SOME NON-HOMOGENEOUS HARDY SPACES ON
LOCALLY COMPACT VILENKIN GROUPS

BY

SHANZHEN LU AND DACHUN YANG (BEIJING)

1. Introduction. Let $0 < q \leq 1 < p < \infty$, $-1 < \alpha \leq 0$ and $G$ be a
locally compact Vilenkin group. In [5], we have introduced certain new homoge-
neous Hardy spaces $HK_{p,\alpha}^q(G)$ associated with the Herz space $K_{p,\alpha}^q(G)$
on a locally compact Vilenkin group $G$. In this paper, we consider their
non-homogeneous versions. More precisely, in Section 2 of this paper we
introduce some non-homogeneous Hardy spaces $HA_{p,\alpha}^q(G)$ associated with
the Beurling algebras $A_{p,\alpha}^q(G)$ on a locally compact Vilenkin group $G$. We
then establish their characterizations in terms of atomic and molecular de-
compositions. Moreover, for the space $HA_{1,0}^p(G)$, we obtain its dual space,
RCMO$_p^q(G)$. Using atomic and molecular characterizations we can show
that $HA_{p,0}^p(G) \subsetneq HK_{p,0}^q(G) \subsetneq H_{p}^q(G)$, $HK_{p,\alpha}^q(G) \cap L_{p}^q(G) \subset HA_{p,\alpha}^q(G)$
and $HK_{1,\alpha}^q(G) \cap L_{1}^q(G) = HA_{1,\alpha}^q(G)$ (see [3, 4, 9, 10] for the definitions
of the Hardy spaces $H_{p}^q(G)$). In [10] Onneweer and Quek introduce a subspace
$Y^*(G)$ of the Hardy space $H(G)$. As an application of the theory of $\S$2, in
Section 3 of this paper we discuss the relation between the spaces $HA_{p,0}^p(G)$
and $Y^*(G)$. In the last section, $\S$4, of this paper, we give another applica-
tion of the theory of $\S$2. We first introduce some general non-homogeneous
Herz spaces $K(\beta, p, q; G)$ on a locally compact Vilenkin group $G$ (see [6, 8]),
where $\beta \in \mathbb{R}$, $0 < p, q \leq \infty$, $K(1/q - 1/p, p, q; G) = A_{p,0}^q(G)$. We then
discuss the relation between the spaces $HA_{p,0}^p(G)$ and $K(\beta, p, q; G)$ with
$\beta > 1/q - 1/p$. Moreover, we state a conjecture about the relation between
$HA_{1,0}^p(G)$ and $A_{p,0}^q(G)$. Our motivation for this paper comes mainly from
case.

Now, let us introduce some notation on locally compact Vilenkin groups;
for more details we refer to [3, 4] and [5–11].

1991 Mathematics Subject Classification: 43A70, 42B30.
Key words and phrases: Vilenkin group, Herz space, Hardy space, atom, molecule,
dual space.

The research was supported by the NNSF of China.
Throughout this paper, $G$ will denote a locally compact Abelian group containing a strictly decreasing sequence of open compact subgroups $\{G_n\}_{n=-\infty}^{\infty}$ such that

(i) $\bigcup_{n=-\infty}^{\infty} G_n = G$ and $\bigcap_{n=-\infty}^{\infty} G_n = \{0\}$,
(ii) $\sup\{\text{order}(G_n/G_{n+1}) : n \in \mathbb{Z}\} < \infty$.

We choose a Haar measure $\mu$ on $G$ so that $\mu(G_0) = 1$ and $\mu(G_n) := (m_n)^{-1}$ for each $n \in \mathbb{Z}$. Then for each $\alpha > 0$ and $k \in \mathbb{Z}$, we have

\[
\sum_{n=k}^{\infty} (m_n)^{-\alpha} \leq C(m_k)^{-\alpha},
\]

\[
\sum_{n=-\infty}^{k} (m_n)^{\alpha} \leq C(m_k)^{\alpha}.
\]

There exists a metric $d$ on $G \times G$ defined by $d(x, x) = 0$ and $d(x, y) = (m_l)^{-1}$ if $x - y \in G_l \setminus G_{l+1}$, for $l \in \mathbb{Z}$. Then the topology on $G$ determined by $d$ coincides with the original topology. For $x \in G$, we set $|x| = d(x, 0)$, and for each $\alpha \in \mathbb{R}$ define the function $v_\alpha$ on $G$ by $v_\alpha(x) = |x|^\alpha$; the corresponding measure $v_\alpha \, d\mu = |x|^\alpha \, d\mu$ is denoted by $d\mu_\alpha$. Moreover, $dx$ will sometimes be used in place of $d\mu$. It is easy to deduce that $\mu_\alpha(G_1) \leq C(m_l)^{-\alpha+1}$ if $\alpha > -1$, and if $l < n$ and $x \in G_l \setminus G_{l+1}$, then $\mu_\alpha(x + G_n) = (m_l)^{-\alpha}(m_n)^{-1}$.

