COLLOQUIUM MATHEMATICUM

VOL. LXIX 1995 FASC. 1

SOME NON-HOMOGENEOUS HARDY SPACES ON LOCALLY COMPACT VILENKIN GROUPS

BY

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1. Introduction. Let $0 < q \le 1 < p < \infty$, $-1 < \alpha \le 0$ and G be a locally compact Vilenkin group. In [5], we have introduced certain new homogeneous Hardy spaces $HK^q_{p,\alpha}(G)$ associated with the Herz space $K^q_{p,\alpha}(G)$ on a locally compact Vilenkin group G. In this paper, we consider their non-homogeneous versions. More precisely, in Section 2 of this paper we introduce some non-homogeneous Hardy spaces $HA^q_{p,\alpha}(G)$ associated with the Beurling algebras $A^q_{p,\alpha}(G)$ on a locally compact Vilenkin group G. We then establish their characterizations in terms of atomic and molecular decompositions. Moreover, for the space $HA^q_{p,0}(G)$, we obtain its dual space, $\mathrm{RCMO}^q_p(G)$. Using atomic and molecular characterizations we can show that $HA^q_{p,\alpha}(G) \subsetneq HK^q_{p,\alpha}(G) \subsetneq H^p_\alpha(G)$, $HK^q_{p,\alpha}(G) \cap L^p_\alpha(G) \subset HA^q_{p,\alpha}(G)$ and $HK^1_{p,\alpha}(G) \cap L^p_\alpha(G) = HA^1_{p,\alpha}(G)$ (see [3, 4, 9, 10] for the definitions of the Hardy spaces $H^p_\alpha(G)$). In [10] Onneweer and Quek introduce a subspace $Y^*(G)$ of the Hardy space $H^1(G)$. As an application of the theory of §2, in Section 3 of this paper we discuss the relation between the spaces $HA^1_{p,0}(G)$ and $Y^*(G)$. In the last section, §4, of this paper, we give another application of the theory of §2. We first introduce some general non-homogeneous Herz spaces $K(\beta, p, q; G)$ on a locally compact Vilenkin group G (see [6, 8]), where $\beta \in \mathbb{R}$, $0 < p, q \le \infty$, $K(1/q - 1/p, p, q; G) = A^q_{p,0}(G)$. We then discuss the relation between the spaces $HA^q_{p,0}(G)$ and $K(\beta, p, q; G)$ with $\beta > 1/q - 1/p$. Moreover, we state a conjecture about the relation between $HA^1_{p,0}(G)$ and $A^1_{p,0}(G)$. Our motivation for this paper comes mainly from Chen and Lau's paper [1] and García-Cuerva's paper [2] in the Euclidean case.

Now, let us introduce some notation on locally compact Vilenkin groups; for more details we refer to [3, 4] and [5–11].

1991 *Mathematics Subject Classification*: 43A70, 42B30.

Key words and phrases: Vilenkin group, Herz space, Hardy space, atom, molecule, dual space.

The research was supported by the NNSF of China.

Throughout this paper, G will denote a locally compact Abelian group containing a strictly decreasing sequence of open compact subgroups $\{G_n\}_{n=-\infty}^{\infty}$ such that

(i) $\bigcup_{n=-\infty}^{\infty} G_n = G$ and $\bigcap_{n=-\infty}^{\infty} G_n = \{0\}$,
(ii) $\sup\{\mathrm{order}\,(G_n/G_{n+1}) : n \in \mathbb{Z}\} < \infty$.

We choose a Haar measure μ on G so that $\mu(G_0) = 1$ and $\mu(G_n) := (m_n)^{-1}$ for each $n \in \mathbb{Z}$. Then for each $\alpha > 0$ and $k \in \mathbb{Z}$, we have

$$\sum_{n=k}^{\infty}(m_n)^{-\alpha} \le C(m_k)^{-\alpha}, \tag{1.1}$$

$$\sum_{n=-\infty}^{k}(m_n)^{\alpha} \le C(m_k)^{\alpha}. \tag{1.2}$$

There exists a metric d on $G \times G$ defined by $d(x,x) = 0$ and $d(x,y) = (m_l)^{-1}$ if $x - y \in G_l \setminus G_{l+1}$, for $l \in \mathbb{Z}$. Then the topology on G determined by d coincides with the original topology. For $x \in G$, we set $|x| = d(x,0)$, and for each $\alpha \in \mathbb{R}$ define the function v_α on G by $v_\alpha(x) = |x|^\alpha$; the corresponding measure $v_\alpha\, d\mu = |x|^\alpha\, d\mu$ is denoted by $d\mu_\alpha$. Moreover, dx will sometimes be used in place of $d\mu$. It is easy to deduce that $\mu_\alpha(G_l) \le C(m_l)^{-(\alpha+1)}$ if $\alpha > -1$, and if $l < n$ and $x \in G_l \setminus G_{l+1}$, then $\mu_\alpha(x + G_n) = (m_l)^{-\alpha}(m_n)^{-1}$.

In this paper, $\mathcal{S}(G)$ and $\mathcal{S}'(G)$ denote the spaces of test functions and distributions on G respectively. For details, see [9] and [11].

Now, let us make the precise definition.

Definition 1.1. Let $0 < q \le 1 \le p < \infty$ and $-1 < \alpha \le 0$. The non-homogeneous Herz space $A^q_{p,\alpha}(G)$ is defined by

$$A^q_{p,\alpha}(G) := \{f : f \text{ is a measurable function on } G \text{ and } \|f\|_{A^q_{p,\alpha}(G)} < \infty\},$$

where

$$\|f\|_{A^q_{p,\alpha}(G)} := \Big\{ \sum_{l=-\infty}^{-1} \mu_\alpha(G_l)^{1-q/p}\|f\chi_{G_l\setminus G_{l+1}}\|^q_{L^p_\alpha(G)} + \|f\chi_{G_0}\|^q_{L^p_\alpha(G)} \Big\}^{1/q}.$$

Here and below we write $L^p_\alpha(G) = \{f : f$ is a measurable function on G and $(\int_G |f(x)|^p\, d\mu_a(x))^{1/p} < \infty\}$.

Obviously, $A^q_{p,\alpha}(G) \subsetneq K^q_{p,\alpha}(G) \subsetneq L^q_\alpha(G)$ (see [5] for the definition of spaces $K^q_{p,\alpha}(G)$; see also [8]). More precisely, we have the following proposition (see also [6]).

Proposition 1.1. *Let* $0 < q \le 1 \le p < \infty$ *and* $-1 < \alpha \le 0$. *Then* $A^q_{p,\alpha}(G) = K^q_{p,\alpha}(G) \cap L^p_\alpha(G)$ *and* $\|f\|_{A^q_{p,\alpha}(G)} \sim \|f\|_{L^p_\alpha(G)} + \|f\|_{K^q_{p,\alpha}(G)}$.

