Let $E$ be a locally convex space and $X$ complex manifold modelled on a locally convex space. A holomorphic map $f$ from $E$ to $X$ is called a map of uniform type if $f$ can be factorized holomorphically through the canonical map $\omega_\rho$ from $E$ to $E_\rho$ for some continuous seminorm $\rho$ on $E$. Here for each continuous seminorm $\rho$ on $E$ we denote by $E_\rho$ the canonical Banach space associated with $\rho$, and by $\omega_\rho$ the canonical map from $E$ to $E_\rho$. Now let $H(E, X)$ and $H_u(E, X)$ denote the sets of holomorphic maps and of holomorphic maps of uniform type from $E$ to $X$ respectively. The aim of the present note is to find some necessary and sufficient conditions for the equality
\[
H(E, X) = H_u(E, X)
\]
to hold. This problem for vector-valued holomorphic maps, i.e. for the case where $X$ is a locally convex space, was investigated by some authors. The first result on this problem belongs to Colombeau and Mujica. In [2] they have shown that the equality (UN) holds when $E$ is a dual Fréchet–Montel space and $X$ a Fréchet space. Next, a necessary and sufficient condition for (UN) to hold in the class of scalar holomorphic functions on a nuclear Fréchet space was established by Meise and Vogt [7]. An important sufficient condition for (UN) for scalar holomorphic functions on such a space was also found recently by those two authors [8]. However, until now, when $X$ does not have a linear structure, the problem has not been investigated.

Here we consider this problem for holomorphic maps with values in a complex manifold of infinite dimension, in particular, in the projective space associated with a Fréchet space (see the definition in §2). In the first section, by the method of [4], we give a characterization of the uniformity of holomorphic maps with values in complex Banach manifolds. The scalar case has been proved by Meise and Vogt [7] by a different method. Section 2 is devoted to proving the main result (Theorem 2.1) of this note: every holomorphic map from a dual space of a nuclear Fréchet space (i.e., from

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a (DFN)-space) to the projective space $\mathbb{CP}(F)$ associated with a Fréchet space $F$ is of uniform type. This is a variant of a result of Colombeau and Mujica [2].

The main tools for the proof of Theorem 2.1 are the solvability of $\partial$-equations for $C^\infty$ closed differential $(0,1)$-forms together with the uniformity of $C^\infty$ functions on a (DFN)-space which have been shown in [1] and [2] respectively. However, the factoriality of the ring of germs of holomorphic functions in infinitely many complex variables is also used here (see [6]).

Finally, we shall use standard notations from the sheaf theory of germs of holomorphic functions as presented in [3] for the finite-dimensional case and in [6] for the infinite-dimensional case, and from the theory of nuclear locally convex spaces in [10].

1. An extension characterization of uniformity. In this section we shall prove the following.

1.1. Theorem. Let $E$ be a nuclear locally convex space and $X$ a complex Banach manifold. Then the following two conditions are equivalent:

(i) Every holomorphic map from $E$ to $X$ is of uniform type.

(ii) If $E$ is a subspace of a locally convex space $F$ then every holomorphic map on $E$ with values in $X$ can be holomorphically extended to $F$.

Proof. (i)⇒(ii). Let $f : E \to X$ be a holomorphic map. By hypothesis there exists a continuous seminorm $\rho$ on $E$ and a holomorphic map $g$ from $E_\rho$ to $X$ such that $f = g_\rho$. Take a continuous seminorm $\rho_1 \geq \rho$ on $E$ such that the canonical map $\omega_{\rho_1,\rho}$ from $E_\rho$ to $E_{\rho_1}$ is nuclear. We can write $\omega_{\rho_1,\rho}$ in the form $\omega_{\rho_1,\rho} = \alpha \circ \beta$, where $\beta : E_{\rho_1} \to \ell^\infty$ and $\alpha : \ell^\infty \to E_\rho$ are continuous linear maps. By the Hahn–Banach theorem, $\beta$ can be extended to a continuous linear map $\hat{\beta} : F_{\rho_1} \to \ell^\infty$, where $\hat{\beta}$ is a continuous seminorm on $F$ such that $\hat{\beta}|E = \rho_1$. Then $g\alpha\hat{\beta}\omega_{\rho_1}$ is a holomorphic extension of $f$ to $F$.

