rangle$ of C such that for each $i = 1, \dots, n$, if Player i's strategy was followed, then $m_i(P_i) \geq \alpha_i$.*

We shall prove the theorem by induction on n. The first meaningful case is $n = 2$. Thus, we assume that there are two players, Player A and Player B (which we shall refer to as “she” and “he” respectively), and corresponding measures m_a and m_b. Let α be a real number with $0 < \alpha < 1$. We must show that there exists a game-theoretic algorithm for fairness with respect to the pair of measures $\langle m_a, m_b \rangle$ and the entitlement sequence $\langle \alpha, 1 - \alpha \rangle$.

We next state the rules and corresponding strategies for our game-theoretic algorithm. Following this, we will show that it is possible for the rules and strategies to be followed, and then we will prove that the given strategies, if followed, produce the desired outcome.

Rules	Player A’s strategy	Player B’s strategy
1. Player B partitions C into $w+1$ pieces $Q_1, \dots, Q_{w+1}$, where w is the unique positive integer satisfying that $w\alpha \leq 1 < (w+1)\alpha$.		Partition C into $w+1$ many pieces $Q_1, \dots, Q_{w+1}$ such that for $i = 1, \dots, w$, $m_b(Q_i) = \alpha$. (Then, $m_b(Q_{w+1}) < \alpha$.)
2. Player A either takes one of $Q_1, \dots, Q_{w+1}$, or says some positive integer s. If she takes one of these sets, let F_a be the chosen set, let F_b equal $C \setminus F_a$, and declare the game to be over.	If, for some $i = 1, \dots, w+1$, $m_a(Q_i) \geq \alpha$, choose the Q_i which is biggest according to m_a. Otherwise, say “s”, where s is the least positive integer such that $\frac{1}{s} \leq \frac{(1-\alpha)[m_a(Q_{w+1})+\alpha w-1]}{\alpha(1-m_a(Q_{w+1}))}$.	

Rules	Player A's strategy	Player B's strategy
3. For each $i = 1, \ldots, w$, Player B partitions Q_i into s many pieces $R_{i1}, \ldots, R_{is}$.		Partition each Q_i so that for all $i, i' = 1, \ldots, w$, and all $j, j' = 1, \ldots, s$, $m_b(R_{ij}) = m_b(R_{i'j'})$.
4. Player A chooses t many of the R_{ij}'s, where t is the greatest non-negative integer such that $\frac{t\alpha}{s} \leq \alpha - 1 + w\alpha$. Let F_a be the union of these R_{ij}'s together with Q_{w+1}, let F_b equal $C \setminus F_a$, and declare the game to be over.	Choose the t many R_{ij}'s that are largest according to m_a.	

We next show that it is possible for the players to follow the rules and their respective strategies.

The only rule where this is not obvious is Rule 4. In order to see that there must be a non-negative integer t such that $t\alpha/s \leq \alpha - 1 + w\alpha$, we simply observe that by Rule 1, $1 < (w+1)\alpha$, and hence $\alpha - 1 + w\alpha > 0$. Thus, it is possible for both players to follow the rules.

The only part of the strategies where it is not obvious that it is possible to follow the given strategy is in Player A's strategy for Rule 2. Suppose that for each $i = 1, \ldots, w+1$, $m_a(Q_i) < \alpha$, and hence Player A's strategy calls for her to pick the least positive integer s such that

$$\frac{1}{s} \leq \frac{(1-\alpha)[m_a(Q_{w+1}) + \alpha w - 1]}{\alpha[1 - m_a(Q_{w+1})]}.$$

To establish that such an s exists, we must show that

$$\frac{(1-\alpha)[m_a(Q_{w+1}) + \alpha w - 1]}{\alpha[1 - m_a(Q_{w+1})]} > 0.$$

Since, by assumption, $0 < \alpha < 1$, it suffices to show that $m_a(Q_{w+1}) + \alpha w - 1 > 0$ and $1 - m_a(Q_{w+1}) > 0$.

Since, for each $i = 1, \ldots, w$, we have $m_a(Q_i) < a$, it follows that $m_a(Q_{w+1}) = 1 - m_a(Q_1 \cup \ldots \cup Q_w) > 1 - \alpha w$. Hence, $m_a(Q_{w+1}) + \alpha w - 1 > 0$. Also, since $m_a(Q_{w+1}) < \alpha < 1$, it follows that $1 - m_a(Q_{w+1}) > 0$. Thus, it is possible for both players to follow their respective strategies.