Proof. Let $f \in A^q_{p,\alpha}(G)$. Since $\mu_\alpha(G_l) \sim (m_l)^{-(1+\alpha)}$ (see [7]) and $f(x) = \sum_{l=-\infty}^{-1} f(x)\chi_{G_l\setminus G_{l+1}}(x) + f(x)\chi_{G_0}(x)$, we have

$$\begin{aligned}\|f\|_{L^p_\alpha(G)} &\le \|f\chi_{G_0}\|_{L^p_\alpha(G)} + \sum_{l=-\infty}^{-1} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^p_\alpha(G)} \\ &\le \|f\|_{A^q_{p,\alpha}(G)} + \sum_{l=-\infty}^{-1} \mu_\alpha(G_l)^{-(1/q-1/p)}\|f\|_{A^q_{p,\alpha}(G)} \\ &\le \|f\|_{A^q_{p,\alpha}(G)}\Big\{1 + C\sum_{l=-\infty}^{-1}(m_l)^{(1+\alpha)(1/q-1/p)}\Big\} \\ &\le C\|f\|_{A^q_{p,\alpha}(G)} < \infty,\end{aligned}$$

where we have used (1.2).

On the other hand, by the definition of $K^q_{p,\alpha}(G)$ (see [5] and [8]), we have

$$\begin{aligned}\|f\|_{K^q_{p,\alpha}(G)} &= \Big\{\sum_{l=-\infty}^{\infty} \mu_\alpha(G_l)^{1-q/p}\|f\chi_{G_l\setminus G_{l+1}}\|^q_{L^p_\alpha(G)}\Big\}^{1/q} \\ &\le \Big\{\sum_{l=-\infty}^{-1} \mu_\alpha(G_l)^{1-q/p}\|f\chi_{G_l\setminus G_{l+1}}\|^q_{L^p_\alpha(G)} \\ &\quad + \|f\chi_{G_0}\|^q_{L^p_\alpha(G)}\sum_{l=0}^{\infty}\mu_\alpha(G_l)^{1-q/p}\Big\}^{1/q} \\ &\le \Big\{\sum_{l=-\infty}^{-1} \mu_\alpha(G_l)^{1-q/p}\|f\chi_{G_l\setminus G_{l+1}}\|^q_{L^p_\alpha(G)} \\ &\quad + C\|f\chi_{G_0}\|_{L^p_\alpha(G)}\sum_{l=0}^{\infty}(m_l)^{-(1+\alpha)(1-q/p)}\Big\}^{1/q} \\ &\le C\Big\{\sum_{l=-\infty}^{-1} \mu_\alpha(G_l)^{1-q/p}\|f\chi_{G_l\setminus G_{l+1}}\|^q_{L^p_\alpha(G)} + \|f\chi_{G_0}\|^q_{L^p_\alpha(G)}\Big\} \\ &= C\|f\|_{A^q_{p,\alpha}(G)} < \infty.\end{aligned}$$

Thus, $f \in K^q_{p,\alpha}(G) \cap L^p_\alpha(G)$, and

$$\|f\|_{L^p_\alpha(G)} + \|f\|_{K^q_{p,\alpha}(G)} \le C\|f\|_{A^q_{p,\alpha}(G)}.$$

Conversely, suppose $f \in L^p_\alpha(G) \cap K^q_{p,\alpha}(G)$. Then

$$\|f\|_{A^q_{p,\alpha}(G)} = \Big\{\sum_{l=-\infty}^{-1} \mu_\alpha(G_l)^{1-q/p}\|f\chi_{G_l\setminus G_{l+1}}\|^q_{L^p_\alpha(G)} + \|f\chi_{G_0}\|^q_{L^p_\alpha(G)}\Big\}^{1/q}$$

$$\leq \{\|f\|^q_{L^p_\alpha(G)} + \|f\|^q_{K^q_{p,\alpha}(G)}\}^{1/q}$$
$$\leq C\{\|f\|_{L^p_\alpha(G)} + \|f\|_{K^q_{p,\alpha}(G)}\} < \infty.$$

That is, $f \in A^q_{p,\alpha}(G)$. The proof of Proposition 1.1 is finished.

2. The Hardy spaces $HA^q_{p,\alpha}(G)$**.** Let $\Delta_n = \mu(G_n)^{-1}\chi_{G_n} = m_n\chi_{G_n}$. For $f \in \mathcal{S}'(G)$, we define $f_n(x) = f * \Delta_n(x)$. Then f_n is a function on G which is constant on the cosets of G_n in G. Moreover, $\lim_{n\to\infty} f_n = f$ in $\mathcal{S}'(G)$ (see [11]). For $f \in \mathcal{S}'(G)$, we define its maximal function $f^*(x)$ by

$$f^*(x) = \sup_{n\in\mathbb{Z}} |f * \Delta_n(x)| = \sup_{n\in\mathbb{Z}} \Big|\mu(G_n)^{-1} \int\limits_{x+G_n} f(y)\, d\mu(y)\Big|.$$

Now we define new Hardy spaces $HA^q_{p,\alpha}(G)$ associated with the non-homogeneous Herz spaces $A^q_{p,\alpha}(G)$.

Definition 2.1. Let $0 < q \leq 1 < p < \infty$ and $-1 < \alpha \leq 0$. The Hardy spaces $HA^q_{p,\alpha}(G)$ are defined by

$$HA^q_{p,\alpha}(G) := \{f \in \mathcal{S}'(G) : f^* \in A^q_{p,\alpha}(G)\}$$

and

$$\|f\|_{HA^q_{p,\alpha}(G)} := \|f^*\|_{A^q_{p,\alpha}(G)}.$$

Evidently, $HA^q_{p,\alpha}(G) \subsetneq HK^q_{p,\alpha}(G) \subsetneq H^q_\alpha(G)$.