(ii)⇒(i). Let $cs(E)$ denote the set of all continuous seminorms on $E$. Consider the locally convex space

$$F = \prod \{ E_{\rho} : \rho \in cs(E) \}$$

containing $E$ as a subspace. By the hypothesis for every holomorphic map $f$ from $E$ to $X$ there exists a holomorphic map $g : F \to X$ such that $g|_E = f$. Let $V$ be a coordinate neighbourhood in $X$ and $U = g^{-1}(V)$. Since $V$ is isomorphic to an open set in a Banach space we can find a finite set $A$ in $cs(E)$ and a non-empty open subset $W$ of $U$ such that

$$g(z) = g(\{\{z_\rho\}_{\rho \in A})$$
for every $z \in W$. Put

$$G = \{ z \in F : \text{there exists a neighbourhood } Z \text{ of } z \text{ in } F \text{ such that } g(y) = g(\{ y_{\varrho} \}_{\varrho \in A}) \text{ for every } y \in Z \}. $$

Since $G$ is a non-empty open subset of $F$, to complete the proof it suffices to show that $G$ is closed in $F$.

Let $z_0 \in \partial G$. Take a connected neighbourhood $W_0$ of $z_0$ in $F$ such that $g(W_0)$ is contained in a coordinate neighbourhood of $X$. Consider a holomorphic map $h : W_0 \to X$ given by

$$h(z) = g(\{ z_{\varrho} \}_{\varrho \in A}) \quad \text{for } z \in W_0.$$ 

Since $h$ and $g$ are holomorphic on $W_0$ with $h = g$ on $G \cap W_0 \neq \emptyset$, we have $h = g|_{W_0}$. Hence $z_0 \in G$ and $G$ is closed.

2. Uniformity of holomorphic maps with values in the projective space associated with a Fréchet space. Before formulating Theorem 2.1 we describe the projective space $\mathbb{C}P(F)$ associated with a locally convex space $F$. As in the case where $\dim F < \infty$, $\mathbb{C}P(F)$ is the space of all complex lines in $F$ passing through $0 \in F$. This space is equipped with the quotient topology under the canonical map $F \setminus \{ 0 \} \to \mathbb{C}P(F) : x \mapsto [x]$, the complex line passing through $x$ and $0 \in F$. For each $\alpha \in F^* \setminus \{ 0 \}$ we consider the open subset $V_\alpha$ of $\mathbb{C}P(F)$ and the map $\theta_\alpha : V_\alpha \to \ker \alpha$ given by

$$V_\alpha = \{ [x] \in \mathbb{C}P(F) : \alpha(x) \neq 0 \} \quad \text{and} \quad \theta_\alpha([x]) = \frac{\alpha(x)e_\alpha - x}{\alpha(x)},$$

where $e_\alpha \in F$ is chosen such that $\alpha(e_\alpha) = 1$. It is easy to see that $\theta_\alpha$ is a homeomorphism between $V_\alpha$ and $\ker \alpha$ with

$$\theta_\alpha^{-1}(z) = [z - e_\alpha] \quad \text{for } z \in \ker \alpha.$$ 

Moreover,

$$\theta_\beta \theta_\alpha^{-1}(z) = \frac{\beta(z - e_\alpha)e_\beta - (z - e_\alpha)}{\beta(z - e_\alpha)}$$

is holomorphic on $\theta_\alpha(V_\alpha \cap V_\beta)$. Thus $\mathbb{C}P(F)$ is a complex manifold with the local coordinate system $\{(V_\alpha, \theta_\alpha) : \alpha \in F^* \setminus \{ 0 \}\}$. From the above relation it follows that $\theta_\alpha([x])$ is meromorphic on $V_\beta$ for every $\beta \in F^* \setminus \{ 0 \}$, $\beta \neq \alpha$. Thus $\theta_\alpha$ can be considered as a meromorphic function on $\mathbb{C}P(F)$ with values in $\ker \alpha \subset F$.

