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ON WEIGHTED INEQUALITIES FOR OPERATORS OF POTENTIAL TYPE

BY

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In this paper, we discuss a class of weighted inequalities for operators of potential type on homogeneous spaces. We give sufficient conditions for the weak and strong type weighted inequalities

$$\sup_{\lambda>0} \lambda |\{x \in X : |T(f\,d\sigma)(x)| > \lambda\}|_{\omega}^{1/q} \le C\Big(\int\limits_X |f|^p\,d\sigma\Big)^{1/p}$$

and

$$\Big(\int\limits_X |T(f\,d\sigma)|^q\,d\omega\Big)^{1/q} \le C\Big(\int\limits_X |f|^p\,d\sigma\Big)^{1/p}$$

in the cases of $0 < q < p \le \infty$ and $1 \le q < p < \infty$, respectively, where T is an operator of potential type, and ω and σ are Borel measures on the homogeneous space X. We show that under certain restrictions on the measures those sufficient conditions are also necessary. A consequence is given for the fractional integrals in Euclidean spaces.

1. Introduction. Weighted norm inequalities for fractional integrals or Riesz potentials have been studied by many authors. Among them, E. T. Sawyer and R. L. Wheeden [8] recently considered a general family of potential-like operators on homogeneous spaces, and characterized two-weight norm inequalities for these operators in the case $1 < p \le q < \infty$ (cf. also [2] and [9]). In this paper, we shall consider the case of $q < p$.

A *homogeneous space* (X, d, μ) is a set X together with a quasi-metric d and a doubling measure μ. We recall that a *quasi-metric* is a mapping $d : X \times X \to [0, \infty)$ which satisfies the same conditions as a metric, except that the triangle inequality is weakened to

$$d(x,y) \le \kappa(d(x,z) + d(z,y)) \quad \text{for all } x, y, z \in X, \tag{1.1}$$

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where $\kappa \geq 1$ is a constant which is independent of x, y, and z. Without loss of generality (see [3]), we may assume that all balls $B(x,r) = \{y \in X : d(x,y) < r\}$ are open. Also, we assume that all annuli $B(x,R) \setminus B(x,r)$ in X are nonempty for $R > r > 0$. As usual, for a ball $B = B(x,r)$ and $c > 0$, we denote by cB the ball $B(x,cr)$. For convenience, we shall call a Borel set $Q \subset X$ a *cube centered at* $x \in X$ if there exists $r > 0$ so that $B(x,r) \subset Q \subset B(x,\vartheta r)$, where $\vartheta \geq 1$ is a fixed constant. We also recall that a *doubling measure* μ on X is a nonnegative measure on the Borel subsets of X so that $|2B|_\mu \leq C_\mu |B|_\mu$ for all balls $B \subset X$, where $|B|_\mu$ denotes the μ-measure of the ball B. For simplicity, we shall assume that all measures considered in this paper are locally finite and vanish at individual points.

Let σ and ω be Borel measures on a homogeneous space X. The operators T studied in this paper have the form

$$T(f\,d\sigma)(x) = \int_X K(x,y) f(y)\, d\sigma(y), \qquad x \in X, \tag{1.2}$$

where the kernel $K(x,y)$ is nonnegative, lower semicontinuous and satisfies the following condition: There are constants $C_1 > 1$ and $C_2 > 1$ such that

$$\begin{aligned} &K(x,y) \leq C_1 K(x',y) \quad \text{whenever} \quad d(x',y) \leq C_2 d(x,y);\\ &K(x,y) \leq C_1 K(x,y') \quad \text{whenever} \quad d(x,y') \leq C_2 d(x,y). \end{aligned} \tag{1.3}$$

We shall denote the adjoint of T by T^*, which is given by

$$T^*(g\,d\omega)(y) = \int_X K(x,y) g(x)\, d\omega(x), \qquad y \in X. \tag{1.4}$$

For a ball B in X, we set

$$\varphi(B) = \sup\{K(x,y) : x, y \in B, \text{ and } d(x,y) \geq \alpha^{-1} r(B)\}, \tag{1.5}$$

for some fixed constant $\alpha \geq 2\kappa$, where κ is the constant in (1.1).

For $0 < p < \infty$, let $L^{p,\infty}(d\omega)$ be the weak-L^p space with respect to the measure ω with the quasi-norm

$$\|f\|_{L^{p,\infty}(d\omega)} = \sup_{\lambda > 0} \lambda |\{x \in X : |f(x)| > \lambda\}|_\omega^{1/p}, \tag{1.6}$$

and, for $1 \leq p \leq \infty$, let $L^p(d\omega)$ be the L^p space with respect to the measure ω with the norm

$$\|f\|_{L^p(d\omega)} = \Big(\int_X |f(x)|^p\, d\omega(x)\Big)^{1/p}, \tag{1.7}$$

where the obvious change is needed when $p = \infty$.

In the case of $1 < p \leq q < \infty$ and $X = \mathbb{R}^n$, the solution to the two-weight problem is due to E. T. Sawyer ([6] and [7], see also [8] and [9] for the counterpart for homogeneous spaces). Sawyer's condition for the weak-type inequality

$$\|T(f\,d\sigma)\|_{L^{q,\infty}(d\omega)} \le C\|f\|_{L^p(d\sigma)} \tag{1.8}$$

is that

$$\Big(\int_Q T^*(\chi_Q\,d\omega)(x)^{p'}\,d\sigma(y)\Big)^{1/p'} \le C|Q|_\omega^{1/q'} \tag{1.9}$$

holds for all cubes Q in $\mathbb{R}^n$, where $p' = p/(p-1)$; while for the strong-type inequality

$$\|T(f\,d\sigma)\|_{L^q(d\omega)} \le C\|f\|_{L^p(d\sigma)} \tag{1.10}$$

is that both

$$\Big(\int_{\mathbb{R}^n} T(\chi_Q\,d\sigma)(x)^q\,d\omega(x)\Big)^{1/q} \le C|Q|_\sigma^{1/p} \tag{1.11}$$

and

$$\Big(\int_{\mathbb{R}^n} T^*(\chi_Q\,d\omega)(x)^{p'}\,d\sigma(x)\Big)^{1/p'} \le C|Q|_\omega^{1/q'} \tag{1.12}$$

hold for all cubes Q in $\mathbb{R}^n$. However, the characterization for the case $q < p$ remains open. In this paper, we give some sufficient conditions in order to have the weak and strong type inequalities (1.8) and (1.10) for this case, and we shall show that, under certain restrictions on measures, those conditions are also necessary. Our conditions are suggested by a recent work [10] of I. E. Verbitsky on the fractional maximal function.

In order to state our results, we need the following dyadic cube decomposition of a homogeneous space X. It has been shown in [8] that for some $r > 1$ (in fact, $r = 8\kappa^5$ will do), and any (large negative) integer m, there are points $\{x_j^k\} \subset X$ and a family $\mathcal{D}_m = \{E_j^k\}$ of cubes in X centered at x_j^k for $k = m, m+1, \ldots$ and $j = 1, 2, \ldots$ such that

(i) $B(x_j^k, r^k) \subset E_j^k \subset B(x_j^k, r^{k+1})$,

(ii) for each $k = m, m+1, \ldots$, $X = \bigcup_j E_j^k$ and $\{E_j^k\}$ is pairwise disjoint in j, and

(iii) if $k < l$ then either $E_j^k \cap E_i^l = \emptyset$ or $E_j^k \subset E_i^l$.

We set $\mathcal{D} = \bigcup_{m\in\mathbb{Z}} \mathcal{D}_m$ and call the cubes in $\mathcal{D}$ *dyadic cubes*. If $Q = E_j^k \in \mathcal{D}_m$, for some $m \in \mathbb{Z}$, we define the *side-length* of Q to be $l(Q) = 2r^k$, and denote by Q^* the containing ball $B(x_j^k, r^{k+1})$ of Q.

Let $1 \le q < p < \infty$. For a given operator of potential type T, we consider the following auxiliary functions:

$$\Psi_p^*(x) = \sup_{Q\in\mathcal{D}:\, x\in Q} \left(\frac{1}{|Q|_\omega} \int_{\eta Q^*} T^*(\chi_Q\,d\omega)^{p'}\,d\sigma\right)^{1/p'}, \tag{1.13}$$

and

$$\Psi_q(x) = \sup_{Q\in\mathcal{D}:\, x\in Q} \left(\frac{1}{|Q|_\sigma} \int\limits_{\eta Q^*} T(\chi_Q\, d\sigma)^q\, d\omega\right)^{1/q}, \tag{1.14}$$

where η is a sufficiently large fixed constant; we shall assume that $\eta \geq 2\kappa$. We also set

$$\Phi_p(x) = \sup_{Q\in\mathcal{D}:\, x\in Q} \{\varphi(Q)|Q|_\omega^{1/p}|Q|_\sigma^{1/p'}\}, \tag{1.15}$$

where by $\varphi(Q)$ we mean $\varphi(Q^*)$.