Remark 2.1. Consider $q = 1 < p < \infty$ and $-1 < \alpha \leq 0$. Since $A^1_{p,\alpha}(G) \subset K^1_{p,\alpha}(G) \subset L^1_\alpha(G)$, if $f^* \in K^1_{p,\alpha}(G)$ then $f \in L^1_\alpha(G)$ by Lemma 3.5 of Kitada [3]. Therefore we can redefine $HA^1_{p,\alpha}(G)$ by

$$HA^1_{p,\alpha}(G) = \{f \in L^1_\alpha(G) : f^* \in A^1_{p,\alpha}(G)\}.$$

From this definition, we immediately deduce that $HA^1_{p,\alpha}(G) = HK^1_{p,\alpha}(G) \cap L^p_\alpha(G)$, and

$$\|f\|_{HA^1_{p,\alpha}(G)} \sim \|f\|_{L^p_\alpha(G)} + \|f\|_{HK^1_{p,\alpha}(G)}. \tag{2.1}$$

In fact, let $f \in HA^1_{p,\alpha}(G)$. Then $f^* \in A^1_{p,\alpha}(G) = K^1_{p,\alpha}(G) \cap L^p_\alpha(G)$ by Proposition 1.1. Since $f \in L^1_\alpha(G)$, it is easy to verify that $f \in L^1_{\mathrm{loc}}(G)$, and therefore $|f(x)| \leq f^*(x)$. Thus, $\|f\|_{L^p_\alpha(G)} \leq \|f^*\|_{L^p_\alpha(G)} < \infty$, that is, $f \in L^p_\alpha(G)$. In addition, since $f^* \in K^1_{p,\alpha}(G)$ we know that $f \in HK^1_{p,\alpha}(G)$. Thus, $f \in HK^1_{p,\alpha}(G) \cap L^p_\alpha(G)$, and

$$\|f\|_{L^p_\alpha(G)} + \|f\|_{HK^1_{p,\alpha}(G)} \leq \|f^*\|_{L^p_\alpha(G)} + \|f^*\|_{K^1_{p,\alpha}(G)}$$
$$\leq C\|f^*\|_{A^1_{p,\alpha}(G)} = C\|f\|_{HA^1_{p,\alpha}(G)} < \infty.$$

Conversely, we have the general fact

$$HK^q_{p,\alpha}(G) \cap L^p_\alpha(G) \subset HA^q_{p,\alpha}(G) \tag{2.2}$$

for $0 < p \le 1 < q < \infty$ and $-1 < \alpha \le 0$.

In fact, if $f \in HK^q_{p,\alpha}(G) \cap L^p_\alpha(G)$, then $f^* \in K^q_{p,\alpha}(G) \cap L^p_\alpha(G) = A^q_{p,\alpha}(G)$. That is, $f \in HA^q_{p,\alpha}(G)$ and

$$\begin{aligned} \|f\|_{HA^q_{p,\alpha}(G)} &= \|f^*\|_{A^q_{p,\alpha}(G)} \le C\{\|f^*\|_{L^p_\alpha(G)} + \|f^*\|_{K^q_{p,\alpha}(G)}\} \\ &\le C\{\|f\|_{L^p_\alpha(G)} + \|f\|_{HK^q_{p,\alpha}(G)}\} < \infty, \end{aligned}$$

where we use Proposition 1.1 and the fact that f^* is bounded on $L^p_\alpha(G)$ for $1 < p < \infty$ and $-1 < \alpha \le 0$ (see Kitada [4]).

In order to establish the characterization of the space $HA^q_{p,\alpha}(G)$ in terms of decompositions, we need to introduce the following concept of a central atom of restricted type.

Definition 2.2. Let $0 < q \le 1 < p < \infty$ and $-1 < \alpha \le 0$. A function $a(x)$ on G is said to be a *central $(q,p)_\alpha$-atom of restricted type* if

(1) $\operatorname{supp} a \subset G_n$ for some $n \in \mathbb{Z} \setminus \mathbb{N}$,
(2) $(\int_{G_n} |a(x)|^p \, d\mu_\alpha(x))^{1/p} \le \mu_\alpha(G_n)^{1/p-1/q}$,
(3) $\int a(x)\, dx = 0$.

Remark 2.2. The definition of a central atom of restricted type is a modification of the definition of a central atom in [5], where (1) reads: $\operatorname{supp} a \subset G_n$ for some $n \in \mathbb{Z}$.

Theorem 2.1. *Suppose* $0 < q \le 1 < p < \infty$ *and* $-1 < \alpha \le 0$. *A distribution f on G is in $HA^q_{p,\alpha}(G)$ if and only if $f = \sum_j \lambda_j a_j$ in $\mathcal{S}'(G)$, where a_j's are central $(q,p)_\alpha$-atoms of restricted type and $\sum |\lambda_j|^q < \infty$. Then*

$$\|f\|_{HA^q_{p,\alpha}(G)} \sim \Big\{\sum |\lambda_j|^q\Big\}^{1/q}.$$

Moreover, for $q = 1$, the equality $f(x) = \sum \lambda_j a_j(x)$ holds pointwise.

Proof. We can prove this theorem by a procedure similar to the proof of Theorem 2.5 in [5]. However, for $q = 1$, using (2.1), we can give a simple proof. In fact, let $f \in HA^1_{p,\alpha}(G)$. Then $f \in HK^1_{p,\alpha}(G) \cap L^1_\alpha(G)$. Therefore, by Theorem 2.5 of [5], we know that $f(x) = \sum_j \overline{\lambda}_j \overline{a}_j(x)$, where each $\overline{a}_j$ is a central $(1,p)_\alpha$-atom with support G_{n_j} for some $n_j \in \mathbb{Z}$, and $\sum |\overline{\lambda}_j| < \infty$. Set $I_1 = \{j \in \mathbb{Z} : n_j \in \mathbb{N} \cup \{0\}\}$ and $I_2 = \mathbb{Z} \setminus I_1$. Write

$$f(x) = \sum_{j \in I_1} \overline{\lambda}_j \overline{a}_j(x) + \sum_{j \in I_2} \overline{\lambda}_j \overline{a}_j(x) =: f_1(x) + f_2(x),$$

where $\operatorname{supp} f_1 \subset G_0$. In addition,

$$\begin{aligned}\|f_2\|_{L^p_\alpha(G)} &\le \sum_{j\in I_2} |\overline{\lambda}_j|\,\|\overline{a}_j(x)\|_{L^p_\alpha(G)} \le \sum_{j\in I_2} |\overline{\lambda}_j|\mu_\alpha(G_{n_j})^{1/p-1/q}\\ &\le C\sum_{j\in I_2} |\overline{\lambda}_j|(m_{n_j})^{-(\alpha+1)(1/p-1/q)}\\ &\le C\sum_{j\in I_2} |\overline{\lambda}_j| \le C\sum_j |\overline{\lambda}_j| < \infty.\end{aligned}$$

