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A MULTIFRACTAL ANALYSIS OF AN INTERESTING CLASS OF MEASURES

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1. Introduction. Let $\mu_n = p_n\delta(0) + (1-p_n)\delta(1/2^n)$, $n = 1, 2, \ldots$, where $p_n \in [0,1]$ and $\delta(x)$ denotes the probability atom at x. The infinite convolution product of the μ_n converges in the weak* sense to a probability measure μ on $[0,1]$ which is known as a coin tossing measure [9],

$$\mu = \mathop{\ast}_{n=1}^{\infty} \mu_n.$$

Let $x = \sum_{n=1}^{\infty} \varepsilon_n(x)/2^n$, where $\varepsilon_n(x) \in \{0,1\}$, be the 2-adic expansion of $x \in [0,1]$. It is not difficult to see that if

$$d\nu_{a,N} = \prod_{n=1}^{N} (1 + a_n r_n(x))d\lambda, \qquad N = 1, 2, \ldots,$$

where $a = (a_n)_{n\ge1}$, λ denotes the Lebesgue measure, $r_n(x) = 1 - 2\varepsilon_n(x)$ is the nth Rademacher function and $p_n = (1 + a_n)/2$, then

$$\lim_{N\to\infty} \nu_{a,N} = \mu_a$$

in the weak* sense and $\mu = \mu_a$ (see also [12]). So we have two ways to describe the same measure. In this work we shall use the second way. The characterizations of the sequences $(a_n)_{n\ge1}$ which give continuous or singular measures are given in [5], [6], [9], [11]. In a previous work [4] we have proved that

$$\liminf_{n\to\infty} \frac{\log \mu_a(E_{n,k}(x))}{-n\log 2} = \delta_a \qquad \mu_a\text{-a.e.},$$

where

$$\delta_a = 1 - \limsup_{N\to\infty} \frac{1}{N\log 4} \sum_{n=1}^{N} \log[(1 + a_n)^{1+a_n}(1 - a_n)^{1-a_n}]$$

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and $E_{n,k}(x)$ is the segment $[k/2^n, (k+1)/2^n)$ containing x, for some $k = 0, 1, 2, \ldots, 2^n - 1$. From this relation we deduce that μ_a is δ_a-dimensional [8] and $\dim \mu_a = \delta_a$, where $\dim \mu_a = \inf\{\dim E : \mu_a(E) = 1\}$ and $\dim E$ denotes the Hausdorff dimension (HD) of the Borel set E (see [1]). If there are infinitely many $c \in \mathbb{R}$ such that $\dim E_c > 0$, where

$$E_c = \left\{x : \liminf_{n\to\infty} \frac{\log \mu_a(E_{n,k}(x))}{-n \log 2} = c\right\},$$

then we say that μ_a is *multifractal* [8], [10]. We have seen [2], [3] that some special cases of Markov measures are multifractal. In Section 2 we shall give a necessary and sufficient condition for μ_a to be multifractal under the condition $\sup_n |a_n| < 1$. In Section 3 we give an application which permits us to give a lower bound for the HD of a set $M_\beta(b)$, where

$$M_\beta(b) = \left\{x : \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \beta_n \varepsilon_n(x) \le b\right\}, \tag{1}$$

$b, \beta_n \in \mathbb{R}$, $\beta = (\beta_n)_{n\ge1}$, $|\beta_n| \le M$, $M > 0$. In some special cases our method gives equality.

2. A multifractal analysis. We need the following lemma, which can be deduced from [4]:

LEMMA 1. *Let* $\gamma = (\gamma_n)_{n\ge1}$ *and* μ_γ *be the corresponding coin tossing measure. If* $\sup_n |a_n| < 1$, *then*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log(1 + a_n r_n(x))$$
$$= \limsup_{N\to\infty} \frac{1}{2N} \sum_{n=1}^{N} \log\left[(1+a_n)^{1+\gamma_n}(1-a_n)^{1-\gamma_n}\right] \quad \mu_\gamma\text{-a.e.}$$

THEOREM 1. *If the sequence* $a = (a_n)_{n\ge1}$ *is such that* $\sup_n |a_n| < 1$, *then* μ_a *is multifractal if and only if*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |a_n| > 0.$$

Proof. (i) Suppose that $\limsup_{N\to\infty}(1/N)\sum_{n=1}^N |a_n| = 0$. Then by the Cauchy–Schwarz inequality we have equivalently $\lim_{N\to\infty}(1/N)\sum_{n=1}^N |a_n|^2 = 0$. Since $\mu_a(E_{N,k}(x)) = \nu_{a,N}(E_{N,k}(x)) = 2^{-N}\prod_{n=1}^N (1 + a_n r_n(x))$ and

$$E_c = \left\{x : 1 - \limsup_{N\to\infty} \frac{1}{N\log 4} \sum_{n=1}^{N} \left[\log(1-a_n^2) + r_n(x)\log\left(\frac{1+a_n}{1-a_n}\right)\right] = c\right\},$$

using the uniform convergence of the Taylor series for the function $\log(1+x)$, $|x| \le \sup_n |a_n|$, we see that $E_c = \emptyset$ if $c \ne 1$ and $E_c = [0,1]$ if $c = 1$ and so μ_a is not multifractal.