THEOREM 1.1. *Let* $0 < q < p \leq \infty$, $p > 1$. *Then in order that the weak-type inequality* (1.8) *holds for all* $f \in L^p(d\sigma)$, *it is sufficient that*

$$\Psi_p^* \in L^{pq/(p-q),\infty}(d\omega). \tag{1.16}$$

Conversely, (1.8) *implies*

$$\Phi_p \in L^{pq/(p-q),\infty}(d\omega). \tag{1.17}$$

THEOREM 1.2. *Let* $1 \leq q < p < \infty$. *Then the strong-type inequality* (1.10) *holds for all* $f \in L^p(d\sigma)$ *if both the following conditions are satisfied*:

$$\Psi_p^* \in L^{pq/(p-q)}(d\omega) \quad \text{and} \quad \Psi_q \in L^{pq/(p-q)}(d\sigma). \tag{1.18}$$

Conversely, (1.10) *implies*

$$\Phi_p \in L^{pq/(p-q)}(d\omega) \quad \text{and} \quad \Phi_q \in L^{pq/(p-q)}(d\sigma). \tag{1.19}$$

We do not know, in general, whether conditions (1.16) and (1.18) are necessary for the weak-type inequality (1.8) and the strong-type inequality (1.10), respectively. However, the next theorem shows that this is the case if both ω and σ are doubling, and ω subjects to the following A_∞-like condition for some range of the exponent ε: There exists a constant $C_\varepsilon > 0$ such that

$$\left(\frac{|B'|_\omega}{|B|_\omega}\right)^\varepsilon \leq C_\varepsilon \frac{\varphi(B)}{\varphi(B')} \tag{1.20}$$

for all pairs of balls $B' \subset B$ in X.

THEOREM 1.3. *Let ω and σ be doubling measures on X. Then condition* (1.17) *is necessary and sufficient in order that the weak-type inequality* (1.8) *holds for* $0 < q < p \leq \infty$ *provided that ω satisfies condition* (1.20) *with the exponent* $0 < \varepsilon < 1$; *and the first condition of* (1.19) *is necessary and sufficient in order that the strong-type inequality* (1.10) *holds for* $1 \leq q < p < \infty$ *provided that ω satisfies condition* (1.20) *with the exponent* $0 < \varepsilon < 1/q$.

We remark that, for the strong-type inequality, if both ω and σ are doubling then for (1.10) the second condition in (1.19) is necessary and sufficient if σ satisfies condition (1.20) with the exponent $0 < \varepsilon < 1/p$, or either one of the conditions in (1.19) is necessary and sufficient if both ω and σ satisfy condition (1.20) with the exponent $0 < \varepsilon < 1$.

A typical example of operators of potential type is the fractional integral, which is defined, in the Euclidean space $\mathbb{R}^n$, by

$$T_\gamma(f\,d\sigma)(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\,d\sigma(y), \tag{1.21}$$

where $0 < \gamma < n$. We note that, for T_γ, we have $\varphi(B) \approx |B|^{\gamma/n-1}$ for all balls B in $\mathbb{R}^n$, where $|B|$ denotes the Lebesgue measure of the ball $B \subset \mathbb{R}^n$.

For the operator T_γ and A_∞ weights ω and σ, we have the following corollary of Theorem 1.2, which was obtained earlier in [10] with a different approach. We recall first that a measure ω belongs to the A_∞ *class of Muckenhoupt* if there are constants $C \geq 1$ and $0 < \delta \leq 1$ such that

$$\frac{1}{C}\left(\frac{|E|}{|Q|}\right)^{1/\delta} \leq \frac{|E|_\omega}{|Q|_\omega} \leq C\left(\frac{|E|}{|Q|}\right)^{\delta} \tag{1.22}$$

for all cubes Q and all measurable subsets E of Q (see [1]).

COROLLARY 1.4. *Let $X = \mathbb{R}^n$, $T = T_\gamma$, and $1 \leq q < p < \infty$. Suppose that both ω and $\sigma \in A_\infty$. Then the strong-type inequality* (1.10) *holds if and only if either $\Phi_p \in L^{pq/(p-q)}(d\omega)$ or $\Phi_q \in L^{pq/(p-q)}(d\sigma)$.*

2. Proof of Theorem 1.1. We shall assume that $0 < q < p < \infty$. Only a few obvious modifications of the following proof are needed when $p = \infty$. We start with the proof of the sufficiency part of the theorem. Let $f \in L^p(d\sigma)$ be given. For $\lambda > 0$, we define $\Omega_\lambda = \{x \in X : T(f\,d\sigma)(x) > \lambda\}$, which is an open set by the lower semicontinuity of the kernel $K(x,y)$. To finish the proof, it is enough to show that

$$\sup_{\lambda>0} \lambda^q |\Omega_\lambda \cap D|_\omega \leq C\|f\|^q_{L^p(d\sigma)} \tag{2.1}$$

for all dyadic cubes $D \in \mathcal{D}$, with the constant C independent of D and f.

We set $\Omega^\Psi_\lambda = \{x \in \Omega_\lambda : \|f\|_{L^p(d\sigma)}\Psi^*_p(x)^{p/(p-q)} \leq \beta\lambda\}$, where $\beta > 0$ is a constant which will be chosen at the end of the proof. Then, for an arbitrarily fixed dyadic cube D such that $\Omega_\lambda \cap D \neq \emptyset$,

$$\Omega_\lambda \cap D \subset (\Omega^\Psi_\lambda \cap D) \cup \{x \in X : \|f\|_{L^p(d\sigma)}\Psi^*_p(x)^{p/(p-q)} > \beta\lambda\},$$

and hence

$$\begin{aligned} \lambda^q|\Omega_\lambda \cap D|_\omega &\leq \lambda^q|\Omega^\Psi_\lambda \cap D|_\omega \\ &\quad + \lambda^q|\{x \in X : \|f\|_{L^p(d\sigma)}\Psi^*_p(x)^{p/(p-q)} > \beta\lambda\}|_\omega. \end{aligned} \tag{2.2}$$

It follows immediately from condition (1.16), which is equivalent to

$(\Psi_p^*)^{p/(p-q)} \in L^{q,\infty}(d\omega)$, that

$$\lambda^q |\{x \in X : \|f\|_{L^p(d\sigma)} \Psi_p^*(x)^{p/(p-q)} > \beta\lambda\}|_\omega \le C(\|f\|_{L^p(d\sigma)}/\beta)^q. \tag{2.3}$$

We now estimate the first term in (2.2). For $m \in \mathbb{Z}$, we denote by $\mathcal{D}_{\lambda,m}$ the dyadic cubes $Q \in \mathcal{D}_m$ with the property that $RQ^* \subset \Omega_\lambda$, where $R = 3\kappa^2$, and let $\Omega_{\lambda,m} = \bigcup_{Q \in \mathcal{D}_{\lambda,m}} Q$. It is obvious that $\lim_{m\to-\infty} \Omega_{\lambda,m} = \Omega_\lambda$.

Let $A > 1$ be a constant which will be chosen shortly. It is easy to observe that $\Omega_\lambda \subset \Omega_{\lambda/A}$ and $\Omega_{\lambda,m} \subset \Omega_{\lambda/A,m}$ for all $m \in \mathbb{Z}$. It is shown in [8] that the sequence $\{Q_j\}$ of maximal dyadic cubes in $\mathcal{D}_{\lambda/A,m}$ has the following properties:

$$\begin{aligned} &\text{(i) } \Omega_{\lambda/A,m} = \textstyle\bigcup_j Q_j \text{ and } Q_i \cap Q_j = \emptyset \text{ for } i \ne j,\\ &\text{(ii) } RQ_j \subset \Omega_{\lambda/A}, \text{ and } 2\kappa R r Q_j^* \cap \Omega_{\lambda/A}^{\mathrm{c}} \ne \emptyset \text{ for all } j, \text{ and}\\ &\text{(iii) } \textstyle\sum_j \chi_{2\kappa Q_j^*} \le C\chi_{\Omega_{\lambda/A}}. \end{aligned} \tag{2.4}$$

Let j be temporarily fixed such that $Q_j \cap \Omega_\lambda^\Psi \ne \emptyset$. It is well known that the operator T satisfies the following maximum principle (see [8]): There is a positive constant C, independent of D, f, λ, m, j and A, such that

$$T(\chi_{(2\kappa Q_j^*)^{\mathrm{c}}} f\, d\sigma)(x) \le C(\lambda/A) \quad \text{for all } x \in Q_j. \tag{2.5}$$