That is, $f_2 \in L^p_\alpha(G)$, and $f_1 = f - f_2 \in L^p_\alpha(G)$. Thus, $b(x) = \|f_1\|^{-1}_{L^p_\alpha(G)} \times \mu_\alpha(G_0)^{1/p-1/q} f_1(x)$ is a central $(1,p)_\alpha$-atom of restricted type with support G_0, and

$$f(x) = f_2(x) + \mu_\alpha(G_0)^{-(1/p-1/q)}\|f_1\|_{L^p_\alpha(G)} b(x).$$

Set $\lambda_0 = \mu_\alpha(G_0)^{-(1/p-1/q)}\|f_1\|_{L^p_\alpha(G)}$, $a_0(x) = b(x)$, $\lambda_j = \overline{\lambda}_j$ and $a_j(x) = \overline{a}_j(x)$ for $j \in I_2$. Then $a_j(x)$'s ($j \in I_2 \cup \{0\}$) are central $(1,p)_\alpha$-atoms of restricted type and

$$\begin{aligned}\sum_{j\in I_2\cup\{0\}} |\overline{\lambda}_j| &= \mu_\alpha(G_0)^{-(1/p-1/q)}\|f_1\|_{L^p_\alpha(G)} + \sum_{j\in I_2}|\lambda_j|\\ &\le C\|f\|_{L^p_\alpha(G)} + C\|f_2\|_{L^p_\alpha(G)} + \sum_{j\in I_2}|\lambda_j|\\ &\le C\Big(\|f\|_{L^p_\alpha(G)} + \sum_j |\lambda_j|\Big)\\ &\le C(\|f\|_{L^p_\alpha(G)} + \|f\|_{HK^1_{p,\alpha}(G)}) \le C\|f\|_{HA^1_{p,\alpha}(G)} < \infty,\end{aligned}$$

where we have used (2.1).

Now suppose $f(x) = \sum_j \lambda_j b_j(x)$ satisfies the hypothesis of the theorem and $\operatorname{supp} b_j \subset G_{n_j}$ for some $n_j \in \mathbb{Z}\setminus\mathbb{N}$. By Theorem 2.5 of [5], we know that $f \in HK^1_{p,\alpha}(G)$ and $\|f\|_{HK^1_{p,\alpha}(G)} \le C\{\sum_j |\lambda_j|\}$. On the other hand, we have

$$\begin{aligned}\|f\|_{L^p_\alpha(G)} &\le \sum_j |\lambda_j|\,\|b_j\|_{L^p_\alpha(G)} \le C\sum_j |\lambda_j|(m_{n_j})^{-(1+\alpha)(1/p-1/q)}\\ &\le C\sum_j |\lambda_j| < \infty.\end{aligned}$$

That is, $f \in L^p_\alpha(G)$. By (2.1), we know $f \in HK^1_{p,\alpha}(G) \cap L^p_\alpha(G) = HA^1_{p,\alpha}(G)$ and

$$\|f\|_{HA^1_{p,\alpha}(G)} \le C(\|f\|_{L^p_\alpha(G)} + \|f\|_{HK^1_{p,\alpha}(G)}) \le C\sum_j |\lambda_j|.$$

This finishes the proof of Theorem 2.1.

Similarly to the spaces $HK^q_{p,\alpha}(G)$, we have the following molecular decomposition characterization of the spaces $HA^q_{p,\alpha}(G)$ (see [5]).

DEFINITION 2.3. Suppose $0 < q \le 1 < p < \infty$, $-1 < \alpha \le 0$ and $b > \max\{(1+\alpha)(1/q-1/p), 1-(1+\alpha)/p\}$. A function $M(x)$ on G is said to be a *central* $(q,p,b)_\alpha$*-molecule of restricted type* if

(1) $\|M\|_{L^p_\alpha(G)} \le 1$,
(2) $\mathfrak{R}_{p,\alpha}(M) := \|M\|^{1-\theta}_{L^p_\alpha(G)} \| \, |x|^b M\|^{\theta}_{L^p_\alpha(G)} < \infty$,
(3) $\int M(x)\,dx = 0$,

where $\theta = (1/q-1/p)(1+\alpha)/b$.

Remark 2.3. The definition of a central molecule of restricted type is a modification of the definition of a central molecule in [5], where condition (1) is absent.

THEOREM 2.2. *Let* $0 < q \le 1 < p < \infty$, $-1 < \alpha \le 0$ *and suppose that* $b > \max\{(1+\alpha)(1/q-1/p), 1-(1+\alpha)/p\}$. *A distribution f on G is in $HA^q_{p,\alpha}(G)$ if and only if $f = \sum_k \lambda_k M_k$, both in $\mathcal{S}'(G)$ and pointwise, where each M_k is a central $(q,p,b)_\alpha$-molecule of restricted type, $\mathfrak{R}_{p,\alpha}(M_k) \le C < \infty$, C is independent of M_k and $\sum_k |\lambda_k|^q < \infty$. Moreover,*

$$\|f\|_{HA^q_{p,\alpha}(G)} \sim \Big(\sum |\lambda_k|^q\Big)^{1/q}.$$

Proof. We can show this theorem using the same procedure as in the proof of Theorem 3.3 of [5]. For $q = 1$ we can once again give a simple proof using (2.1). In fact, let $f \in HA^1_{p,\alpha}(G)$. Then by (2.1), $f \in HK^1_{p,\alpha}(G) \cap L^p_\alpha(G)$. Thus, from Theorem 3.3 of [5], we deduce that $f(x) = \sum_{k=1}^\infty \overline{\lambda}_k \overline{M}_k(x)$, where each $\overline{M}_k(x)$ is a central $(1,p,b)_\alpha$-molecule, $\mathfrak{R}_{p,\alpha}(\overline{M}_k) \le C_0 < \infty$ with C_0 independent of $\overline{M}_k$ and $\sum_{k=1}^\infty |\overline{\lambda}_k| \le C\|f\|_{HK^1_{p,\alpha}(G)}$. Let $I_1 = \{k \in \mathbb{N} : \|\overline{M}_k\|_{L^p_\alpha(G)} \le 1\}$ and $I_2 = \mathbb{N} \setminus I_1$. Write

$$f(x) = \sum_{k=1}^\infty \overline{\lambda}_k \overline{M}_k(x) = \sum_{k\in I_1} \overline{\lambda}_k \overline{M}_k(x) + \sum_{k\in I_2} \overline{\lambda}_k \overline{M}_k(x) =: f_1(x) + f_2(x).$$

We have

$$\|f_1\|_{L^p_\alpha(G)} \le \sum_{k\in I_1} |\overline{\lambda}_k| \, \|\overline{M}_k\|_{L^p_\alpha(G)} \le \sum_{k\in I_1} |\overline{\lambda}_k| < \infty,$$

that is, $f_1 \in L^p_\alpha(G)$. Therefore, $f_2 = f - f_1 \in L^p_\alpha(G)$ and

$$\|f_2\|_{L^p_\alpha(G)} \le \|f\|_{L^p_\alpha(G)} + \|f_1\|_{L^p_\alpha(G)} \le \|f\|_{L^p_\alpha(G)} + \sum_{k=1}^\infty |\overline{\lambda}_k| < \infty.$$

