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HOLOMORPHIC MAPS OF UNIFORM TYPE

BY

LE MAU HAI AND THAI THUAN QUANG (HANOI)

Let E be a locally convex space and X complex manifold modelled on a locally convex space. A holomorphic map f from E to X is called a *map of uniform type* if f can be factorized holomorphically through the canonical map ω_ϱ from E to E_ϱ for some continuous seminorm ϱ on E. Here for each continuous seminorm ϱ on E we denote by E_ϱ the canonical Banach space associated with ϱ, and by ω_ϱ the canonical map from E to E_ϱ. Now let $H(E,X)$ and $H_{\mathrm{u}}(E,X)$ denote the sets of holomorphic maps and of holomorphic maps of uniform type from E to X respectively. The aim of the present note is to find some necessary and sufficient conditions for the equality

$$H(E,X) = H_{\mathrm{u}}(E,X) \tag{UN}$$

to hold. This problem for vector-valued holomorphic maps, i.e. for the case where X is a locally convex space, was investigated by some authors. The first result on this problem belongs to Colombeau and Mujica. In [2] they have shown that the equality (UN) holds when E is a dual Fréchet–Montel space and X a Fréchet space. Next, a necessary and sufficient condition for (UN) to hold in the class of scalar holomorphic functions on a nuclear Fréchet space was established by Meise and Vogt [7]. An important sufficient condition for (UN) for scalar holomorphic functions on such a space was also found recently by those two authors [8]. However, until now, when X does not have a linear structure, the problem has not been investigated.

Here we consider this problem for holomorphic maps with values in a complex manifold of infinite dimension, in particular, in the projective space associated with a Fréchet space (see the definition in §2). In the first section, by the method of [4], we give a characterization of the uniformity of holomorphic maps with values in complex Banach manifolds. The scalar case has been proved by Meise and Vogt [7] by a different method. Section 2 is devoted to proving the main result (Theorem 2.1) of this note: every holomorphic map from a dual space of a nuclear Fréchet space (i.e., from

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a (DFN)-space) to the projective space $\mathbb{CP}(F)$ associated with a Fréchet space F is of uniform type. This is a variant of a result of Colombeau and Mujica [2].

The main tools for the proof of Theorem 2.1 are the solvability of $\overline{\partial}$-equations for C^∞ closed differential $(0,1)$-forms together with the uniformity of C^∞ functions on a (DFN)-space which have been shown in [1] and [2] respectively. However, the factoriality of the ring of germs of holomorphic functions in infinitely many complex variables is also used here (see [6]).

Finally, we shall use standard notations from the sheaf theory of germs of holomorphic functions as presented in [3] for the finite-dimensional case and in [6] for the infinite-dimensional case, and from the theory of nuclear locally convex spaces in [10].

1. An extension characterization of uniformity. In this section we shall prove the following.

1.1. THEOREM. *Let E be a nuclear locally convex space and X a complex Banach manifold. Then the following two conditions are equivalent*:

(i) *Every holomorphic map from E to X is of uniform type.*

(ii) *If E is a subspace of a locally convex space F then every holomorphic map on E with values in X can be holomorphically extended to F.*

Proof. (i)⇒(ii). Let $f : E \to X$ be a holomorphic map. By hypothesis there exists a continuous seminorm ϱ on E and a holomorphic map g from E_ϱ to X such that $f = g\omega_\varrho$. Take a continuous seminorm $\varrho_1 \geq \varrho$ on E such that the canonical map $\omega_{\varrho_1\varrho}$ from E_{ϱ_1} to E_ϱ is nuclear. We can write $\omega_{\varrho_1\varrho}$ in the form $\omega_{\varrho_1\varrho} = \alpha \circ \beta$, where $\beta : E_{\varrho_1} \to \ell^\infty$ and $\alpha : \ell^\infty \to E_\varrho$ are continuous linear maps. By the Hahn–Banach theorem, β can be extended to a continuous linear map $\widehat{\beta} : F_{\widehat{\varrho}_1} \to \ell^\infty$, where $\widehat{\varrho}_1$ is a continuous seminorm on F such that $\widehat{\varrho}_1|_E = \varrho_1$. Then $g\alpha\widehat{\beta}\omega_{\widehat{\varrho}_1}$ is a holomorphic extension of f to F.