With C as in (2.5), we now choose $A = 2C$, and then it follows that

$$\int\limits_{2\kappa Q_j^*} K(x,y) f(y)\, d\sigma(y) = T(f\, d\sigma)(x) - T(\chi_{(2\kappa Q_j^*)^{\mathrm{c}}} f\, d\sigma)(x) > \lambda/2$$

for all $x \in Q_j \cap \Omega_{\lambda,m}$. Let $x_j \in Q_j \cap \Omega_\lambda^\Psi$. Then, by using Hölder's inequality, we have

$$\begin{aligned} \frac{\lambda}{2}|Q_j \cap \Omega_{\lambda,m}|_\omega &< \int\limits_{Q_j} \int\limits_{2\kappa Q_j^*} K(x,y) f(y)\, d\sigma(y)\, d\omega(x)\\ &= \int\limits_{2\kappa Q_j^*} T^*(\chi_{Q_j}\, d\omega) f\, d\sigma\\ &\le \Big(\int\limits_{\eta Q_j^*} T^*(\chi_{Q_j}\, d\omega)^{p'} d\sigma\Big)^{1/p'} \Big(\int\limits_{2\kappa Q_j^*} f^p\, d\sigma\Big)^{1/p}\\ &\le |Q_j|_\omega^{1/p'} \Psi_p^*(x_j) \Big(\int\limits_{2\kappa Q_j^*} f^p\, d\sigma\Big)^{1/p}\\ &\le |Q_j|_\omega^{1/p'} \Big(\frac{\beta\lambda}{\|f\|_{L^p(d\sigma)}}\Big)^{1-q/p} \Big(\int\limits_{2\kappa Q_j^*} f(y)^p\, d\sigma(y)\Big)^{1/p}. \end{aligned}$$

Summing over the family of all maximal cubes Q_j in $\mathcal{D}_{\lambda/A,m}$ which are contained in D, and then using Hölder's inequality again, we obtain

$$\begin{aligned}
&\lambda|\Omega_\lambda^\Psi \cap \Omega_{\lambda,m} \cap D|_\omega \\
&\quad \le 2\bigg(\frac{\beta\lambda}{\|f\|_{L^p(d\sigma)}}\bigg)^{1-q/p} \sum_{j:\, Q_j\subset D} |Q_j|_\omega^{1/p'} \Big(\int\limits_{2\kappa Q_j^*} f^p\, d\sigma\Big)^{1/p} \\
&\quad \le 2\bigg(\frac{\beta\lambda}{\|f\|_{L^p(d\sigma)}}\bigg)^{1-q/p} \Big(\sum_{j:\, Q_j\subset D} |Q_j|_\omega\Big)^{1/p'} \Big(\sum_j \int\limits_{2\kappa Q_j^*} f^p\, d\sigma\Big)^{1/p} \\
&\quad \le C(\beta\lambda)^{1-q/p}(|\Omega_{\lambda/A} \cap D|_\omega)^{1/p'} \|f\|_{L^p(d\sigma)}^{q/p},
\end{aligned}$$

where we have used (2.4)(iii).

Since the constant C in the last inequality is independent of m, by letting $m \to -\infty$ on the left-hand side, we obtain

$$\lambda^q|\Omega_\lambda^\Psi \cap D|_\omega \le C\beta^{1-q/p}(\lambda^q|\Omega_{\lambda/A} \cap D|_\omega)^{1/p'} \|f\|_{L^p(d\sigma)}^{q/p}. \tag{2.6}$$

Combining the estimates (2.3) and (2.6) in (2.2), and then taking the supremum in λ for $0 < \lambda < N$, we obtain

$$\begin{aligned}
&\sup_{0<\lambda<N} \lambda^q|\Omega_\lambda \cap D|_\omega \\
&\quad \le C\bigg(\beta^{1-q/p}(\sup_{0<\lambda<N} \lambda^q|\Omega_\lambda \cap D|_\omega)^{1/p'} \|f\|_{L^p(d\sigma)}^{q/p} + \frac{\|f\|_{L^p(d\sigma)}^q}{\beta^q}\bigg).
\end{aligned}$$

Since $0 < \sup_{0<\lambda<N} \lambda^p|\Omega_\lambda \cap D|_\omega \le N^p|D|_\omega < \infty$, we are able to choose

$$\beta = \bigg(\frac{\|f\|_{L^p(d\sigma)}^q}{\sup_{0<\lambda<N} \lambda^q|\Omega_\lambda \cap D|_\omega}\bigg)^{1/(q+p')}.$$

With this value of β, the last inequality becomes

$$\sup_{0<\lambda<N} \lambda^q|\Omega_\lambda \cap D|_\omega \le C(\sup_{0<\lambda<N} \lambda^q|\Omega_\lambda \cap D|_\omega)^{q/(q+p')} \|f\|_{L^p(d\sigma)}^{qp'/(q+p')}.$$

Therefore, (2.1) follows from division and then letting $N \to \infty$.

The necessity part of the theorem is an easy consequence of the following result [10, Theorem 1.1], which can be proved for homogeneous spaces in a similar way as in [10].

LEMMA 2.1. *Let $0 < q < p < \infty$, and ω be a Borel measure on X. Let $\mathcal{L}^p(d\omega)$ be either the space $L^{p,\infty}(d\omega)$ or the space $L^p(d\omega)$. Suppose that $\varrho : \mathcal{D} \to [0,\infty)$ is a nonnegative set function. Then the weighted inequality*

$$\|\sup_{Q\in\mathcal{D}}\{\lambda_Q\varrho(Q)\chi_Q\}\|_{\mathcal{L}^q(d\omega)} \le C\Big(\sum_{Q\in\mathcal{D}} \lambda_Q^p\Big)^{1/p} \tag{2.7}$$

holds for all $\{\lambda_Q\} \in l^p$ *with* $\lambda_Q \geq 0$ *if and only if*

$$\sup_{Q\in\mathcal{D}} \{\varrho(Q)|Q|_\omega^{1/p}\chi_Q\} \in \mathcal{L}^{pq/(p-q)}(d\omega). \tag{2.8}$$

We now consider the following test function:

$$f = \Big(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q^p}{|Q|_\sigma}\chi_Q\Big)^{1/p} \qquad \text{with } \lambda_Q \geq 0 \text{ and } \lambda_Q = 0 \text{ if } |Q|_\sigma = 0. \tag{2.9}$$

As shown in [8], there exists a constant C depending only on κ in (1.1) and C_1, C_2 in (1.3) so that if B is a ball in X then

$$\varphi(B) \leq CK(x,y) \quad \text{for all } x, y \in B. \tag{2.10}$$

We then have the estimate

$$T(f\,d\sigma)(x) \geq \frac{\lambda_Q}{|Q|_\sigma^{1/p}} \int_Q K(x,y)\,d\sigma(y) \geq C^{-1}\lambda_Q\varphi(Q)|Q|_\sigma^{1/p'}\chi_Q(x)$$

for all $Q \in \mathcal{D}$. Therefore, (1.8) implies that

$$\|\sup_{Q\in\mathcal{D}}\{\lambda_Q\varphi(Q)|Q|_\sigma^{1/p'}\chi_Q\}\|_{L^{q,\infty}(d\omega)} \leq C\Big(\sum_{Q\in\mathcal{D}}\lambda_Q^p\Big)^{1/p},$$

and hence $\|\Phi_p\|_{L^{pq/(p-q),\infty}(d\omega)} \leq C$ according to Lemma 2.1 (with $\varrho(Q) = \varphi(Q)|Q|_\sigma^{1/p'}$). This concludes the proof of Theorem 1.1.

3. Proof of Theorem 1.2. The necessity part of the theorem follows from Lemma 2.1 in the same way as in the last section. The proof of the sufficiency part is a modification of the proof in [7] (see also [8] and [9]) for the $1 < p \leq q < \infty$ case. For the reader's convenience, we shall include most of details.

Without loss of generality, we suppose that f is nonnegative and bounded with compact support. For each $k \in \mathbb{Z}$, we set $\Omega_k = \{x \in X : T(f\,d\sigma)(x) > 2^k\}$. For each $m \in \mathbb{Z}$, let $\mathcal{D}_{k,m}$ denote the dyadic cubes $Q \in \mathcal{D}_m$ with the property that $RQ^* \subset \Omega_k$ for a fixed constant R which will be chosen later. Let $\{Q_j^k\}_j$ be the maximal (with respect to inclusion) cubes in $\mathcal{D}_{k,m}$. It is not difficult to check that the following Whitney-type properties are valid (cf. [8]): For any fixed constant $\eta \geq 2\kappa$ (the value will be determined during the proof), there exists R (equal to a large multiple of $\kappa\eta$) such that

(3.1)
- (i) $\Omega_{k,m} = \bigcup_j Q_j^k$ and $Q_j^k \cap Q_i^k = \emptyset$ for $i \neq j$,
- (ii) $RQ_j^{k^*} \subset \Omega_k$ and $2\kappa RrQ_j^{k^*} \cap \Omega_k^c \neq \emptyset$ for all k and j,
- (iii) $\sum_j \chi_{\eta Q_j^{k^*}} \leq C\chi_{\Omega_k}$ for all k,
- (iv) the number of Q_s^k intersecting a fixed $\eta Q_j^{k^*}$ is at most C,
- (v) $Q_j^k \subsetneq Q_i^l$ implies $k > l$.