We must show that each player's strategy is successful. Thus, we must show that:

(a) $m_a(F_a) \geq \alpha$ if Player A follows her strategy, and
(b) $m_b(F_b) \geq 1 - \alpha$ if Player B follows his strategy.

Proof that Player A's strategy is successful. We assume that Player A follows her strategy. We must show that $m_a(F_a) \geq \alpha$. There are two cases to consider.

Suppose first that F_a was defined by Rule 2. Then, by Player A's strategy for Rule 2, $m_a(Q_i) \geq \alpha$ for some $i = 1, \ldots, w+1$. But in this case $F_a = Q_i$ for the Q_i which is biggest according to m_a, and hence $m_a(F_a) \geq \alpha$, as desired.

Now suppose that F_a was defined by Rule 4. Let R be the union of the t many R_{ij}'s chosen by Player A, and let $Q = Q_{w+1}$. Then $F_a = Q \cup R$.

Since $m_a(Q_1 \cup \ldots \cup Q_w) = 1 - m_a(Q)$, it follows that according to m_a, the average size of each of the sw many R_{ij}'s created by Player B in Rule 3 is $(1 - m_a(Q))/(sw)$. Then, since Player A's strategy for Rule 4 is to choose the t largest of the R_{ij}'s, it follows that

$$m_a(R) \geq \frac{t[1 - m_a(Q)]}{sw}.$$

Since $F_a = R \cup Q$, we have

$$m_a(F_a) \geq \frac{t[1 - m_a(Q)]}{sw} + m_a(Q).$$

Hence, to establish that $m_a(F_a) \geq \alpha$, it suffices to show that

$$\frac{t[1 - m_a(Q)]}{sw} + m_a(Q) \geq \alpha.$$

Suppose, by way of contradiction, that

$$\frac{t[1 - m_a(Q)]}{sw} + m_a(Q) < \alpha, \quad \text{i.e.,} \quad \frac{t}{sw} < \frac{\alpha - m_a(Q)}{1 - m_a(Q)}.$$

Consider the expression

$$\frac{(1-\alpha)[m_a(Q) + \alpha w - 1]}{\alpha[1 - m_a(Q)]}$$

from Player A's strategy for Rule 2. Straightforward algebra yields

$$\frac{(1-\alpha)[m_a(Q) + \alpha w - 1]}{\alpha[1 - m_a(Q)]} = \frac{\alpha - 1 + \alpha w}{\alpha} - \frac{w[\alpha - m_a(Q)]}{1 - m_a(Q)}.$$

Then, Player A's strategy for Rule 2 implies that

$$\frac{1}{sw} \leq \frac{\alpha - 1 + \alpha w}{\alpha w} - \frac{\alpha - m_a(Q)}{1 - m_a(Q)}.$$

This, together with our assumption above, tells us that

$$\frac{t}{sw} < \frac{\alpha - 1 + \alpha w}{\alpha w} - \frac{1}{sw}.$$

This implies that $t\alpha < s(\alpha - 1 + \alpha w) - \alpha$, and hence

$$\frac{(t+1)\alpha}{s} < \alpha - 1 + \alpha w.$$

But this contradicts the definition of t given in Rule 4. This establishes that $m_a(F_1) \geq \alpha$, as desired.

Proof that Player B's strategy is successful. We now assume that Player B follows his strategy. We must show that $m_b(F_b) \geq 1 - \alpha$. There are two cases to consider.

Suppose first that F_b was defined by Rule 2. Let Q_i be the piece chosen by Player A. By Player B's strategy for Rule 1, we must have $m_b(Q_i) = \alpha$. But $F_a = Q_i$, and hence $m_b(F_a) = \alpha$. It follows that $m_b(F_b) = m_b(C \backslash F_a) = 1 - \alpha$.