(ii) Suppose that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}|a_n| > 0.$$

It is clear that c must be such that

$$1 - \limsup_{N\to\infty} \lambda_N \le c \le 1 - \limsup_{N\to\infty} \kappa_N, \tag{2}$$

where

$$\lambda_N = \frac{1}{N\log 2}\sum_{n=1}^{N}\log(1+|a_n|) \quad \text{and} \quad \kappa_N = \frac{1}{N\log 2}\sum_{n=1}^{N}\log(1-|a_n|),$$

otherwise the set E_c is empty. We define the function

$$f(y) = \limsup_{N\to\infty}[\kappa_N + y(\lambda_N - \kappa_N)], \qquad y \in [0,1].$$

If $0 \le y_0 < y \le 1$, then using the properties of lim sup we obtain

$$0 \le f(y) - f(y_0) \le (y - y_0)\limsup_{N\to\infty}(\lambda_N - \kappa_N).$$

This implies that $f(y)$ is continuous on $[0,1]$. Since $f(0) = \limsup_{N\to\infty}\kappa_N$ and $f(1) = \limsup_{N\to\infty}\lambda_N$, from the intermediate value theorem there is $\gamma_0 \in (-1,1)$ such that $f((1+\gamma_0)/2) = 1 - c \in (f(0), f(1))$, $f(0) \le 0 < f(1)$. We consider the measure μ_γ, where $\gamma = (\gamma_n)_{n\ge1}$ with

$$\gamma_n = \gamma_0 \operatorname{sgn}\log\left(\frac{1+a_n}{1-a_n}\right)$$

(sgn is the sign function, $\operatorname{sgn} 0 = 0$). From Lemma 1 we have

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N\log 4}\sum_{n=1}^{N}&\left[\log(1-a_n^2) + r_n(x)\log\left(\frac{1+a_n}{1-a_n}\right)\right] \\
&= \limsup_{N\to\infty}\frac{1}{N\log 4}\sum_{n=1}^{N}\left[\log\,(1-a_n^2) + \gamma_0\log\left(\frac{1+|a_n|}{1-|a_n|}\right)\right] \\
&= \limsup_{N\to\infty}\left[\kappa_N + \frac{1+\gamma_0}{2}(\lambda_N - \kappa_N)\right] = 1 - c \qquad \mu_\gamma\text{-a.e.}
\end{aligned}$$

From this we get $\mu_\gamma(E_c) = 1$ and so $\dim E_c \ge \dim \mu_\gamma = \delta_\gamma > 0$, for infinitely many $c \in \mathbb{R}$ ($f(1) > 0$). This means that μ_a is multifractal.

3. Application. We consider the set of (1). It is clear that we can find a sequence $a = (a_n)_{n\geq 1}$ such that

$$\beta_n = \log\left(\frac{1+a_n}{1-a_n}\right),$$

with $\sup_n |a_n| < 1$. We also have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}|\beta_n| = 0 \Leftrightarrow \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}|a_n| = 0.$$

If $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}|\beta_n| = 0$, then $\dim M_\beta(b)$ is 0 if $b < 0$ and is 1 if $b \geq 0$.

Suppose that $\limsup_{N\to\infty}(1/N)\sum_{n=1}^{N}|a_n| > 0$. From (1) we see that b must be such that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1\\ \beta_n<0}}^{N} \beta_n \leq b \leq \liminf_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1\\ \beta_n>0}}^{N} \beta_n,$$

or equivalently,

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\log\left(\frac{1+a_n}{1+|a_n|}\right) \leq b \leq \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\log\left(\frac{1+a_n}{1-|a_n|}\right), \tag{3}$$

Otherwise $\dim M_\beta(b) = 0$ or 1.

Let $b > 0$ and

$$c = 1 + \frac{b}{\log 2} - \limsup_{N\to\infty} \frac{1}{N\log 2}\sum_{n=1}^{N}\log(1+a_n).$$

Using elementary properties of lim sup, lim inf and (3) we easily see that c satisfies (2). From the proof of Theorem 1 we have

$$\dim E_c \geq \dim \mu_\gamma,$$

where

$$E_c = \left\{x : 1 - \limsup_{N\to\infty} \frac{1}{N\log 4}\sum_{n=1}^{N}[\log(1-a_n^2) + (1-2\varepsilon_n(x))\beta_n] = c\right\}$$

and $\gamma = (\gamma_n)_{n\geq 1}$, $\gamma_n = \gamma_0 \operatorname{sgn}\beta_n$ with

$$\limsup_{N\to\infty}\left[\kappa_N + \frac{1+\gamma_0}{2}(\lambda_N - \kappa_N)\right] = 1 - c.$$

If $x \in E_c$ then

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\beta_n\varepsilon_n(x) \leq b$$

and so $E_c \subset M_\beta(b)$, which means that $\dim M_\beta(b) \geq \dim \mu_\gamma$.

Remark. If $\beta_n > \beta_0 \neq 0$, $n = 1, 2, \ldots$, $b > 0$, using the above method we get

$$\beta_0 = \log\left(\frac{1+a_0}{1-a_0}\right),$$
$$c = 1 + \frac{b}{\log 2} - \frac{1}{\log 2}\log(1+a_0),$$
$$\gamma_n = \gamma_0 \operatorname{sgn}\beta_n$$

and

$$1 - c = \frac{\log(1-a_0)}{\log 2} + \frac{\beta_0}{\log 4}(1+\gamma_0).$$

This gives

$$\frac{b}{\beta_0} = \frac{1-\gamma_0}{2}.$$

Since

$$M_\beta(b) = \left\{x : \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \varepsilon_n(x) \le \frac{b}{\beta_0}\right\}$$

and

$$\dim \mu_\gamma = 1 - \frac{1}{\log 2}\left[\frac{b}{\beta_0}\log\left(\frac{2b}{\beta_0}\right) + \left(1 - \frac{b}{\beta_0}\right)\log\left[2\left(1 - \frac{b}{\beta_0}\right)\right]\right],$$

using Eggleston's Theorem [7] we get $\dim M_\beta(b) = \dim \mu_\gamma$ for $b \le \beta_0/2$.

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