Let $n > 2$ be an integer so that $2^{n-2} > C$, where the constant C is in the maximum principle (2.5). We then have

$$\int_X T(f\,d\sigma)^q\,d\omega \le \lim_{m\to-\infty} \sum_{k,j} (2^{k+n})^q |E_j^k|_\omega, \tag{3.2}$$

where $E_j^k = Q_j^k \cap (\Omega_{k+n-1,m} \setminus \Omega_{k+n})$. For $x \in E_j^k \subset \Omega_{k+n-1,m}$, by applying the maximum principle (2.5) to each dyadic cube Q_j^k (with $\lambda = 2^k$ and $A = 1$ there), we obtain

$$\begin{aligned} T(\chi_{2\kappa Q_j^{k*}} f\,d\sigma)(x) &= T(f\,d\sigma)(x) - T(\chi_{(2\kappa Q_j^{k*})^c} f\,d\sigma)(x) \\ &> 2^{k+n-1} - C2^k > 2^{k+n-1} - 2^{k+n-2} = 2^{k+n-2} \ge 2^k, \end{aligned}$$

and so

$$\begin{aligned} |E_j^k|_\omega &\le 2^{-k} \int_{E_j^k} T(\chi_{2\kappa Q_j^{k*}} f\,d\sigma)\,d\omega = 2^{-k} \int_{2\kappa Q_j^{k*}} fT^*(\chi_{E_j^k} d\omega)\,d\sigma \\ &= 2^{-k} \int_{2\kappa Q_j^{k*}\setminus\Omega_{k+n}} fT^*(\chi_{E_j^k} d\omega)\,d\sigma + 2^{-k} \int_{2\kappa Q_j^{k*}\cap\Omega_{k+n}} fT^*(\chi_{E_j^k}\,d\omega)\,d\sigma \\ &= 2^{-k}(\theta_j^k + \tau_j^k). \end{aligned}$$

We next define the sets E, F and G of indices (j,k) as in [7], that is,

$$\begin{aligned} E &= \{(k,j) : |E_j^k|_\omega \le \beta|\eta Q_j^{k*}|_\omega\}, \\ F &= \{(k,j) : |E_j^k|_\omega > \beta|\eta Q_j^{k*}|_\omega \text{ and } \theta_j^k > \tau_j^k\}, \\ G &= \{(k,j) : |E_j^k|_\omega > \beta|\eta Q_j^{k*}|_\omega \text{ and } \theta_j^k \le \tau_j^k\}, \end{aligned}$$

where β $(0 < \beta < 1)$ is a constant to be chosen at the end of the proof. Then the sum on the right-hand side of (3.2) can be split into three parts, the sums over the sets E, F and G, respectively. We have

$$\begin{aligned} \sum_{(k,j)\in E} (2^{k+n})^q |E_j^k|_\omega &\le C\beta \sum_{k,j} 2^{kq} |\eta Q_j^{k*}|_\omega \\ &\le C\beta \sum_k 2^{kq} |\Omega_k|_\omega \quad \text{by (3.1)(iii)} \\ &\le C\beta \int_X \Big(\sum_k 2^{kq}\chi_{\Omega_k}\Big)\,d\omega \le C\beta \int_X T(f\,d\sigma)^q\,d\omega. \end{aligned}$$

For notational convenience, we set

$$A_j^k = \frac{1}{|Q_j^k|_\sigma} \int_{Q_j^k} f\,d\sigma,$$

$$\Theta_j^k = \left(\frac{1}{|Q_j^k|_\omega} \int\limits_{\eta Q_j^{k*}} T^*(\chi_{Q_j^k}\, d\omega)^{p'} d\sigma \right)^{1/p'},$$

$$\Psi_j^k = \left(\frac{1}{|Q_j^k|_\sigma} \int\limits_{\eta Q_j^{k*}} T(\chi_{Q_j^k}\, d\sigma)^q d\omega \right)^{1/q}.$$

By Hölder's inequality,

$$\begin{aligned}
&\sum_{(k,j)\in F} (2^{k+n})^q |E_j^k|_\omega \le C \sum_{k,j} |E_j^k|_\omega \left(\frac{2\theta_j^k}{|E_j^k|_\omega} \right)^q \\
&\quad \le C\beta^{-q} \sum_{k,j} |E_j^k|_\omega \left(\frac{1}{|\eta Q_j^{k*}|_\omega} \int\limits_{2\kappa Q_j^{k*}\setminus \Omega_{k+n}} f T(\chi_{E_j^k}\, d\omega)\, d\sigma \right)^q \\
&\quad \le C\beta^{-q} \sum_{k,j} \frac{|E_j^k|_\omega}{|\eta Q_j^{k*}|_\omega^q} \Big(\int\limits_{\eta Q_j^{k*}} T^*(\chi_{Q_j^k}\, d\omega)^{p'} d\sigma \Big)^{q/p'} \Big(\int\limits_{\eta Q_j^{k*}\setminus \Omega_{k+n}} f^p\, d\sigma \Big)^{q/p} \\
&\quad \le C\beta^{-q} \sum_{k,j} (|E_j^k|_\omega (\Theta_j^k)^{pq/(p-q)})^{1-q/p} \Big(\int\limits_{\eta Q_j^{k*}\setminus \Omega_{k+n}} f^p\, d\sigma \Big)^{q/p} \\
&\quad \le C\beta^{-q} \Big(\sum_{k,j} |E_j^k|_\omega (\Theta_j^k)^{pq/(p-q)} \Big)^{1-q/p} \Big(\sum_{k,j} \int\limits_{\eta Q_j^{k*}\setminus \Omega_{k+n}} f^p\, d\sigma \Big)^{q/p} \\
&\quad \le C\beta^{-q} \Big(\int\limits_X (\Psi_p^*)^{pq/(p-q)}\, d\omega \Big)^{1-q/p} \Big(\int\limits_X f^p\, d\sigma \Big)^{q/p} \\
&\quad \le C\beta^{-q} \Big(\int\limits_X f^p\, d\sigma \Big)^{q/p} \qquad \text{by (1.18)},
\end{aligned}$$

where we have used the following estimates:

$$\begin{aligned}
\sum_{k,j} |E_j^k|_\omega (\Theta_j^k)^{pq/(p-q)} &\le \int\limits_X \Big(\sum_{k,j} (\Theta_j^k)^{pq/(p-q)} \chi_{E_j^k} \Big)\, d\omega \\
&\le \int\limits_X \Big(\sum_{k,j} \chi_{\eta Q_j^{k*}\setminus \Omega_{k+n}} \Big) (\Psi_p^*)^{pq/(p-q)}\, d\omega \le C \int\limits_X (\Psi_p^*)^{pq/(p-q)}\, d\omega,
\end{aligned}$$

and

$$\sum_{k,j} \int\limits_{\eta Q_j^{k*}\setminus \Omega_{k+n}} f^p\, d\sigma \le \int\limits_X \Big(\sum_{k,j} \chi_{\eta Q_j^{k*}\setminus \Omega_{k+n}} \Big) f^p\, d\sigma \le C \int\limits_X f^p\, d\sigma,$$

since

$$\sum_{j,k} \chi_{\eta Q_j^{k*} \setminus \Omega_{k+n}} \leq C \sum_k \chi_{\Omega_k \setminus \Omega_{k+n}} \leq Cn.$$

We now estimate the remaining part of the sum on the right-hand side of (3.2), that is, the sum over the set G. Following [7], we define

$$H_j^k = \{i : Q_i^{k+n} \cap 2\kappa Q_j^{k^*} \neq \emptyset\}, \qquad L_j^k = \{s : Q_s^k \cap 2\kappa Q_j^{k^*} \neq \emptyset\}.$$