Suppose next that F_b was defined by Rule 4. There are sw many R_{ij}'s. Each R_{ij} is one piece of a partition of Q_i into s many pieces. By Player B's strategy for Rule 3, each of these pieces is the same size, according to m_b. Then, since for each $i = 1, \ldots, w$, $m_b(Q_i) = \alpha$, it must be that for each R_{ij}, $m_b(R_{ij}) = \alpha/s$. F_b is the union of the $sw - t$ many R_{ij}'s not chosen by Player A. Hence,

$$m_b(F_b) = (sw - t)\frac{\alpha}{s} = w\alpha - \frac{t\alpha}{s}.$$

By the definition of t given in Rule 4,

$$\frac{t\alpha}{s} \leq \alpha - 1 + w\alpha.$$

It follows that

$$m_b(F_b) = w\alpha - \frac{t\alpha}{s} \geq w\alpha - \alpha + 1 - w\alpha = 1 - \alpha,$$

as desired. This establishes that the partition $\langle F_a, F_b \rangle$ of C is fair with respect to the pair of measures $\langle m_a, m_b \rangle$ and the entitlement sequence $\langle \alpha, 1 - \alpha \rangle$.

We now assume that the theorem is true for $n = k$, and we wish to show that it holds for $n = k+1$. Hence, we suppose that there are $k+1$ individuals, which we shall refer to as Player 1, ..., Player $k+1$, with corresponding measures $m_1, \ldots, m_{k+1}$. Let $\langle a_1, \ldots, a_{k+1} \rangle$ be an entitlement sequence. We must show that there is a game-theoretic algorithm for fairness with respect to the sequence of measures $\langle m_1, \ldots, m_{k+1} \rangle$ and the entitlement sequence $\langle a_1, \ldots, a_{k+1} \rangle$. For convenience, we shall refer to each of Player 1, ..., Player k as "she" and shall refer to Player $k+1$ as "he".

We describe the desired game-theoretic algorithm in two parts.

PART 1. Let $\beta = \alpha_1 + \ldots + \alpha_k$. Then, $\langle \alpha_1/\beta, \ldots, \alpha_k/\beta \rangle$ is an entitlement sequence. By assumption, there exists a game-theoretic algorithm for producing a partition $\langle P_1', \ldots, P_k' \rangle$ of C that is fair with respect to the sequence of measures $\langle m_1, \ldots, m_k \rangle$ and the entitlement sequence $\langle \alpha_1/\beta, \ldots, \alpha_k/\beta \rangle$. Part 1 of the game-theoretic algorithm for producing the desired partition among Player 1, ..., Player $k+1$ consists of playing this n-person fairness game to obtain $\langle P'_1, \ldots, P'_k \rangle$.

PART 2. We proceed by repeatedly playing the two-player fairness game, with Player $k+1$ playing this game with each of Player 1, ..., Player k successively. In this way, if relevant strategies are followed, Player $k+1$ receives an appropriate portion of P'_1, ..., and P'_k.

We now describe precisely how each of the k many two-player games is played. For each $i = 1, \ldots, k$, we define

$$m'_i = \frac{m_i}{m_i(P'_i)} \quad \text{and} \quad m'_{k+1} = \frac{m_{k+1}}{m_{k+1}(P'_i)}.$$

For each such i, the two-person game is now played with Player i playing the role of Player A, Player $k+1$ playing the role of Player B, with Player i and Player $k+1$ having measures m'_i and m'_{k+1} respectively on the cake P'_i, and with $\alpha = \beta$. Let P_i be the resultant subset of P'_i which the game awards to Player i (i.e., P_i is what was called F_a in the two-person game), and let P^i be the resultant subset of P'_i which the game awards to Player $k+1$ (i.e., P^i is what was called F_b in the two-person game). Set $P_{k+1} = \bigcup_{i=1}^{k} P^i$. Then $\langle P_1, \ldots, P_{k+1} \rangle$ is a partition of C.

This completes our description of the game.

In summary, what we have done is to describe a game-theoretic algorithm which consists of first playing the k-person fairness game (which exists by our induction hypothesis), and then playing the two-player fairness game (which we have previously described explicitly) k many times.

By induction hypothesis, it is possible for each player to follow the rules and to follow their respective strategies for Part 1. We have already shown that each player can follow the rules and their respective strategies for the two-person fairness game, and hence the rules and strategies for Part 2 can be followed.

We must show that each player's strategy is successful. Hence, we must show that for each $i = 1, \ldots, k+1$, if Player i follows his or her strategy, then $m_i(P_i) \geq \alpha_i$.

Proof that Player 1's, ..., Player k's strategies are successful. Fix some $i = 1, \ldots, k$, and assume that Player i follows her strategy.

Our induction hypothesis tells us that the set P'_i that results from the k-person fairness game of Part 1 is such that $m_i(P'_i) \geq \alpha_i / \beta$.