(ii)⇒(i). Let $\mathrm{cs}(E)$ denote the set of all continuous seminorms on E. Consider the locally convex space

$$F = \prod \{E_\varrho : \varrho \in \mathrm{cs}(E)\}$$

containing E as a subspace. By the hypothesis for every holomorphic map f from E to X there exists a holomorphic map $g : F \to X$ such that $g|_E = f$. Let V be a coordinate neighbourhood in X and $U = g^{-1}(V)$. Since V is isomorphic to an open set in a Banach space we can find a finite set A in $\mathrm{cs}(E)$ and a non-empty open subset W of U such that

$$g(z) = g(\{z_\varrho\}_{\varrho \in A})$$

for every $z \in W$. Put

$$G = \{z \in F : \text{there exists a neighbourhood } Z \text{ of } z \text{ in } F \text{ such that } g(y) = g(\{y_\varrho\}_{\varrho \in A}) \text{ for every } y \in Z\}.$$

Since G is a non-empty open subset of F, to complete the proof it suffices to show that G is closed in F.

Let $z_0 \in \partial G$. Take a connected neighbourhood W_0 of z_0 in F such that $g(W_0)$ is contained in a coordinate neighbourhood of X. Consider a holomorphic map $h : W_0 \to X$ given by

$$h(z) = g(\{z_\varrho\}_{\varrho \in A}) \quad \text{for } z \in W_0.$$

Since h and g are holomorphic on W_0 with $h = g$ on $G \cap W \neq \emptyset$, we have $h = g|_{W_0}$. Hence $z_0 \in G$ and G is closed.

2. Uniformity of holomorphic maps with values in the projective space associated with a Fréchet space. Before formulating Theorem 2.1 we describe the projective space $\mathbb{CP}(F)$ associated with a locally convex space F. As in the case where $\dim F < \infty$, $\mathbb{CP}(F)$ is the space of all complex lines in F passing through $0 \in F$. This space is equipped with the quotient topology under the canonical map $F \setminus \{0\} \to \mathbb{CP}(F) : x \mapsto [x]$, the complex line passing through x and $0 \in F$. For each $\alpha \in F^* \setminus \{0\}$ we consider the open subset V_α of $\mathbb{CP}(F)$ and the map $\theta_\alpha : V_\alpha \to \ker \alpha$ given by

$$V_\alpha = \{[x] \in \mathbb{CP}(F) : \alpha(x) \neq 0\} \quad \text{and} \quad \theta_\alpha([x]) = \frac{\alpha(x)e_\alpha - x}{\alpha(x)},$$

where $e_\alpha \in F$ is chosen such that $\alpha(e_\alpha) = 1$. It is easy to see that θ_α is a homeomorphism between V_α and $\ker \alpha$ with

$$\theta_\alpha^{-1}(z) = [z - e_\alpha] \quad \text{for } z \in \ker \alpha.$$

Moreover,

$$\theta_\beta \theta_\alpha^{-1}(z) = \frac{\beta(z - e_\alpha)e_\beta - (z - e_\alpha)}{\beta(z - e_\alpha)}$$

is holomorphic on $\theta_\alpha(V_\alpha \cap V_\beta)$. Thus $\mathbb{CP}(F)$ is a complex manifold with the local coordinate system $\{(V_\alpha, \theta_\alpha) : \alpha \in F^* \setminus \{0\}\}$. From the above relation it follows that $\theta_\alpha([x])$ is meromorphic on V_β for every $\beta \in F^* \setminus \{0\}$, $\beta \neq \alpha$. Thus θ_α can be considered as a meromorphic function on $\mathbb{CP}(F)$ with values in $\ker \alpha \subset F$.