2.1. Theorem. Let $E$ be a (DFN)-space and $F$ a Fréchet space. Then every holomorphic map from $E$ to $\mathbb{C}P(F)$ is of uniform type.

For proving the theorem we need the following two lemmas.
2.2. Lemma. Let $f : D \to \mathbb{CP}(F)$ be a holomorphic map, where $D$ is an open set in a locally convex space $E$ and $F$ is a Fréchet space. Then for each $z \in D$ there exists a neighbourhood $U$ of $z$ in $D$ and two holomorphic functions $h$ and $\sigma$ on $U$ with values in $F$ and $\mathbb{C}$ respectively such that

$$Z(h, \sigma) = \emptyset \quad \text{and} \quad f|_U = [h : \sigma],$$

where $Z(h, \sigma)$ denotes the common zero-set of $h$ and $\sigma$.

Proof. For each $z \in D$ we can find a neighbourhood $U$ of $z$ in $D$ such that if we consider $f$ as a meromorphic function on $U$ with values in $F$ then $f$ can be written in the form

$$f|_U = \frac{h}{\sigma}$$

with $z \notin Z(h, \sigma)$, where $h$ and $\sigma$ are holomorphic functions on $U$. Then $Z(h, \sigma) = \emptyset$ in a neighbourhood of $z$ in $U$.

2.3. Lemma. Let $\beta$ and $\sigma$ be scalar holomorphic functions on an open set $D$ in a locally convex space $E$ and let $g$ be a holomorphic function on $D$ with values in a locally convex space. Assume that $\beta g / \sigma$ is holomorphic on $D$ and $Z(g, \sigma) = \emptyset$. Then $\beta / \sigma$ is holomorphic on $D$.

Proof. Let $z_0 \in D$. Since the local ring $O_{E, z_0}$ of germs of holomorphic functions at $z_0$ is factorial [6, Proposition 5.15] we can write

$$\sigma = \sigma_1^{p_1} \ldots \sigma_n^{p_n}$$

in a neighbourhood $U$ of $z_0$ with the germs $(\sigma_1)_{z_0}, \ldots, (\sigma_n)_{z_0}$ being irreducible. By hypothesis and from the equality $\beta g / \sigma_1 = (\beta g / \sigma) \sigma_1^{p_1-1} \ldots \sigma_n^{p_n}$ it follows that $\beta g / \sigma_1$ is holomorphic at $z_0$. On the other hand, since by hypothesis $Z(g, \sigma) = \emptyset$ and $Z(\sigma) = \bigcup_{i=1}^{p_n} Z(\sigma_i)$ it follows that $Z(g, \sigma_i) = \emptyset$ for $i = 1, \ldots, n$. Hence, from the irreducibility of $\sigma_1$ we infer that $Z(\sigma_1)_{z_0} \subseteq Z(\beta)_{z_0}$. Since $(\sigma_1)_{z_0}$ is irreducible, it follows that $\beta = \beta_1 \sigma_1$ in a neighbourhood $U_1$ of $z_0$ in $U$, where $\beta_1$ and $\sigma_1$ are holomorphic on $U_1$. Hence $\beta / \sigma_1$ is holomorphic on $U_1$. Applying the above argument to $\beta_1, \sigma_1$ and $g$ we get the holomorphy of $\beta / \sigma_1^2$ at $z_0$. Continuing this process we infer that $\beta / \sigma$ is holomorphic at $z_0$.

Now we can prove Theorem 2.1 as follows.