The two-person fairness game of Part 2 played between Player 1 and Player $k+1$ results in a set P_i satisfying $m'_i(P_i) \geq \beta$. Recalling that m'_i was defined by $m'_i = m_i / m_i(P'_i)$, we have

$$m_i(P_i) = m'_i(P_i) m_i(P'_i) \geq \beta \frac{\alpha_i}{\beta} = \alpha_i.$$

Proof that Player $k+1$'s strategy is successful. We assume that Player $k+1$ follows his strategy in each of the k many two-person fairness games that make up Part 2. Then

$$\begin{aligned} m_{k+1}(P_{k+1}) &= m_{k+1}\Big(\bigcup_{i=1}^{k} P^i\Big) = \sum_{i=1}^{k} m_{k+1}(P^i) \\ &= \sum_{i=1}^{k} m'_{k+1}(P^i) m_{k+1}(P'_i) \geq (1-\beta) \sum_{i=1}^{k} m_{k+1}(P'_i) \\ &= 1-\beta = \alpha_{k+1}. \end{aligned}$$

We have shown that the partition $\langle P_1, \ldots, P_{k+1}\rangle$ of C is fair with respect to the sequence of measures $\langle m_1, \ldots, m_{k+1}\rangle$ and the entitlement sequence $\langle \alpha_1, \ldots, \alpha_{k+1}\rangle$. This establishes the theorem.

3. Strong fairness. As noted previously, Woodall's algorithm for strong fairness requires not only that the measures not be identical, but also requires that we be given a witness of the disagreement of some two measures (Brams and Taylor make the same assumption in their game-theoretic algorithm for strong envy-freeness in [2]). We make this same type of assumption to obtain a strongly fair partition in our present context involving entitlements.

We point out that when we say that we are "given" a witness of the disagreement of two measures, our intention is that this set and the values of these measures on this set can be mentioned in the rules of the game.

THEOREM. *Let C be a non-empty set and suppose n individuals, which we shall refer to as Player 1, …, Player n, have measures $m_1, \ldots, m_n$ respectively that each uses to evaluate subsets of C. Let $\langle \alpha_1, \ldots, \alpha_n\rangle$ be an entitlement sequence. Suppose that the n measures are not identical and, in particular, we are given a 5-tuple $\langle i, j, D, \gamma, \delta\rangle$ such that $D \subseteq C$ and $m_i(D) = \gamma > \delta = m_j(D)$. Then there is a game-theoretic algorithm for strong fairness with respect to these measures and this entitlement sequence. In other words, there is a game-theoretic algorithm that produces a partition $\langle P_1, \ldots, P_n\rangle$ of C such that for each $i = 1, \ldots, n$, if Player i's strategy is followed, then $m_i(P_i) > \alpha_i$.*

We shall prove the theorem by induction on n. As in the previous section, the first meaningful case is $n = 2$. We again assume that there are two players, Player A and Player B (which we shall refer to as "she" and "he" respectively), and corresponding measures m_a and m_b. Let α be a real number with $0 < \alpha < 1$. We must show that there exists a game-theoretic algorithm for strong fairness with respect to the pair of measures $\langle m_a, m_b\rangle$ and the entitlement sequence $\langle \alpha, 1-\alpha \rangle$. We assume, without loss of generality, that $m_a(D) = \gamma > \delta = m_b(D)$.

Before stating the rules and strategies for our game, we fix some constants that will be used. It is important to note that these constants could easily be specified within the rules. We specify them here simply for ease of presentation.

We wish to let r be any rational number such that $\gamma > r > \delta$. For definiteness (in keeping with the fact that r will be used in an algorithmic procedure in which "arbitrary choices" of numbers are prohibited), we define r to be the rational number satisfying $\gamma > r > \delta$ and with the property that among all rational numbers in this interval, r is expressible with the least denominator and the least numerator corresponding to this denominator.

Next, we let q be the least positive integer such that for some positive integer p,

$$\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{r}{1-r}\right)\left(\frac{1-\gamma}{\gamma}\right) < \frac{p}{q} < \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{r}{1-r}\right)\left(\frac{1-\delta}{\delta}\right),$$

and we let p be the least such positive integer corresponding to q. We note that since $\gamma > \delta$, it follows that

$$\frac{1-\gamma}{\gamma} < \frac{1-\delta}{\delta},$$

and hence our definition of p and q makes sense.