2.1. THEOREM. *Let E be a (DFN)-space and F a Fréchet space. Then every holomorphic map from E to $\mathbb{CP}(F)$ is of uniform type.*

For proving the theorem we need the following two lemmas.

2.2. LEMMA. *Let $f : D \to \mathbb{CP}(F)$ be a holomorphic map, where D is an open set in a locally convex space E and F is a Fréchet space. Then for each $z \in D$ there exists a neighbourhood U of z in D and two holomorphic functions h and σ on U with values in F and $\mathbb{C}$ respectively such that*

$$Z(h, \sigma) = \emptyset \quad \textit{and} \quad f|_U = [h : \sigma],$$

where $Z(h, \sigma)$ denotes the common zero-set of h and σ.

Proof. For each $z \in D$ we can find a neighbourhood U of z in D such that if we consider f as a meromorphic function on U with values in F then f can be written in the form

$$f|_U = \frac{h}{\sigma}$$

with $z \notin Z(h, \sigma)$, where h and σ are holomorphic functions on U. Then $Z(h, \sigma) = \emptyset$ in a neighbourhood of z in U.

2.3. LEMMA. *Let β and σ be scalar holomorphic functions on an open set D in a locally convex space E and let g be a holomorphic function on D with values in a locally convex space. Assume that $\beta g/\sigma$ is holomorphic on D and $Z(g, \sigma) = \emptyset$. Then β/σ is holomorphic on D.*

Proof. Let $z_0 \in D$. Since the local ring $\mathcal{O}_{E,z_0}$ of germs of holomorphic functions at z_0 is factorial [6, Proposition 5.15] we can write

$$\sigma = \sigma_1^{p_1} \dots \sigma_n^{p_n}$$

in a neighbourhood U of z_0 with the germs $(\sigma_1)_{z_0}, \dots, (\sigma_n)_{z_0}$ being irreducible. By hypothesis and from the equality $\beta g/\sigma_1 = (\beta g/\sigma)\sigma_1^{p_1-1} \dots \sigma_n^{p_n}$ it follows that $\beta g/\sigma_1$ is holomorphic at z_0. On the other hand, since by hypothesis $Z(g, \sigma) = \emptyset$ and $Z(\sigma) = \bigcup_{i=1}^n Z(\sigma_i)$ it follows that $Z(g, \sigma_i) = \emptyset$ for $i = 1, \dots, n$. Hence, from the irreducibility of σ_1 we infer that $Z(\sigma_1)_{z_0} \subseteq Z(\beta)_{z_0}$. Since $(\sigma_1)_{z_0}$ is irreducible, it follows that $\beta = \beta_1 \sigma_1$ in a neighbourhood U_1 of z_0 in U, where β_1 and σ_1 are holomorphic on U_1. Hence β/σ_1 is holomorphic on U_1. Applying the above argument to β_1, σ_1 and g we get the holomorphy of β/σ_1^2 at z_0. Continuing this process we infer that β/σ is holomorphic at z_0.

Now we can prove Theorem 2.1 as follows.

Let $f : E \to \mathbb{CP}(E)$ be a holomorphic map, where E is a (DFN)-space. We denote by $\mathcal{O}_E$ (resp. M_E) the sheaf of germs of holomorphic (resp. meromorphic) functions on E. Let

$$\mathcal{O}_E^* = \{\sigma \in \mathcal{O}_E : \sigma \text{ is invertible}\},$$
$$M_E^* = M_E \setminus \{0\} \quad \text{and} \quad D_E = M_E^*/\mathcal{O}_E^*.$$