We observe that the growth condition (1.3) on the kernel $K(x,y)$ implies that, for $x \notin 2\kappa Q_i^{k+n^*}$,

$$\max_{y \in Q_i^{k+n}} K(x,y) \leq C \min_{y \in Q_i^{k+n}} K(x,y),$$

and hence

$$\max_{y \in Q_i^{k+n}} T^*(\chi_{E_j^k}\, d\omega)(y) \leq C \min_{y \in Q_i^{k+n}} T^*(\chi_{E_j^k}\, d\omega)(y) \quad \text{for all } i \in H_j^k,$$

since $2\kappa Q_i^{k+n^*} \subset \Omega_{k+n}$ by (3.1)(ii) and $E_j^k \cap \Omega_{k+n} = \emptyset$. It then follows that

$$\begin{aligned} \tau_j^k &= \int\limits_{2\kappa Q_j^{k^*} \cap \Omega_{k+n}} f T^*(\chi_{E_j^k} d\omega)\, d\sigma \\ &\leq C \sum_{i \in H_j^k} \big(\min_{y \in Q_i^{k+n}} T^*(\chi_{E_j^k}\, d\omega)(y)\big) \int\limits_{Q_i^{k+n}} f\, d\sigma \\ &\leq C \sum_{i \in H_j^k} \Big(\int\limits_{Q_i^{k+n}} T^*(\chi_{E_j^k}\, d\omega)\, d\sigma\Big)\Big(\frac{1}{|Q_i^{k+n}|_\sigma} \int\limits_{Q_i^{k+n}} f\, d\sigma\Big) \\ &\leq C \sum_{s \in L_j^k} \Big(\sum_{i:\, Q_i^{k+n} \subset Q_s^k} \Big(\int\limits_{Q_i^{k+n}} T^*(\chi_{E_j^k}\, d\omega)\, d\sigma\Big) A_i^{k+n}\Big). \end{aligned}$$

Let K and N be integers such that $-\infty < K < \infty$ and $0 \leq N < n$. We set

$$G_{K,N} = \{(k,j) \in G : k \geq K,\ k \equiv N \pmod n\}. \tag{3.3}$$

We now claim that

$$\sum_{(k,j) \in G_{K,N}} (2^{k+n})^q |E_j^k|_\omega \leq C\Big(\int\limits_X f^p\, d\sigma\Big)^{q/p} \tag{3.4}$$

with a constant C independent of the integers K and N.

Let K and N be temporarily fixed. We shall use the so-called "principal" cubes which are defined as follows: Denote by $I_{K,N}$ all of indices (j,k) so that $k \geq K$ and $k \equiv N \pmod n$. Let $\Gamma_{K,N}^{(0)}$ consist of those indices $(k,j) \in I_{K,N}$

for which Q_j^k is maximal. If $\Gamma_{K,N}^{(s)}$ has been defined, let $\Gamma_{K,N}^{(s+1)}$ consist of those $(k,j) \in I_{K,N}$ for which there is $(t,u) \in \Gamma_{K,N}^{(s)}$ with $Q_j^k \subset Q_u^t$ and

$$\text{(3.5)} \qquad \begin{aligned} &\text{(i) } A_j^k > 2A_u^t, \\ &\text{(ii) } A_i^l \le 2A_u^t \text{ whenever } Q_j^k \subsetneq Q_i^l \subset Q_u^t. \end{aligned}$$

Define $\Gamma_{K,N} = \bigcup_{s=0}^{\infty} \Gamma_{K,N}^{(s)}$, and for each $(k,j) \in I_{K,N}$, define $P(Q_j^k)$ to be the smallest cube Q_u^t containing Q_j^k and with $(t,u) \in \Gamma_{K,N}$. We then have

$$\text{(3.6)} \qquad \begin{aligned} &\text{(i) } P(Q_j^k) = Q_u^t \text{ implies } A_j^k \le 2A_u^t, \\ &\text{(ii) } Q_j^k \subsetneq Q_u^t \text{ and } (t,u) \in \Gamma_{K,N} \text{ imply } A_j^k > 2A_u^t. \end{aligned}$$

We note that if $Q_i^{k+n} \subset Q_s^k$ and $(k+n,i) \notin \Gamma_{K,N}$ then $P(Q_i^{k+n}) = P(Q_s^k)$, and therefore

$$\begin{aligned} &\sum_{(k,j)\in G_{K,N}} (2^{k+n})^q |E_j^k|_\omega \le C \sum_{k,j} |E_j^k|_\omega \bigg(\frac{2\tau_j^k}{|E_j^k|_\omega}\bigg)^q \le C\beta^{-q} \sum_{k,j} \frac{|E_j^k|_\omega}{|\eta Q_j^{k*}|_\omega^q} (\tau_j^k)^q \\ &\le C\beta^{-q} \sum_{k,j} \sum_{s\in L_j^k} \frac{|E_j^k|_\omega}{|\eta Q_j^{k*}|_\omega^q} \Big(\sum_{\substack{i:P(Q_i^{k+n})=P(Q_s^k) \\ Q_i^{k+n}\subset Q_s^k}} \Big(\int_{Q_i^{k+n}} T^*(\chi_{E_j^k} d\omega)\, d\sigma \Big) A_i^{k+n} \Big)^q \\ &\quad + C\beta^{-q} \sum_{k,j} \frac{|E_j^k|_\omega}{|\eta Q_j^{k*}|_\omega^q} \Big(\sum_{\substack{i\in H_j^k \\ (k+n,i)\in\Gamma_{K,N}}} \Big(\int_{Q_i^{k+n}} T^*(\chi_{E_j^k}\, d\omega)\, d\sigma \Big) A_i^{k+n} \Big)^q \\ &= I + II. \end{aligned}$$

For a fixed $(t,u) \in \Gamma_{K,N}$, we claim that there is a large constant η independent of (t,u) such that $Q_j^k \subset \eta {Q_u^t}^*$ for all indices (k,j) such that $Q_s^k \subset Q_u^t$ for some $s \in L_j^k$. To see this, we will use the Whitney properties. If $s \in L_j^k$, then $Q_s^k \cap 2\kappa {Q_j^k}^* \ne \emptyset$, by the definition, and hence there is $z \in Q_s^k \cap 2\kappa {Q_j^k}^*$. Then, by the Whitney structure, $Rl(Q_s^k) \approx d(z, \Omega_k^{\mathrm{c}}) \approx Rl(Q_j^k)$, where R is a large multiple of $\kappa\eta$ for $\eta \ge 2\kappa$ to be chosen; recall also that $l(Q)$ denotes the side-length of the dyadic cube Q, and therefore $l(Q_j^k) \le Cl(Q_s^k)$ with C depending only on κ. On the other hand, $Q_s^k \subset Q_u^t$ implies that $Q_t^u \cap 2\kappa {Q_j^k}^* \ne \emptyset$, and $l(Q_s^k) \le l(Q_u^t)$. This shows that the constant η can be chosen so that it is independent of (t,u). We also note that the cardinality of L_j^k is at most C by (3.1)(iv). Thus,

$$\sum_{(k,j)\in G_{K,N}} \sum_{\substack{s\in L_j^k\\ P(Q_s^k)=Q_u^t}} \frac{|E_j^k|_\omega}{|\eta Q_j^{k*}|_\omega^q}\Big(\sum_{\substack{i:\,P(Q_i^{k+n})=P(Q_s^k)\\ Q_i^{k+n}\subset Q_s^k}}\Big(\int\limits_{Q_i^{k+n}} T^*(\chi_{E_j^k}\,d\omega)d\sigma\Big)A_i^{k+n}\Big)^q$$

$$\le \sum_{k,j} \sum_{\substack{s\in L_j^k\\ P(Q_s^k)=Q_u^t}} \frac{|E_j^k|_\omega}{|\eta Q_j^{k*}|_\omega^q}\Big(\int\limits_{Q_s^k} T^*(\chi_{E_j^k}\,d\omega)\,d\sigma\Big)^q(2A_u^t)^q \quad \text{by (3.1)(i)}$$

$$\le C(A_u^t)^q\sum_{k,j} \sum_{\substack{s\in L_j^k\\ P(Q_s^k)=Q_u^t}} \frac{|E_j^k|_\omega}{|\eta Q_j^{k*}|_\omega^q}\Big(\int\limits_{E_j^k} T(\chi_{Q_u^t}\,d\sigma)\,d\omega\Big)^q$$

$$\le C(A_u^t)^q\sum_{k,j} \sum_{\substack{s\in L_j^k\\ P(Q_s^k)=Q_u^t}} \int\limits_{E_j^k} T(\chi_{Q_u^t}\,d\sigma)^q\,d\omega \quad \text{by Hölder's inequality}$$

$$\le C(A_u^t)^q \int\limits_{\eta Q_u^{t*}} T(\chi_{Q_u^t}\,d\sigma)^q\,d\omega \quad \text{by the comments above and (3.1)(i)}$$

$$\le CA_u^t|Q_u^t|_\sigma(\Psi_u^t)^q.$$

Summing the last inequality over $(t,u)\in\Gamma_{K,N}$ yields

$$\begin{aligned}
I &\le C\beta^{-q}\sum_{(t,u)\in\Gamma_{K,N}}(A_u^t)^q|Q_u^t|_\sigma(\Psi_u^t)^q\\
&\le C\beta^{-q}\int\limits_X\Big(\sum_{(t,u)\in\Gamma_{K,N}}(\Psi_u^t)^q(A_u^t)^q\chi_{Q_u^t}(y)\Big)\,d\sigma(y)\\
&\le C\beta^{-q}\int\limits_X(\Psi_q)^q M_\sigma^{\mathrm{dy}}(f)^q\,d\sigma\\
&\le C\beta^{-q}\Big(\int\limits_X(\Psi_q)^{pq/(p-q)}\,d\sigma\Big)^{1-q/p}\Big(\int\limits_X M_\sigma^{\mathrm{dy}}(f)^p\,d\sigma\Big)^{q/p}\\
&\le C\beta^{-q}\Big(\int\limits_X M_\sigma^{\mathrm{dy}}(f)^p\,d\sigma\Big)^{q/p} \quad \text{by (1.18),}
\end{aligned}$$