Finally, we let s and t be the least positive integers such that $r = \frac{s}{t}$, $p \leq s$, and $q \leq t - s$.

We are now ready to describe our game. We do this in two parts.

Part 1

Rules	Player A's strategy	Player B's strategy
1. Player B partitions D into s many pieces.		Partition D into s many pieces which are equal according to m_b.
2. Player A partitions $C \setminus D$ into $t-s$ many pieces.	Partition $C \setminus D$ into $t-s$ many pieces which are equal with respect to m_a.	
3. Player A chooses p many of the s sets created in Rule 1. Let S_a^1 be the union of these sets.	Choose the p many sets which are largest according to m_a.	
4. Player B chooses q many of the $t-s$ sets created in Rule 2. Let S_b^1 be the union of these sets.		Choose the q many sets which are largest according to m_b.

PART 2. Define measures m'_a and m'_b on $C \setminus (S_a^1 \cup S_b^1)$ as follows:

$$m'_a = \frac{m_a}{m_a[C \setminus (S_a^1 \cup S_b^1)]} \quad \text{and} \quad m'_b = \frac{m_b}{m_b[C \setminus (S_a^1 \cup S_b^1)]}.$$

Now play the two-person fairness game from the previous section, with $C \setminus (S_a^1 \cup S_b^1)$ playing the role of C, with α playing the role of α, and with Player A and Player B using measures m'_a and m'_b respectively. Let S_a^2 be the subset of $C \setminus (S_a^1 \cup S_b^1)$ which the game awards to Player A (i.e., S_a^2 is what was called F_a in the two-person fairness game) and let S_b^2 be the subset of $C \setminus (S_a^1 \cup S_b^1)$ which the game awards to Player B (i.e., S_b^2 is what was called F_b in the two-person fairness game).

Finally, let $S_a = S_a^1 \cup S_a^2$ and let $S_b = S_b^1 \cup S_b^2$. This completes our description of the game.

Clearly, Parts 1 and 2 together define a game-theoretic algorithm. There is certainly no question that the rules can be followed. We must show that each player's strategy is successful. Thus, we must show that:

(a) $m_a(S_a) > \alpha$ if Player A follows her strategy, and
(b) $m_b(S_b) > 1 - \alpha$ if Player B follows his strategy.

Proof that Player A's and Player B's strategies are successful. We assume that Player A follows her strategy. We must show that $m_a(S_a) > \alpha$. We first consider $m_a(S_a^1)$. We claim that $m_a(S_a^1) > \alpha m_a(S_a^1 \cup S_b^1)$. By assumption,

$$\frac{p}{q} > \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{r}{1-r}\right)\left(\frac{1-\gamma}{\gamma}\right).$$

We note that since $r = s/t$, we have $r/(1-r) = s/(t-s)$. Hence,

$$\frac{p}{q} > \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{s}{t-s}\right)\left(\frac{1-\gamma}{\gamma}\right).$$

This implies that

$$\frac{(p/s)\gamma}{(q/(t-s))(1-\gamma)} > \frac{\alpha}{1-\alpha}.$$

By following her strategy for Rule 3 of Part 1, Player A guarantees that $m_a(S_a^1) \geq (p/s)m_a(D)$. Recalling that $m_a(D) = \gamma$, we have $m_a(S_a^1) \geq (p/s)\gamma$. Also, in following her strategy for Rule 2 of Part 1, Player A guarantees that

$$m_a(S_b^1) = \frac{q}{t-s} m_a(C \setminus D) = \frac{q}{t-s}(1 - m_a(D)) = \frac{q}{t-s}(1-\gamma).$$

Then

$$\frac{m_a(S_a^1)}{m_a(S_b^1)} \geq \frac{(p/s)\gamma}{(q/(t-s))(1-\gamma)}$$

and so, by the inequality above,

$$\frac{m_a(S_a^1)}{m_b(S_b^1)} > \frac{\alpha}{1-\alpha}.$$

This implies that $m_a(S_a^1) > \alpha m_a(S_a^1 \cup S_b^1)$, as desired.

Next, we consider $m_a(S_a^2)$. Recall that by definition

$$m_a' = \frac{m_a}{m_a[C \setminus (S_a^1 \cup S_b^1)]}.$$

In following her strategy for Part 2, Player A guarantees that $m_a'(S_a^2) > \alpha$. Hence, $m_a(S_a^2) \geq \alpha m_a[C \setminus (S_a^1 \cup S_b^1)]$.