Let $f : E \to \mathbb{CP}(E)$ be a holomorphic map, where $E$ is a (DFN)-space. We denote by $O_E$ (resp. $M_E$) the sheaf of germs of holomorphic (resp. meromorphic) functions on $E$. Let

$$O^*_E = \{ \sigma \in O_E : \sigma \text{ is invertible} \},$$

$$M^*_E = M_E \setminus \{0\} \quad \text{and} \quad D_E = M^*_E / O^*_E.$$
Here as in the finite-dimensional case, $D_E$ is called the sheaf of germs of divisors on $E$. We denote by $Z$ the sheaf of integers on $E$. Then we have two exact sheaf sequences on $E$:

$$0 \to Z \to \mathcal{O}_E \xrightarrow{\exp} \mathcal{O}_E^* \to 0,$$

$$0 \to \mathcal{O}_E^* \to M_E^* \xrightarrow{\eta} D_E \to 0,$$

where $\exp(\sigma) = e^{2\pi i \sigma}$ and $\eta$ is the canonical projection. By [6, p. 266, Proposition 3.6] we have $H^1(E, \mathcal{O}_E) = 0$. On the other hand, since $H^2(E, \mathbb{Z}) = 0$, considering the exact cohomology sequences associated with the above exact sheaf sequences it follows that for every divisor $d \in H^0(E, D_E)$ there exists a meromorphic function $\sigma \in H^0(E, M_E^*)$ such that $\eta(\sigma) = d$, where $\eta$ is the map from $H^0(E, M_E^*)$ to $H^0(E, D_E)$ induced by $\eta$. By applying Lemma 2.2 to $f$ we can find an open cover $\{U_j\}$ of $E$ and holomorphic functions $h_j$ and $\sigma_j$ on $U_j$ such that

$$Z(h_j, \sigma_j) = \emptyset \quad \text{and} \quad f|_{U_j} = [h_j : \sigma_j]$$

for every $j$. Since $h_j/\sigma_j = h_j/\sigma_j$ on $U_i \cap U_j$, Lemma 2.3 implies that the formula

$$z \mapsto (\sigma_j)_z \mathcal{O}_{E,z}$$

for $z \in U_j$ defines a divisor $d$ on $E$. Thus there exists a meromorphic function $\beta$ on $E$ with $\beta \neq 0$ such that $\beta_j/d_j \in \mathcal{O}_{E,z}$ for $z \in E$.

These relations imply that $\beta$ is holomorphic on $E$ and $h = \beta f$ is holomorphic on $E$ with $Z(h, \beta) = \emptyset$. Let $\{x^*_j\}$ be a sequence of continuous linear functionals on $F$ which separates the points of $h(E)$. Since $Z(\{x^*_j\}, \beta) = \emptyset$, it follows from [6, p. 247, Proposition 3.2] that there exist $C^\infty$ functions $\varphi_j$, $j \geq 0$, such that

$$\sum_{j \geq 1} \varphi_j |x^*_j h|^2 + \varphi_0 |\beta|^2 = 1.$$

By applying a result of Colombeau and Mujica [2] we find a continuous seminorm $\varrho$ on $E$ and $C^\infty$ functions $\tilde{\varphi}_j$, $j \geq 0$, together with holomorphic functions $\tilde{h}_j$, $j \geq 1$, $\tilde{\beta}$ and $\tilde{h}$ on $E_\varrho$ such that $\varphi_j = \tilde{\varphi}_j \omega_\varrho$, $x^*_j h = \tilde{h}_j \omega_\varrho$ for $j \geq 1$ and $\varphi_0 = \tilde{\varphi}_0 \omega_\varrho$, $\beta = \tilde{\beta} \omega_\varrho$, $h = \tilde{h} \omega_\varrho$. Then $\sum_{j \geq 1} \tilde{\varphi}_j |\tilde{h}_j|^2 + \varphi_0 |\tilde{\beta}|^2 = 1$ on $E_\varrho$. Thus $Z(\{\tilde{h}_j\}_{j \geq 1}, \tilde{\beta}) = \emptyset$ and, hence, $Z(\tilde{h}, \tilde{\beta}) = \emptyset$. Consequently, the formula

$$\hat{f}(z) = [\hat{h}(z) : \hat{\beta}(z)] \quad \text{for} \quad z \in E_\varrho$$

defines a holomorphic map $\hat{f}$ from $E_\varrho$ to $\mathbb{CP}(F)$ such that $f = \hat{f} \omega_\varrho$. Theorem 2.1 is proved.

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