since (3.6)(ii) and the geometric series imply that for each fixed y,

$$\begin{aligned}
\sum_{(t,u)\in\Gamma_{K,N}}(\Psi_u^t)^q(A_u^t)^q\chi_{Q_u^t}(y) &\le \sum_{(t,u)\in\Gamma_{K,N}}\Psi_q(y)^q(A_u^t)^q\chi_{Q_u^t}(y)\\
&\le 2^q\Psi_q(y)^q\sup_{Q_u^t\ni y}(A_u^t)^q \le 2^q\Psi_q(y)^qM_\sigma^{\mathrm{dy}}(f)^q,
\end{aligned}$$

where M_σ^{dy} is the dyadic maximal operator with respect to the measure σ, which is defined by

$$M_\sigma^{\mathrm{dy}}(g)(x) = \sup_{x\in Q:\, Q\in\mathcal{D}} \frac{1}{|Q|_\sigma} \int_Q |g(y)|\, d\sigma(y).$$

For a fixed $(k,j) \in G_{K,N}$, it follows from Hölder's inequality that

$$\begin{aligned}
&\Big(\sum_{\substack{i\in H_j^k \\ (k+n,i)\in\Gamma_{K,N}}} \Big(\int_{Q_i^{k+n}} T^*(\chi_{E_j^k}\, d\omega)\, d\sigma\Big) A_i^{k+n}\Big)^q \\
&\le \Big(\sum_{\substack{i\in H_j^k \\ (k+n,i)\in\Gamma_{K,N}}} \Big(\int_{Q_i^{k+n}} T^*(\chi_{Q_j^k}\, d\omega)^{p'}\, d\sigma\Big)^{1/p'} |Q_i^{k+n}|_\sigma^{1/p}\, A_i^{k+n}\Big)^q \\
&\le \Big(\sum_{i\in H_j^k} \int_{Q_i^{k+n}} T^*(\chi_{Q_j^k}\, d\omega)^{p'}\, d\sigma\Big)^{q/p'} \Big(\sum_{\substack{i\in H_j^k \\ (k+n,i)\in\Gamma_{K,N}}} |Q_i^{k+n}|_\sigma (A_i^{k+n})^p\Big)^{q/p}.
\end{aligned}$$

By the same argument as above, we can show that if $i \in H_j^k$ then $Q_i^{k+n} \subset \eta Q_j^{k*}$ for sufficiently large η (larger than its earlier value). Indeed, if $i \in H_j^k$ then Q_i^{k+n} touches $2\kappa Q_j^{k*}$. If $z \in Q_i^{k+n} \cap 2\kappa Q_j^{k*}$, then, by the Whitney structure, $d(z, \Omega_{k+n}^{\mathrm{c}}) \approx Rl(Q_i^{k+n})$ and $d(z,\Omega_k^{\mathrm{c}}) \approx Rl(Q_j^k)$ (R is a large multiple of $\kappa\eta$ for $\eta > 2\kappa$ to be chosen). But $d(z,\Omega_{k+n}^{\mathrm{c}}) \le d(z,\Omega_k^{\mathrm{c}})$ since $\Omega_{k+n} \subset \Omega_k$. Thus, $l(Q_i^{k+n}) \le Cl(Q_j^k)$ with C depending only on κ, and it follows that η can be chosen as desired. Therefore, the first factor of the last expression is at most

$$\Big(\int_{\eta Q_j^{k*}} T^*(\chi_{Q_j^k}\, d\omega)^{p'} d\sigma\Big)^{q/p'} = |Q_j^k|_\omega^{q/p'} (\Theta_j^k)^q.$$

On the other hand, for a fixed $(t,u) \in \Gamma_{K,N}$, if $u \in H_j^{t-n}$ then, by the definition of H_j^{t-n}, we have $Q_u^t \cap 2\kappa Q_j^{t-n*} \neq \emptyset$, and it then follows from the last observation that $Q_u^t \subset \eta Q_j^{t-n*}$. Thus, if z is any point of Q_u^t,

$$\sum_j \chi_{H_j^{t-n}}(u) \le \sum_j \chi_{\eta Q_j^{t-n*}}(z) \le C, \tag{3.7}$$

by the Whitney property (3.1)(i).

Therefore, II is estimated by

$$C\beta^{-q} \sum_{(k,j)\in G_{K,N}} \frac{|E_j^k|_\omega}{|\eta Q_j^{k*}|_\omega^q} |Q_j^k|_\omega^{q/p'} (\Theta_j^k)^q \Big(\sum_{\substack{i\in H_j^k \\ (k+n,i)\in\Gamma_{K,N}}} |Q_i^{k+n}|_\sigma (A_i^{k+n})^p\Big)^{q/p}$$

$$\le C\beta^{-q}\sum_{k,j}(|E_j^k|_\omega(\Theta_j^k)^{pq/(p-q)})^{1-q/p}\Big(\sum_{\substack{i\in H_j^k\\(k+n,i)\in\Gamma_{K,N}}}|Q_i^{k+n}|_\sigma(A_j^{k+n})^p\Big)^{q/p}$$

$$\le C\beta^{-q}\Big(\sum_{k,j}|E_j^k|_\omega(\Theta_j^k)^{pq/(p-q)}\Big)^{1-q/p}\Big(\sum_{k,j}\sum_{\substack{i\in H_j^k\\(k+n,i)\in\Gamma_{K,N}}}|Q_i^{k+n}|_\sigma(A_j^{k+n})^p\Big)^{q/p}$$

$$\le C\beta^{-q}\Big(\int_X(\Psi_p^*)^{pq/(p-q)}\,d\omega\Big)^{1-q/p}\Big(\sum_{k,j}\sum_{\substack{i\in H_j^k\\(k+n,i)\in\Gamma_{K,N}}}|Q_i^{k+n}|_\sigma(A_j^{k+n})^p\Big)^{q/p}$$

$$\le C\beta^{-q}\Big(\sum_{k,j}\sum_{\substack{i\in H_j^k\\(k+n,i)\in\Gamma_{K,N}}}|Q_i^{k+n}|_\sigma(A_j^{k+n})^p\Big)^{q/p}\qquad\text{by (1.18)}$$

$$= C\beta^{-q}\Big(\sum_{k,j,i}\chi_{H_j^k}(i)\chi_{\Gamma_{K,N}}(k+n,i)|Q_i^{k+n}|_\sigma(A_j^{k+n})^p\Big)^{q/p}$$

$$= C\beta^{-q}\Big(\sum_{t,j,u}\chi_{H_j^{t-n}}(u)\chi_{\Gamma_{K,N}}(t,u)|Q_u^t|_\sigma(A_u^t)^p\Big)^{q/p}$$

$$= C\beta^{-q}\Big(\sum_{(t,u)\in\Gamma_{K,N}}\Big(\sum_j\chi_{H_j^{t-n}}(u)\Big)|Q_u^t|_\sigma(A_u^t)^p\Big)^{q/p}$$

$$\le C\beta^{-q}\Big(\sum_{(t,u)\in\Gamma_{K,N}}|Q_u^t|_\sigma(A_u^t)^p\Big)^{q/p}\qquad\text{by (3.7)}$$

$$\le C\beta^{-q}\Big(\int_X M_\sigma^{\mathrm{dy}}(f)^p\,d\sigma\Big)^{q/p}.$$

Combining inequalities, we obtain

$$\sum_{(k,j)\in G_{K,N}}(2^{k+n})^q|E_j^k|_\omega\le C\beta^{-q}\Big(\int_X M_\sigma^{\mathrm{dy}}(f)^p\,d\sigma\Big)^{q/p}\le C\beta^{-q}\Big(\int_X f^p\,d\sigma\Big)^{q/p};$$

the last inequality follows from the fact that the dyadic maximal operator M_σ^{dy} is strong-type $(L^p(d\sigma),L^p(d\sigma))$ for $1<p<\infty$. This proves (3.4). Now let $K\to-\infty$ in (3.4) and then sum over $N=0,1,2,\ldots,n-1$ to get

$$\sum_{(k,j)\in G}(2^{k+n})^q|E_j^k|_\omega\le C\beta^{-q}\Big(\int_X f^p\,d\sigma\Big)^{q/p}.$$

We have now proved that

$$\sum_{k,j}(2^{k+n})^q|E_j^k|_\omega=\Big(\sum_{(k,j)\in E}+\sum_{(k,j)\in F}+\sum_{(k,j)\in G}\Big)(2^{k+n})^q|E_j^k|_\omega$$

$$\leq C\beta \int_X T(f\,d\sigma)^q\,d\omega + C\beta^{-q}\Big(\int_X f^p\,d\sigma\Big)^{q/p}$$

with the constant C independent of m. By letting $m \to \infty$, we obtain

$$\int_X T(f\,d\sigma)^q\,d\omega \leq C\beta \int_X T(f\,d\sigma)^q\,d\omega + C\beta^{-q}\Big(\int_X f^p\,d\sigma\Big)^{q/p}. \tag{3.8}$$

We claim that the first term on the right-hand side of (3.8) is finite. Accepting the claim for the moment, we are then able to choose β so small that $C\beta < 1/2$. Subtracting the first term on the right-hand side of (3.8) from both sides, we obtain (1.10) for $f \geq 0$ bounded with compact support, and hence for arbitrary $f \geq 0$ by the monotone convergence theorem.