Finally, we have

$$\begin{aligned} m_a(S_a) &= m_a(S_a^1 \cup S_a^2) = m_a(S_a^1) + m_a(S_a^2) \\ &> \alpha m_a(S_a^1 \cup S_a^2) + \alpha m_a[C \setminus (S_a^1 \cup S_a^2)] \\ &= \alpha[m_a(S_a^1 \cup S_a^2) + m_a(C \setminus (S_a^1 \cup S_a^2))] = \alpha m_a(C) = \alpha, \end{aligned}$$

as desired.

The proof that Player B's strategy is successful is similar, and we omit it. This establishes that the partition $\langle S_a, S_b\rangle$ of C is strongly fair with respect to the pair of measures $\langle m_a, m_b\rangle$ and the entitlement sequence $\langle \alpha, 1-\alpha\rangle$.

Assume now that the theorem is true for $n = k$. We wish to show that it holds for $n = k+1$. We suppose that there are $k+1$ individuals, which we shall refer to as Player 1, …, Player $k+1$, with corresponding measures $m_1, \ldots, m_{k+1}$. Assume that these measures are not identical and that we are given $\langle i, j, D, \gamma, \delta\rangle$ such that $D \subseteq C$ and $m_i(D) = \gamma > \delta = m_j(D)$. Let $\langle \alpha_1, \ldots, \alpha_{k+1}\rangle$ be an entitlement sequence. We must show that there is a game-theoretic algorithm for strong fairness with respect to the sequence of measures $\langle m_1, \ldots, m_{k+1}\rangle$ and the entitlement sequence $\langle \alpha_1, \ldots, \alpha_{k+1}\rangle$. As in the previous section, we shall refer to each of Player 1, …, Player k as "she" and shall refer to Player $k+1$ as "he".

We may assume, without loss of generality, that $i = 1$ and $j = 2$. We define the desired game-theoretic algorithm in three parts.

Part 1. As in the previous section, if we let $\beta = \alpha_1 + \ldots + \alpha_k$, then $\langle \alpha_1/\beta, \ldots, \alpha_k/\beta\rangle$ is an entitlement sequence. By assumption, there exists a game-theoretic algorithm for producing a partition $\langle P_1', \ldots, P_k'\rangle$ of C that is strongly fair with respect to the sequence of measures $\langle m_1, \ldots, m_k\rangle$ and the entitlement sequence $\langle \alpha_1/\beta, \ldots, \alpha_k/\beta\rangle$. Part 1 of the desired game-theoretic algorithm for producing a partition which is strongly fair with respect to the sequence of measures $\langle m_1, \ldots, m_{k+1}\rangle$ and the entitlement sequence $\langle \alpha_1, \ldots, \alpha_{k+1}\rangle$ consists of playing this n-person strong fairness game.

PART 2. We proceed in precisely the same manner as we did in Part 2 of the induction step in the previous section. Thus, Part 2 consists of playing the two-person fairness game k times, with Player $k+1$ successively playing with Player 1, ..., Player k. As in the previous section, for each $i = 1, \dots, k$, the ith game is played with Player i using measure $m_i' = m_i/m_i(P_i')$, with Player $k+1$ using measure $m_{k+1}' = m_{k+1}/m_{k+1}(P_i')$, with P_i' in the role of C, and with $\alpha = \beta$. For each such i, let P_i'' be the resultant subset of P_i' which the game awards to Player i and let P^i be the resultant subset of P_i' which the game awards to Player $k+1$. Set $P_{k+1}'' = \bigcup_{i=1}^{k} P^i$. Then $\langle P_1'', \dots, P_{k+1}'' \rangle$ is a partition of C.

As we shall show, the partition $\langle P_1'', \dots, P_{k+1}'' \rangle$ is "almost" strongly fair. More specifically, we shall see that for $i = 1, \dots, k$, $m_i(P_i'') > \alpha_i$ and that $m_{k+1}(P_{k+1}'') \geq \alpha_{k+1}$. However, it need not be true that $m_{k+1}(P_{k+1}'') > \alpha_{k+1}$. What Part 3 will do is to allow Player $k+1$ to grab a small piece of P_j'' for some $j = 1, \dots, k$. This piece will increase the size of Player $k+1$'s portion so that he will believe that his piece has measure strictly greater than α_{k+1}, but will be sufficiently small so that Player j will still believe that her piece has measure strictly greater than α_j. Any $j = 1, \dots, k$ will be sufficient, as long as $m_{k+1}(Pj'') > 0$.