Thus, $M_0(x) := (\|f\|_{L^p_\alpha(G)} + \sum_{k=1}^\infty |\overline{\lambda}_k|)^{-1} f_2$ is a $(1,p,b)_\alpha$-molecule of restricted type, and

$$\begin{aligned}\mathfrak{R}_{p,\alpha}(M_0) &= \Big(\|f\|_{L^p_\alpha(G)} + \sum_{k=1}^\infty |\overline{\lambda}_k|\Big)^{-1} \|f_2\|^{1-\theta}_{L^p_\alpha(G)} \| \, |x|^b f_2\|^\theta_{L^p_\alpha(G)} \\ &\le \Big(\|f\|_{L^p_\alpha(G)} + \sum_{k=1}^\infty |\overline{\lambda}_k|\Big)^{-\theta} \Big(\sum_{k\in I_2} |\overline{\lambda}_k| \, \| \, |x|^b \overline{M}_k \|_{L^p_\alpha(G)}\Big)^\theta \\ &\le C_0 < \infty.\end{aligned}$$

If we set $\lambda_0 = \|f\|_{L^p_\alpha(G)} + \sum_{k=1}^\infty |\overline{\lambda}_k|$, $\lambda_k = \overline{\lambda}_k$ and $M_k = \overline{M}_k$ for $k \in I_1$, then $f(x) = \sum_{k\in I_1\cup\{0\}} \lambda_k M_k(x)$, where each M_k is a $(1,p,b)_\alpha$-molecule of restricted type, $\mathfrak{R}_{p,\alpha}(M_k) \le C_0 < \infty$ with C_0 independent of M_k, and

$$\begin{aligned}\sum_{k\in I_1\cup\{0\}} |\lambda_k| &\le C\Big(\|f\|_{L^p_\alpha(G)} + \sum_{k=1}^\infty |\overline{\lambda}_k|\Big) \\ &\le C(\|f\|_{L^p_\alpha(G)} + \|f\|_{HK^1_{p,\alpha}(G)}) \le C\|f\|_{HA^q_{p,\alpha}(G)}\end{aligned}$$

by (2.1).

Conversely, suppose $f(x) = \sum_{k=1}^\infty \lambda_k M_k(x)$ satisfies the hypothesis of the theorem. Then, by Theorem 3.3 of [5], we know that $f \in HK^1_{p,\alpha}(G)$ and

$$\|f\|_{HK^1_{p,\alpha}(G)} \le C\Big(\sum_{k=1}^\infty |\lambda_k|\Big) < \infty.$$

On the other hand, we have

$$\|f\|_{L^p_\alpha(G)} \le \sum_{k=1}^\infty |\lambda_k| \, \|M_k\|_{L^p_\alpha(G)} \le \sum_{k=1}^\infty |\lambda_k| < \infty.$$

That is, $f \in L^p_\alpha(G)$. Thus, by (2.1), $f \in HK^1_{p,\alpha}(G) \cap L^p_\alpha(G) = HA^1_{p,\alpha}(G)$ and

$$\|f\|_{HA^1_{p,\alpha}(G)} \le C\{\|f\|_{L^p_\alpha(G)} + \|f\|_{HK^1_{p,\alpha}(G)}\} \le C\Big(\sum_{k=1}^\infty |\lambda_k|\Big).$$

This finishes the proof of the theorem.

Similarly to the case of the space $HK^q_p(G)$, when $\alpha = 0$, for $HA^q_p(G) := HA^q_{p,0}(G)$, we can also obtain the dual space $\mathrm{RCMO}^q_p(G)$ consisting of functions of central mean oscillation of restricted type.

Definition 2.4. Let $0 < q \le 1 < p < \infty$. A function $f \in L^p_{\mathrm{loc}}(G)$ is said to belong to $\mathrm{RCMO}^q_p(G)$ if and only if for every $n \in \mathbb{Z}\setminus\mathbb{N}$, there exists a constant C_n such that

$$\sup_{n\in\mathbb{Z}\setminus\mathbb{N}} \mu(G_n)^{1-1/q} \Big(\mu(G_n)^{-1} \int_{G_n} |f(x) - C_n|^p \, dx\Big)^{1/p} < \infty.$$

It is easy to verify that, if such C_n's exist, we can take $C_n = m_{G_n}(f) = \mu(G_n)^{-1}\int_{G_n} f(x)\,dx$. Set

$$\|f\|_{\mathrm{RCMO}_p^q(G)} := \sup_{n\in\mathbb{Z}\setminus\mathbb{N}} \mu(G_n)^{1-1/q}\Big(\mu(G_n)^{-1}\int\limits_{G_n} |f(x)-m_{G_n}(f)|^p\,dx\Big)^{1/p}.$$

Remark 2.4. The definition of the space $\mathrm{RCMO}_p^q(G)$ is a modification of the definition of $\mathrm{CMO}_p^q(G)$, the space of functions of central mean oscillation, where the supremum is taken over $\mathbb{Z}$ instead of $\mathbb{Z}\setminus\mathbb{N}$.

Similarly to Theorem 2.9 of [5], we can prove the following duality theorem (see [5] for the details).

THEOREM 2.3. *Let* $0<q\le 1<p<\infty$ *and* $1/p+1/p'=1$. *Then*

$$(HA_p^q(G))^* = \mathrm{RCMO}_{p'}^q(G)$$

in the following sense: given $g\in \mathrm{RCMO}_{p'}^q(G)$, *the functional* Λ_g *defined for finite combinations of atoms* $f=\sum_{finite}\lambda_j a_j \in HA_p^q(G)$ *by*

$$\Lambda_q(f) = \int\limits_G f(x)g(x)\,dx$$

extends uniquely to a continuous linear functional $\Lambda_g\in (HA_p^q(G))^*$, *whose* $(HA_p^q(G))^*$ *norm satisfies*

$$\|\Lambda_g\| \le C\|g\|_{\mathrm{RCMO}_{p'}^q(G)}.$$

Conversely, given $\Lambda\in (HA_p^q(G))^*$, *there exists a unique (up to a constant)* $g\in\mathrm{RCMO}_{p'}^q(G)$ *such that* $\Lambda=\Lambda_q$. *Moreover,*

$$\|g\|_{\mathrm{RCMO}_{p'}^q(G)} \le C\|\Lambda\|.$$

In addition, as an application of the theory of atomic-molecular decompositions of $HA_{p,\alpha}^q(G)$, we can establish certain interpolation theorems and prove boundedness theorems on multipliers (see Theorems 2.6–2.7 and Theorems 4.1–4.4 of [5]). We omit the details. In the following sections, we obtain some other applications of the atomic theory of the spaces $HA_{p,0}^q(G)$.