We finally show that the claim we made above follows from Theorem 1.1. We first note that, since $1 < q < p < \infty$, by virtue of Theorem 1.1 and duality, the condition $\Psi_q \in L^{pq/(p-q)}(d\sigma)$ implies that $\|T^*(g\,d\omega)\|_{L^{p',\infty}(d\sigma)} \leq C\|g\|_{L^{q'}(d\omega)}$ for all $g \in L^{q'}(d\omega)$. We now let g be a nonnegative function on X so that $\|g\|_{L^{q'}(d\omega)} \leq 1$, and let B be a ball in X. We then have

$$\begin{aligned}
\int_X T(\chi_B\,d\sigma)g\,d\omega &= \int_B T^*(g\,d\omega)\,d\sigma \\
&= \int_0^\infty |B \cap \{y \in X : T^*(g\,d\omega)(y) > \lambda\}|_\sigma\,d\lambda \\
&\leq \int_0^\infty \min\{|B|_\sigma, |\{y \in X : T^*(g\,d\omega)(y) > \lambda\}|_\sigma\}\,d\lambda \\
&\leq \int_0^\infty \min\Big\{|B|_\sigma, \frac{C}{\lambda^{p'}}\|g\|^{p'}_{L^{q'}(d\omega)}\Big\}\,d\lambda \\
&\leq \int_0^\infty \min\Big\{|B|_\sigma, \frac{C}{\lambda^{p'}}\Big\}\,d\lambda \leq C|B|_\sigma^{1/p}.
\end{aligned}$$

By taking the supremum over all such g, we see that $\|T(\chi_B\,d\sigma)\|_{L^q(d\omega)} \leq C|B|_\sigma^{1/p}$, which is finite by the local finiteness of the measure σ. Therefore, if f is bounded by a constant A and supported in a ball B, then $T(f\,d\sigma) \leq AT(\chi_B\,d\sigma)$, which belongs to the class $L^q(d\sigma)$. This completes the proof of the claim and hence the proof of Theorem 1.2.

4. Proofs of Theorem 1.3 and Corollary 1.4.

Both proofs of Theorem 1.3 and Corollary 1.4 are based on the following result, the proof is adapted from [5]. (See also [10, Corollary 1.2] for a similar result.)

PROPOSITION 4.1. *Let $\mathcal{D}$ be the family of dyadic cubes in X, and let $\varrho(Q) \geq 0$ $(Q \in \mathcal{D})$ be such that*

$$\sum_{Q'} \varrho(Q') \leq C_\varrho \varrho(Q), \tag{4.1}$$

where the sum is taken over any family of disjoint dyadic cubes $Q' \in \mathcal{D}$ such that $Q' \subset Q$. Assume that $0 < r < \infty$ and ω satisfies condition (1.20) *with the exponent $0 < \varepsilon \leq 1$. Then*

$$\int_X \sup_{Q \ni x} \{\varphi(Q)\varrho(Q)\}^r \, d\sigma(x) \leq C \int_X \sup_{Q \ni x} \left\{\varphi(Q)\varrho(Q)\left(\frac{|Q|_\sigma}{|Q|_\omega}\right)^{1/r}\right\}^r d\omega(x). \tag{4.2}$$

In the case of $X = \mathbb{R}^n$ and $\varphi(Q) \approx |Q|^{\gamma/n-1}$, (4.2) *is also valid for $\omega \in A_\infty$.*

Proof. Let λ be a fixed positive constant which will be chosen shortly. For each nonnegative integer k, we define

$$\Omega_k = \{x \in X : \sup_{Q \ni x} \{\varphi(Q)\varrho(Q)\} > \lambda^k\}$$

and, for each $m \in \mathbb{Z}$, let $\{Q_j^k\}_j$ be the family of maximal dyadic cubes in $Q \in \mathcal{D}_m$ such that

$$\varphi(Q_j^k)\varrho(Q_j^k) > \lambda^k.$$

Then $\Omega_{k,m} = \bigcup_j Q_j^k$ is a disjoint union and $\lim_{m \to -\infty} \Omega_{\lambda,m} = \Omega_\lambda$. Moreover, if $\widetilde{Q}_j^k \in \mathcal{D}_m$ is the smallest dyadic cube which contains Q_j^k properly, then, by (1.3), it is not hard to verify (see [11]) that $\varphi(\widetilde{Q}_j^k) \leq C_\varphi \varphi(Q_j^k)$, since Q_j^k and $\widetilde{Q}_j^k$ have compatible side-lengths. We also have $\varrho(Q_j^k) \leq C_\varrho \varrho(\widetilde{Q}_j^k)$ by (4.1). Therefore

$$\lambda^k < \varphi(Q_j^k)\varrho(Q_j^k) \leq C_\varphi C_\varrho \varphi(\widetilde{Q}_j^k)\varrho(\widetilde{Q}_j^k) \leq C_\varphi C_\varrho \lambda^k.$$

We now choose $\lambda > C_\varepsilon C_\varphi C_\varrho^2$. Let $E_j^k = Q_j^k \cap (\Omega_{k,m} \setminus \Omega_{k+1})$. Then $\{E_j^k\}_{j,k}$ is disjoint, and $\Omega_{k,m} \setminus \Omega_{k+1} = \bigcup_j E_j^k$. We claim that

$$|Q_j^k|_\omega < \beta |E_j^k|_\omega, \tag{4.3}$$

for some constant $\beta > 1$ independent of j and k.

To prove the claim, we estimate

$$\frac{|Q_j^k \cap \Omega_{k+1}|_\omega}{|Q_j^k|_\omega} = \sum_i \frac{|Q_j^k \cap Q_i^{k+1}|_\omega}{|Q_j^k|_\omega} = \sum_{i:\, Q_i^{k+1} \subset Q_j^k} \frac{|Q_i^{k+1}|_\omega}{|Q_j^k|_\omega}$$

$$\begin{aligned}
&\le \sum_{i:\, Q_i^{k+1}\subset Q_j^k} \left(C_\varepsilon \frac{\varphi(Q_j^k)}{\varphi(Q_i^{k+1})}\right)^{1/\varepsilon} \quad \text{by (1.20)}\\
&\le \left(C_\varepsilon \frac{\varphi(Q_j^k)}{\lambda^{k+1}}\right)^{1/\varepsilon} \sum_{i:\, Q_i^{k+1}\subset Q_j^k} \varrho(Q_i^{k+1})^{1/\varepsilon}\\
&\le \left(C_\varepsilon \frac{\varphi(Q_j^k)}{\lambda^{k+1}} \sum_{i:\, Q_i^{k+1}\subset Q_j^k} \varrho(Q_i^{k+1})\right)^{1/\varepsilon} \quad \text{since } 0<\varepsilon\le 1\\
&\le \left(C_\varepsilon \frac{\varphi(Q_j^k)\varrho(Q_j^k)}{\lambda^{k+1}}\right)^{1/\varepsilon} \quad \text{by (4.1)}\\
&\le \left(\frac{C_\varepsilon C_\varphi C_\varrho^2}{\lambda}\right)^{1/\varepsilon} < 1.
\end{aligned}$$

This shows that

$$|E_j^k|_\omega = |Q_j^k|_\omega - |Q_j^k \cap \Omega_{k+1}|_\omega > \left(1 - \left(\frac{C_\varepsilon C_\varphi C_\varrho^2}{\lambda}\right)^{1/\varepsilon}\right)|Q_j^k|_\omega,$$

and hence the claim is proved. In the case of $X=\mathbb{R}^n$ and $\varphi(Q)\approx |Q|^{\gamma/n-1}$, one can estimate $|Q_j^k \cap \Omega_{k+1}|/|Q_j^k|$ instead of $|Q_j^k \cap \Omega_{k+1,m}|_\omega/|Q_j^k|_\omega$, and then use the definition of A_∞ weights (see also [5]).