PART 3

Rules	Player j's strategy	Player $k+1$'s strategy
1. Player $k+1$ says some positive integer $j = 1, \dots, k$.		Say "j" where P_j'' is the biggest set among $P_1'', \dots, P_k''$ according to m_{k+1}.
2. Player j chooses a positive integer w and partitions P_j'' into w many pieces.	Let w be the least positive integer such that $m_j(P_j'')/w < m_j(P_j'') - \alpha_j$. Partition P_j'' into w many pieces which are equal with respect to m_j.	
3. Player $k+1$ chooses one of the w many pieces created in Rule 2.		Choose the set which is biggest according to m_{k+1}.
4. Let P_j be the union of the $w-1$ sets which were not chosen by Player $k+1$ in Rule 3. For $i = 1, \dots, j-1, j+1, \dots, k$, let P_i equal P_i''. Let P_{k+1} be the union of P_{k+1}'' and the piece chosen by Player $k+1$ in Rule 3.		

This completes our description of the game.

To show that all rules and strategies can be followed, we first note that there is nothing to show for Part 1, since the required result follows from our induction hypothesis. There is also nothing to show for Part 2, since Part 2 involves a repeated use of a game-theoretic algorithm whose rules and strategies were previously analyzed. For Part 3, there is no difficulty in following the rules and strategies as long as we know that if Player j follows her strategy, then $m_j(P''_j) - \alpha_j > 0$. We will establish this as part of proving that Player j's strategy is successful.

Next, we show that each player's strategy is successful. Thus, we must show that for each $i = 1, \ldots, k+1$, $m_i(P_i) > \alpha_i$.

Proof that Player j's strategy is successful. We assume that Player j follows her strategy.

By induction hypothesis, the set P'_j that results from the k-person strong fairness game of Part 1 is such that $m_j(P'_j) > \alpha_j/\beta$.

The two-person fairness game of Part 2 played between Player j and Player $k+1$ results in a set P''_j satisfying that $m'_j(P''_j) \geq \beta$. But m'_j was defined by $m'_j = m_j/m_j(P'_j)$. Hence, we have

$$m_j(P''_j) = m'_j(P''_j)m_j(P'_j) > \beta\frac{\alpha_j}{\beta} = \alpha_j.$$

Let Q be the set chosen by Player $k+1$ in Rule 3 of the two-player game of Part 3. Then $P_j = P''_j \setminus Q$. By Player j's strategy for Rule 2,

$$m_j(Q) = \frac{m_j(P''_j)}{w} < m_j(P''_j) - \alpha_j.$$

Hence,

$$m_j(P_j) = m_j(P''_j) - m_j(Q) > \alpha_j.$$

Proof that Player 1's, ..., Player $j-1$'s, Player $j+1$'s, ..., Player k's strategies are successful. Fix any $i = 1, \ldots, j-1, j+1, \ldots, k$. We assume that Player i follows her strategy. Then, precisely as in the proof above that Player j's strategy is successful, we have $m_i(P''_i) > \alpha_i$. By Rule 4 of Part 3, $P_i = P''_i$. Hence, $m_i(P_i) > \alpha_i$.

Proof that Player $k+1$'s strategy is successful. We assume that Player $k+1$ follows his strategy. Then, precisely as in Part 2 of the fairness game of the previous section, we have $m_{k+1}(P''_{k+1}) \geq \alpha_{k+1}$.

As above, we assume that Q is the set chosen by Player $k+1$ in Rule 3 of the two-player game of Part 3. Then $P_{k+1} = P''_{k+1} \cup Q$. By Player $k+1$'s strategy for Rule 1, $m_{k+1}(P''_j) > 0$, and by Player $k+1$'s strategy for Rule 3, $m_{k+1}(Q) > 0$. Hence,

$$m_{k+1}(P_{k+1}) = m_{k+1}(P''_{k+1}) + m_{k+1}(Q) > \alpha_{k+1}.$$

We have shown that the partition $\langle P_1, \ldots, P_{k+1}\rangle$ of C is strongly fair with respect to the sequence of measures $\langle m_1, \ldots, m_{k+1}\rangle$ and the entitlement sequence $\langle \alpha_1, \ldots, \alpha_{k+1}\rangle$. This establishes the theorem.

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