3. The relation between $HA_{p,0}^1(G)$ and $Y^*(G)$.

In order to establish certain $H^p(G)$ multiplier results, Onneweer and Quek [10] introduce a subspace $Y^*(G)$ of $H^1(G)$ as follows:

$$Y^*(G) = \Big\{ f\in L^1(G) : \int f(x)\,dx = 0 \text{ and } \|f\|_{L^1(G)} + \Big\| f\log^+\Big(\frac{|f|}{\|f\|_{L^1(G)}}\Big)\Big\|_{L^1(G)} + \|\,|f(x)|\log^+|x|\,\|_{L^1(G)} < \infty \Big\}.$$

On the other hand, as we point out in §2, $HA^1_{p,0}(G)$ is also a subspace of $H^1(G)$. In this section, as another application of atomic theory for the space $HA^1_{p,0}(G)$, we discuss the relation between $HA^1_{p,0}(G)$ and $Y^*(G)$. First, we have $HA^1_{p,0}(G) \not\subset Y^*(G)$ for $1 < p < \infty$.

Let Γ denote the dual group of G and for each $n \in \mathbb{Z}$ let $\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}$. Take $\gamma_0 \in \Gamma_1 \setminus \Gamma_0$ and define

$$f(x) = \begin{cases} \dfrac{m_k}{|k|^{3/2}}\gamma_0(x), & x \in G_k \setminus G_{k+1} \text{ and } k = -1, -2, -3, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

For $k \in \mathbb{Z} \setminus (\{0\} \cup \mathbb{N})$, let $b_k(x) = m_k \gamma_0(x) \chi_{G_k \setminus G_{k+1}}(x)$. Then $\operatorname{supp} b_k \subset G_k$ and

$$\begin{aligned} \int b_k(x)\,dx &= m_k \int_{G_k \setminus G_{k+1}} \gamma_0(x)\,dx \\ &= m_k \Big(\int_{G_k} \gamma_0(x)\,dx - \int_{G_{k+1}} \gamma_0(x)\,dx \Big) = 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \|b_k\|_{L^p(G)} &= m_k \Big(\int_{G_k \setminus G_{k+1}} |\gamma_0(x)|^p\,dx \Big)^{1/p} \\ &\le m_k (m_k)^{-1/p} \le \mu(G_k)^{1/p-1}. \end{aligned}$$

Thus, $b_k(x)$ is a central $(1,p)_0$-atom of restricted type. Since $\sum_{k=-\infty}^{-1} 1/|k|^{3/2} < \infty$ and $f(x) = \sum_{k=-\infty}^{-1} b_k(x)/|k|^{3/2}$, by Theorem 2.1 we know that $f \in HA^1_{p,0}(G)$ and

$$\|f\|_{HA^1_{p,0}(G)} \le C \sum_{k=-\infty}^{-1} \frac{1}{|k|^{3/2}} < \infty.$$

But note that $|y| = (m_k)^{-1} \ge 2^{-k}$ if $y \in G_k \setminus G_{k+1}$, and so $\log^+ |y| \ge C(-k)$. Therefore,

$$\begin{aligned} \int |f(y)| \log^+ |y|\,dy &= \sum_{k=-\infty}^{-1} \int_{G_k \setminus G_{k+1}} |f(y)| \log^+ |y|\,dy \\ &\ge \sum_{k=-\infty}^{-1} \frac{m_k}{|k|^{3/2}} (-k) \mu(G_k \setminus G_{k+1}) = \infty. \end{aligned}$$

Thus, $f \notin Y^*(G)$.

However, we have the following theorem:

THEOREM 3.1. *Let* $1 < p < \infty$. *Then*

$$\Big\{f \in A^1_{p,0}(G) : \int f(x)\,dx = 0 \text{ and } \|f\|_{L^1(G)} + \|f(x)\log^+|x|\,\|_{L^1(G)} < \infty\Big\} \subset HA^1_{p,0}(G).$$

From this theorem we can deduce that $(Y^*(G) \cap A^1_{p,0}(G)) \subset HA^1_{p,0}(G)$.

Proof of Theorem 3.1. Let $f^*(x) = \sup_{n\in\mathbb{Z}} |f * \Delta_n(x)|$. Write

$$\|f\|_{HA^1_{p,0}(G)} = \|f^*\|_{A^1_{p,0}(G)} = \sum_{l=-\infty}^{-1} (m_l)^{-(1-1/p)} \|f^*\chi_{G_l\setminus G_{l+1}}\|_{L^p(G)} + \|f^*\chi_{G_0}\|_{L^p(G)} =: I_1 + I_2.$$

We first estimate I_1. Suppose $x \in G_l\setminus G_{l+1}$ for some $l \in \mathbb{Z}\setminus(\mathbb{N}\cup\{0\})$. Let $n \le l$. Then $x \in G_l \subset G_n$, and $\Delta_n(x) = m_n$. In addition, since $\int_G f(y)\,dy = 0$, we have

$$\begin{aligned} |f * \Delta_n(x)| &= \Big|\int_G f(y)(\Delta_n(x-y) - \Delta_n(x))\,dy\Big| \\ &\le \int_{G\setminus G_l} |f(y)|\,|\Delta_n(x-y) - \Delta_n(x)|\,dy, \end{aligned}$$

where we have used the fact that if $y \in G_l$, then $x - y \in G_n$ and therefore $\Delta_n(x-y) = \Delta_n(x) = m_n$. Further, we have

$$|f * \Delta_n(x)| \le 2m_n \int_{G\setminus G_l} |f(y)|\,dy \le 2m_l \int_{G\setminus G_l} |f(y)|\,dy.$$

Thus,

$$\begin{aligned} I_1 &= \sum_{l=-\infty}^{-1} (m_l)^{-(1-1/p)} \|f^*\chi_{G_l\setminus G_{l+1}}\|_{L^p(G)} \\ &\le C \sum_{l=-\infty}^{-1} \int_{G\setminus G_l} |f(y)|\,dy \\ &\quad + \sum_{l=-\infty}^{-1} (m_l)^{-(1-1/p)} \|(\sup_{n>l} |f * \Delta_n(x)|)\chi_{G_l\setminus G_{l+1}}(x)\|_{L^p(G)} \\ &=: CII_1 + II_2. \end{aligned}$$

Note that if $y \in G_j \setminus G_{j+1}$, we have $|y| = (m_j)^{-1} \ge 2^{-j}$, so that