Now, by the definition of E_j^k and the estimate (4.3), we obtain

$$\begin{aligned}
\sum_{j,k} (\lambda^{k+1})^r |E_j^k|_\sigma &\le \sum_{j,k} \lambda^r (\varphi(Q_j^k)\varrho(Q_j^k))^r |Q_j^k|_\sigma\\
&\le \beta\lambda^r \sum_{j,k} (\varphi(Q_j^k)\varrho(Q_j^k))^r \frac{|Q_j^k|_\sigma}{|Q_j^k|_\omega}|E_j^k|_\omega \quad \text{by (4.3)}\\
&\le C \int_X \sup_{Q\ni x} \left\{ \varphi(Q_j^k)\varrho(Q) \left(\frac{|Q|_\sigma}{|Q|_\omega}\right)^{1/r}\right\}^r d\omega(x),
\end{aligned}$$

where the constant C is independent of m. It then follows that

$$\begin{aligned}
\int_X \sup_{Q\ni x} \{\varphi(Q)\varrho(Q)\}^r \, d\sigma(x) &\le \lim_{m\to -\infty} \sum_{j,k} (\lambda^{k+1})^r |E_j^k|_\sigma\\
&\le C \int_X \sup_{Q\ni x} \left\{ \varphi(Q_j^k)\varrho(Q) \left(\frac{|Q|_\sigma}{|Q|_\omega}\right)^{1/r}\right\}^r d\omega(x).
\end{aligned}$$

This completes the proof of the proposition. ∎

We now prove Theorem 1.3. It is easy to see from (2.10) that $\Phi_p \le C\Psi_p^*$ and $\Phi_q \le C\Psi_q$. We now prove that the inverse is also true if ω and σ satisfy the hypothesis of the theorem. We first recall that the doubling property of ω implies that there exist α, $\beta > 1$ such that $|\alpha B|_\omega \ge \beta |B|_\omega$ for all balls B (see [11]). Without loss of generality, we can assume $\alpha \ge 2\kappa$. Now, let $Q \in \mathcal{D}$ be fixed. For each fixed $y \in Q$, we choose a decreasing sequence of balls $B_y^0 \supset B_y^1 \supset B_y^2 \supset \ldots$ so that $B_y^0 = B(y, \alpha r(Q^*))$ and $B_y^k = B(y, \alpha^{-k+1} r(B_y^0))$ for $k \ge 1$. We then have $Q^* \subset B_y^0 \subset \vartheta Q^*$, where $\vartheta = \kappa(\alpha+1)$. We also note that, by virtue of (2.10), we have

$$\varphi(B) \le C\varphi(B') \quad \text{for all pairs of balls } B' \subset B. \tag{4.4}$$

Then, since $|\{x\}|_\omega = 0$, we obtain

$$\begin{aligned}
\int_Q K(x,y)\,d\omega(x) &\le \int_{Q^*} K(x,y)\,d\omega(x) \le \sum_{k=1}^{\infty} \varphi(B_y^k)|B_y^k|_\omega \\
&\le C\varphi(B_y^0)|B_y^0|_\omega \sum_{k=1}^{\infty} \left(\frac{|B_y^k|_\omega}{|B_y^0|_\omega}\right)^{1-\varepsilon} \qquad \text{by (1.20)} \\
&\le C\varphi(B_y^0)|B_y^0|_\omega \sum_{k=1}^{\infty} \left(\frac{1}{\beta^k}\right)^{1-\varepsilon} \\
&\le C\varphi(Q)|\vartheta Q^*|_\omega \quad \text{by (4.4)} \\
&\le C\varphi(Q)|Q|_\omega \qquad \text{since } \omega \text{ is doubling.}
\end{aligned}$$

Therefore, by using the doubling property of σ, we have

$$\left(\frac{1}{|Q|_\omega} \int_{\eta Q^*} \left(\int_Q K(x,y)\,d\omega(x)\right)^{p'} d\sigma(y)\right)^{1/p'} \le C\varphi(Q)|Q|_\omega^{1/p}|Q|_\sigma^{1/p'},$$

for all $Q \in \mathcal{D}$, and hence $\Psi_p^* \le C\Phi_p$. Then the first statement of the theorem follows from Theorem 1.1.

To prove the second statement, we also need to show $\Psi_q \le C\Phi_q$. To do this, we first apply Minkowski's inequality to obtain

$$\begin{aligned}
&\left(\frac{1}{|Q|_\sigma} \int_{\eta Q^*} \left(\int_Q K(x,y)\,d\sigma(y)\right)^{q} d\omega(x)\right)^{1/q} \\
&\qquad\le \frac{1}{|Q|_\sigma^{1/q}} \int_Q \left(\int_{\eta Q^*} K(x,y)^q\,d\omega(x)\right)^{1/q} d\sigma(y).
\end{aligned}$$

Then we repeat the same argument as above and use the assumption $0 < \varepsilon < 1/q$.

Next, by applying Proposition 4.1 with $r = pq/(p-q)$ and $\varrho(Q) = |Q|_\omega^{1/q}|Q|_\sigma^{1/q'}$ (we note that (4.1) is satisfied by using Hölder's inequality), we have

$$
\begin{aligned}
&\|\Phi_q\|_{L^{pq/(p-q)}(d\sigma)} \\
&\qquad \le C\Big\|\sup_{Q\ni x}\Big\{\varphi(Q)|Q|_\omega^{1/q}|Q|_\sigma^{1/q'}\Big(\frac{|Q|_\sigma}{|Q|_\omega}\Big)^{(p-q)/pq}\Big\}\Big\|_{L^{pq/(p-q)}(d\omega)} \\
&\qquad = C\|\Phi_p\|_{L^{pq/(p-q)}(d\omega)}.
\end{aligned}
$$

Thus, the second statement follows from Theorem 1.2. This concludes the proof of Theorem 1.3.

Finally, we sketch the proof of Corollary 1.4. First of all, since both $\omega, \sigma \in A_\infty$ by assumption, and in particular ω is absolutely continuous with respect to the Lebesgue measure, it follows from a theorem of Muckenhoupt and Wheeden (see [4]) that

$$
\int_{\eta Q^*} T_\gamma(\chi_Q\,d\omega)^{p'}\,d\sigma \le C\int_{\eta Q^*}\sup_{Q'\ni y}\Big\{\frac{|Q'\cap Q|_\omega}{|Q'|^{1-\gamma/n}}\Big\}^{p'}d\sigma.
$$

Then, by using (4.2) with $r = p'$ and $\varrho(Q) = |Q|_\sigma$, and also noting that $T_\gamma^* = T_\gamma$, the right-hand side of last inequality is bounded by

$$
C\int_{\eta Q^*}\sup_{Q'\ni y}\Big\{\frac{|Q'|_\omega}{|Q'|^{1-\gamma/n}}\Big(\frac{|Q'|_\sigma}{|Q'|_\omega}\Big)^{1/p'}\Big\}^{p'}d\omega(y) = C\int_{\eta Q^*}\Phi_p(y)^{p'}\,d\omega(y).
$$

Therefore, since ω is doubling and $pq/(p-q) = p'q'/(q'-p')$,

$$
\begin{aligned}
\|\Psi_p^*\|_{L^{pq/(p-q)}(d\omega)} &\le C\Big\|\sup_{Q\ni x}\Big(\frac{1}{|Q|_\omega}\int_{\eta Q^*}\Phi_p^{p'}\,d\omega\Big)^{1/p'}\Big\|_{L^{p'q'/(q'-p')}(d\omega)} \\
&\le C\|M_\omega(\Phi_p^{p'})\|^{1/p'}_{L^{q'/(q'-p')}(d\omega)} \le C\|\Phi_p\|_{L^{pq/(p-q)}(d\omega)},
\end{aligned}
$$

where M_ω is the Hardy–Littlewood maximal operator with respect to the doubling measure ω, which is strong-type $(L^{q'/(q'-p')}(d\omega), L^{q'/(q'-p')}(d\omega))$.

The same arguments as above and in the proof of Theorem 1.3 show that

$$
\begin{aligned}
\|\Psi_q\|_{L^{pq/(p-q)}(d\sigma)} &\le C\|\Phi_q\|_{L^{pq/(p-q)}(d\sigma)}, \\
\|\Phi_q\|_{L^{pq/(p-q)}(d\omega)} &\le C\|\Phi_p\|_{L^{pq/(p-q)}(d\sigma)},
\end{aligned}
$$

and

$$
\|\Phi_p\|_{L^{pq/(p-q)}(d\omega)} \le C\|\Phi_q\|_{L^{pq/(p-q)}(d\sigma)}.
$$

Thus, the corollary follows from Theorem 1.2.

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