Here as in the finite-dimensional case, D_E is called the sheaf of germs of divisors on E. We denote by $\mathbb{Z}$ the sheaf of integers on E. Then we have two exact sheaf sequences on E:

$$0 \to \mathbb{Z} \to \mathcal{O}_E \xrightarrow{\exp} \mathcal{O}_E^* \to 0,$$
$$0 \to \mathcal{O}_E^* \to M_E^* \xrightarrow{\eta} D_E \to 0,$$

where $\exp(\sigma) = e^{2\pi i\sigma}$ and η is the canonical projection. By [6, p. 266, Proposition 3.6] we have $H^1(E, \mathcal{O}_E) = 0$. On the other hand, since $H^2(E, \mathbb{Z}) = 0$, considering the exact cohomology sequences associated with the above exact sheaf sequences it follows that for every divisor $d \in H^0(E, D_E)$ there exists a meromorphic function $\sigma \in H^0(E, M_E^*)$ such that $\widehat{\eta}(\sigma) = d$, where $\widehat{\eta}$ is the map from $H^0(E, M_E^*)$ to $H^0(E, D_E)$ induced by η. By applying Lemma 2.2 to f we can find an open cover $\{U_j\}$ of E and holomorphic functions h_j and σ_j on U_j such that

$$Z(h_j, \sigma_j) = \emptyset \quad \text{and} \quad f|_{U_j} = [h_j : \sigma_j]$$

for every j. Since $h_i/\sigma_i = h_j/\sigma_j$ on $U_i \cap U_j$, Lemma 2.3 implies that the formula

$$z \mapsto (\sigma_j)_z \mathcal{O}_{E,z}^*$$

for $z \in U_j$ defines a divisor d on E. Thus there exists a meromorphic function β on E with $\beta \neq 0$ such that $\beta_z/d_z \in \mathcal{O}_{E,z}^*$ for $z \in E$.

These relations imply that β is holomorphic on E and $h = \beta f$ is holomorphic on E with $Z(h, \beta) = \emptyset$. Let $\{x_j^*\}$ be a sequence of continuous linear functionals on F which separates the points of $h(E)$. Since $Z(\{x_j^* h\}, \beta) = \emptyset$, it follows from [6, p. 247, Proposition 3.2] that there exist C^∞ functions $\varphi_j,\ j \geq 0$, such that

$$\sum_{j\geq 1} \varphi_j |x_j^* h|^2 + \varphi_0 |\beta|^2 = 1.$$

By applying a result of Colombeau and Mujica [2] we find a continuous seminorm ϱ on E and C^∞ functions $\widehat{\varphi}_j,\ j \geq 0$, together with holomorphic functions $\widehat{h}_j,\ j \geq 1$, $\widehat{\beta}$ and $\widehat{h}$ on E_ϱ such that $\varphi_j = \widehat{\varphi}_j \omega_\varrho$, $x_j^* h = \widehat{h}_j \omega_\varrho$ for $j \geq 1$ and $\varphi_0 = \widehat{\varphi}_0 \omega_\varrho$, $\beta = \widehat{\beta} \omega_\varrho$, $h = \widehat{h} \omega_\varrho$. Then $\sum_{j\geq 1} \widehat{\varphi}_j |\widehat{h}_j|^2 + \widehat{\varphi}_0 |\widehat{\beta}|^2 = 1$ on E_ϱ. Thus $Z(\{\widehat{h}_j\}_{j\geq 1}, \widehat{\beta}) = \emptyset$ and, hence, $Z(\widehat{h}, \widehat{\beta}) = \emptyset$. Consequently, the formula

$$\widehat{f}(z) = [\widehat{h}(z) : \widehat{\beta}(z)] \quad \text{for } z \in E_\varrho$$

defines a holomorphic map $\widehat{f}$ from E_ϱ to $\mathbb{CP}(F)$ such that $f = \widehat{f} \omega_\varrho$. Theorem 2.1 is proved.

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DEPARTMENT OF MATHEMATICS
PEDAGOGICAL INSTITUTE 1 HANOI
TULIEM, HANOI, VIETNAM

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