$-j \leq C \log |y| = C \log^+ |y|$ as long as $j \leq -1$. From this, we deduce that

$$\begin{aligned} II_1 &= \sum_{l=-\infty}^{-1} \int_{G \setminus G_l} |f(y)|\, dy = \sum_{l=-\infty}^{-1} \sum_{j=-\infty}^{l-1} \int_{G_j \setminus G_{j+1}} |f(y)|\, dy \\ &= \sum_{j=-\infty}^{-2} \sum_{l=j+1}^{-1} \int_{G_j \setminus G_{j+1}} |f(y)|\, dy \\ &= \sum_{j=-\infty}^{-2} (-j) \int_{G_j \setminus G_{j+1}} |f(y)|\, dy \leq C \sum_{j=-\infty}^{-2} \int_{G_j \setminus G_{j+1}} |f(y)| \log |y|\, dy \\ &\leq C \| \, |f(x)| \log^+ |x| \, \|_{L^1(G)} < \infty. \end{aligned}$$

Now we estimate II_2. Set $E_l(x) = (f \chi_{G_l \setminus G_{l+1}})(x)$. Note that

$$|f * \Delta_n(x)| \leq \int_G |f(y)| \Delta_n(x-y)\, dy.$$

If $n > l$ and $x \in G_l \setminus G_{l+1}$, then $x \notin G_n$. Note that if $y \notin x + G_n$, then $\Delta_n(x-y) = 0$ and $x + G_n \subset x + G_{l+1} \subset (G_l \setminus G_{l+1})$. Thus, $\Delta_n(x-y) \neq 0$ only if $y \in G_l \setminus G_{l+1}$. From this it follows that for $x \in G_l \setminus G_{l+1}$ and $l \leq -1$,

$$\sup_{n>l} |f * \Delta_n(x)| \leq \sup_{n>l} \int |E_l(y)| \Delta_n(x-y)\, dy \leq (E_l)^*(x).$$

Therefore,

$$\begin{aligned} II_2 &= \sum_{l=-\infty}^{-1} (m_l)^{-(1-1/p)} \| (\sup_{n>l} |f * \Delta_n(x)|) \chi_{G_l \setminus G_{l+1}}(x) \|_{L^p(G)} \\ &\leq C \sum_{l=-\infty}^{-1} (m_l)^{-(1-1/p)} \| (E_l)^*(x) \|_{L^p(G)} \\ &\leq C \sum_{l=-\infty}^{-1} (m_l)^{-(1-1/p)} \| E_l \|_{L^p(G)} \leq C \| f \|_{A^1_{p,0}(G)} < \infty, \end{aligned}$$

where we have used the $L^p(G)$-boundedness of $(E_l)^*$ (see Kitada [4]).

For I_2, we first deduce that if $x \in G_0$ and $n \leq 0$, then

$$|f * \Delta_n(x)| \leq m_n \int_{x+G_n} |f(y)|\, dy \leq \|f\|_{L^1(G)}.$$

If $n > 0$ and $x \in G_0$, then $\Delta_n(x-y) = 0$ for $y \notin G_0$. Thus,

$$|\Delta_n * f(x)| \le \int_G |f(y)|\Delta_n(x-y)\,dy \le \int_{G_0} |f(y)|\Delta_n(x-y)\,dy = \int_G E_0(y)\Delta_n(x-y)\,dy,$$

where we set $E_0(y) := f(y)\chi_{G_0}(y)$. Therefore, if $x \in G_0$, we have

$$f^*(x) = \sup_{n\in\mathbb{Z}} |f * \Delta_n(x)| \le \sup_{n>0} |f * \Delta_n(x)| + \sup_{n\le 0} |f * \Delta_n(x)| \le \|f\|_{L^1(G)} + (E_0)^*(x).$$

Thus,

$$I_2 = \|f^*(x)\chi_{G_0}(x)\|_{L^p(G)} \le \|f\|_{L^1(G)} + \|(E_0)^*\|_{L^p(G)} \le \|f\|_{L^1(G)} + C\|E_0\|_{L^p(G)} \le \|f\|_{L^1(G)} + C\|f\|_{A^1_{p,0}(G)} < \infty.$$

That is, $f \in HA^1_{p,0}(G)$ and we finish the proof of Theorem 3.1.

4. The relation between $HA^q_{p,0}(G)$ and the general non-homogeneous Herz spaces. In this section, we first introduce general non-homogeneous Herz spaces $K(\beta,p,q;G)$ on locally compact Vilenkin groups. For the definitions of the general homogeneous Herz spaces on locally compact Vilenkin groups see [8] and [6].

Definition 4.1. Let $\beta \in \mathbb{R}$ and $0 < q, p \le \infty$. The non-homogeneous Herz space $K(\beta,p,q;G)$ is defined by

$$K(\beta,p,q;G) := \{f : f \text{ is a measurable function on } G \text{ and } \|f\|_{K(\beta,p,q;G)} < \infty\},$$

where

$$\|f\|_{K(\beta,p,q;G)} := \Big\{ \sum_{l=-\infty}^{-1} \mu(G_l)^{\beta q} \|f\chi_{G_l\setminus G_{l+1}}\|^q_{L^p(G)} + \|f\chi_{G_0}\|^q_{L^p(G)} \Big\}^{1/q}.$$

Obviously, $K(1/q-1/p,p,q;G) = A^q_{p,0}(G)$ for $0 < q \le 1 \le p < \infty$. Concerning the relation between the spaces $K(\beta,p,q;G)$ and $HA^q_{p,0}(G)$, we have the following fact.

Theorem 4.1. *Let $0 < q \le 1 < p < \infty$. If $\beta > 1/q - 1/p$, then*

$$K(\beta,p,q;G) \cap \Big\{ f \in L^1(G) : \int f(x)\,dx = 0 \Big\} \subset HA^q_{p,0}(G).$$

Remark 4.1. If $q = 1 < p < \infty$, by Theorem 2.1 and the definition of the space $HA^1_{p,0}(G)$, we obviously have

$$HA^1_{p,0}(G) \subset K\Big(1-\frac{1}{p}, p, 1; G\Big) \cap \Big\{f \in L^1(G) : \int f(x)\,dx = 0\Big\}$$
$$= A^1_{p,0}(G) \cap \Big\{f \in L^1(G) : \int f(x)\,dx = 0\Big\}.$$

Thus, a natural question is whether the following equality holds:

$$HA^1_{p,0}(G) = A^1_{p,0}(G) \cap \Big\{f \in L^1(G) : \int f(x)\,dx = 0\Big\}.$$

Proof of Theorem 4.1. Let $f \in K(\beta, p, q; G) \cap \{f \in L^1(G) : \int f(x)\,dx = 0\}$. Set $C_0 = G_0$, $C_i = G_{-i} \setminus G_{-i+1}$, $i = 1, 2, \ldots$ Write

$$f(x) = \sum_{n=0}^{\infty} (f(x) - f_{C_n})\chi_{C_n}(x) + \sum_{n=0}^{\infty} f_{C_n}\chi_{C_n}(x) =: E(x) + F(x),$$

where $f_{C_n} = \mu(C_n)^{-1}\int_{C_n} f(t)\,dt$.

If we set

$$M = \sup\{\mathrm{order}(G_k/G_{k+1})\} < \infty,$$

then from $\int f(y)\,dy = 0$ we deduce that

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty}\Big(\int_{C_n} f(t)\,dt\Big)\frac{\chi_{C_n}(x)}{\mu(C_n)} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{j=n+1}^{\infty}\int_{C_j} f(t)\,dt\Big)\Big(\frac{\chi_{C_{n+1}}(x)}{\mu(C_{n+1})} - \frac{\chi_{C_n}(x)}{\mu(C_n)}\Big) \\ &= \sum_{n=0}^{\infty}\lambda_n^{(1)}a_n^{(1)}(x), \end{aligned}$$

where

$$\begin{aligned} a_n^{(1)}(x) = M^{1/p-1}&(1+M^{1-1/p})^{-1}\mu(G_{-n-1})^{1-1/q} \\ &\times\Big(\frac{\chi_{C_{n+1}}(x)}{\mu(C_{n+1})} - \frac{\chi_{C_n}(x)}{\mu(C_n)}\Big) \end{aligned}$$

and

$$\lambda_n^{(1)} = M^{1-1/p}(1+M^{1-1/p})\mu(G_{-n-1})^{1/q-1}\Big(\sum_{j=n+1}^{\infty}\int_{C_j} f(t)\,dt\Big).$$

It is easy to see that $\operatorname{supp} a_n^{(1)} \subset G_{-n-1}$, $\int a_n^{(1)}(x)\,dx = 0$ and

$$\begin{aligned}\|a_n^{(1)}\|_{L^p(G)} &\le \frac{1}{M^{1-1/p}(1+M^{1-1/p})\mu(G_{-n-1})^{1/q-1}} \\ &\quad \times \left(\frac{1}{\mu(G_{-n})^{1-1/p}} + \frac{1}{\mu(G_{-n+1})^{1-1/p}}\right) \\ &\le \frac{1}{\mu(G_{-n-1})^{1/q-1/p}}.\end{aligned}$$

That is, $a_n^{(1)}(x)$ is a central $(q,p)_0$-atom. Also, by (1.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} |\lambda_n^{(1)}|^q &\le C \sum_{n=0}^{\infty} \mu(G_{-n-1})^{1-q} \Big| \sum_{j=n+1}^{\infty} \int_{C_j} f(t)\,dt \Big|^q \\ &\le C \sum_{n=0}^{\infty} \sum_{j=n+1}^{\infty} \Big(\int_{C_j} |f(t)|\,dt \Big)^q \mu(G_{-n-1})^{1-q} \\ &\le C \sum_{n=0}^{\infty} \sum_{j=n+1}^{\infty} \Big(\int_{C_j} |f(t)|^p\,dt \Big)^{q/p} \mu(C_j)^{(1-1/p)q} \mu(G_{-n-1})^{1-q} \\ &\le C \sum_{n=0}^{\infty} \sum_{j=n+1}^{\infty} \Big(\int_{C_j} |f(t)|^p |t|^{\beta p}\,dt \Big)^{q/p} \\ &\quad \times (m_{-j})^{\beta q-(1-1/p)q} (m_{-n-1})^{q-1} \\ &\le C \sum_{j=1}^{\infty} \Big(\int_{C_j} |f(t)|^p |t|^{\beta p}\,dt \Big)^{q/p} (m_{-j})^{\beta q-(1-1/p)q} \\ &\quad \times \Big(\sum_{n=0}^{j-1} (m_{-n-1})^{q-1} \Big) \\ &\le C \sum_{j=1}^{\infty} \Big(\int_{C_j} |f(t)|^p |t|^{\beta p}\,dt \Big)^{q/p} (m_{-j})^{q(\beta-1/q+1/p)} \\ &\le C \Big(\int |f(t)|^p |t|^{\beta p}\,dt \Big)^{q/p} \\ &\le C \Big\{ \sum_{l=-\infty}^{-1} \mu(G_l)^{\beta q} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^p(G)}^q + \|f\chi_{G_0}\|_{L^p(G)}^q \Big\} \\ &= C\|f\|_{K(\beta,p,q;G)}^q < \infty.\end{aligned}$$

Thus, by Theorem 2.1 we know that $F \in HA_{p,0}^1(G)$.

Now, we turn to $E(x)$. Write

$$\begin{aligned}E(x) &= \sum_{n=0}^{\infty}(f(x)-f_{C_n})\chi_{C_n}(x)\\ &= \sum_{n=0}^{\infty}\mu(G_{-n})^{1/q-1/p}\|(f-f_{C_n})\chi_{C_n}\|_{L^p(G)}\\ &\quad\times\frac{\{f(x)-f_{C_n}(x)\}\chi_{C_n}(x)}{\|(f-f_{C_n})\chi_{C_n}\|_{L^p(G)}\mu(G_{-n})^{1/q-1/p}}\\ &=: \sum_{n=0}^{\infty}\lambda_n^{(2)}a_n^{(2)}(x).\end{aligned}$$

It is easy to see that

$$a_n^{(2)}(x) = \frac{\{f(x)-f_{C_n}\}\chi_{C_n}(x)}{\|(f-f_{C_n})\chi_{C_n}\|_{L^p(G)}\mu(G_{-n})^{1/q-1/p}}$$

is a central $(q,p)_0$-atom, and

$$\begin{aligned}\sum_{n=0}^{\infty}|\lambda_n^{(2)}|^q &= \sum_{n=0}^{\infty}\mu(G_{-n})^{1-q/p}\|(f-f_{C_n})\chi_{C_n}\|_{L^p(G)}^q\\ &\le \sum_{n=0}^{\infty}\mu(G_{-n})^{1-q/p}(\|f\chi_{C_n}\|_{L^p(G)}+|f_{C_n}|\mu(C_n)^{1/p})^q\\ &\le C\sum_{n=0}^{\infty}\mu(G_{-n})^{1-q/p}\|f\chi_{C_n}\|_{L^p(G)}^q\\ &= C\|f\|_{K(1/q-1/p,p,q;G)}^q \le C\|f\|_{K(\beta,p,q;G)}^q < \infty.\end{aligned}$$

Thus, by Theorem 2.1 we know that $E \in HA_{p,0}^1(G)$. Therefore, $f = E+F \in HA_{p,0}^1(G)$. This finishes the proof of Theorem 4.1.

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Reçu par la Rédaction le 13.12.1993