In this paper, $S(G)$ and $S'(G)$ denote the spaces of test functions and distributions on $G$ respectively. For details, see [9] and [11].

Now, let us make the precise definition.

**Definition 1.1.** Let $0 < q \leq 1 \leq p < \infty$ and $-1 < \alpha \leq 0$. The non-homogeneous Herz space $A_p^{q, \alpha}(G)$ is defined by

$A_p^{q, \alpha}(G) := \{ f : f$ is a measurable function on $G$ and $\|f\|_{A_p^{q, \alpha}(G)} < \infty \}$,

where

$\|f\|_{A_p^{q, \alpha}(G)} := \left\{ \sum_{l=-\infty}^{0} \mu_\alpha(G_l)^{1-q/p} \|f1_{G_l \setminus G_{l+1}}\|_{L_p^q(G)}^q + \|f1_{G}\|_{L_p^q(G)}^q \right\}^{1/q}$.

Here and below we write $L_p^q(G) = \{ f : f$ is a measurable function on $G$ and $\left( \int_{G} |f(x)|^p \, d\mu_\alpha(x) \right)^{1/p} < \infty \}$.

Obviously, $A_p^{q, \alpha}(G) \subseteq K_p^{q, \alpha}(G) \subseteq L_p^q(G)$ (see [5] for the definition of spaces $K_p^{q, \alpha}(G)$; see also [8]). More precisely, we have the following proposition (see also [6]).

**Proposition 1.1.** Let $0 < q \leq 1 \leq p < \infty$ and $-1 < \alpha \leq 0$. Then $A_p^{q, \alpha}(G) = K_p^{q, \alpha}(G) \cap L_p^q(G)$ and $\|f\|_{A_p^{q, \alpha}(G)} \sim \|f\|_{L_p^q(G)} + \|f\|_{K_p^{q, \alpha}(G)}$. 

Proof. Let $f \in A^q_{p,\alpha}(G)$. Since $\mu_a(G_l) \sim (m_l)^{-(1+\alpha)}$ (see [7]) and
\[
f(x) = \sum_{l=-\infty}^{-1} f(x)\chi_{G_l\setminus G_{l+1}}(x) + f(x)\chi_{G_0}(x),
\]
we have
\[
\|f\|_{L^p(G)} \leq \|f\chi_{G_0}\|_{L^p(G)} + \sum_{l=-\infty}^{-1} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^p(G)}
\]
\[
\leq \|f\|_{A^q_{p,\alpha}(G)} + \sum_{l=-\infty}^{-1} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^p(G)}
\]
\[
\leq \|f\|_{A^q_{p,\alpha}(G)} \left\{ 1 + C \sum_{l=-\infty}^{-1} \frac{m_l}{(1+\alpha)(1-q/p)} \right\}
\]
where we have used (1.2).

On the other hand, by the definition of $K^q_{p,\alpha}(G)$ (see [5] and [8]), we have
\[
\|f\|_{K^q_{p,\alpha}(G)} = \left\{ \sum_{l=-\infty}^{-1} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^p(G)}^q \right\}^{1/q}
\]
\[
\leq \left\{ \sum_{l=-\infty}^{-1} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^p(G)}^q + \|f\chi_{G_0}\|_{L^p(G)}^q \right\}^{1/q}
\]
\[
\leq \left\{ \sum_{l=-\infty}^{-1} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^p(G)}^q + C \|f\chi_{G_0}\|_{L^p(G)}^q \right\}^{1/q}
\]
\[
= C \|f\|_{A^q_{p,\alpha}(G)} < \infty.
\]
Thus, $f \in K^q_{p,\alpha}(G)$. Conversely, suppose $f \in L^p(G)$ and $K^q_{p,\alpha}(G)$.

Conversely, suppose $f \in L^p(G) \cap K^q_{p,\alpha}(G)$. Then
\[
\|f\|_{A^q_{p,\alpha}(G)} = \left\{ \sum_{l=-\infty}^{-1} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^p(G)}^q \right\}^{1/q}
\]
Proposition 1.1. Since \( f \in L^q(G) \) and \( f \in \mathcal{S}'(G) \). Therefore we can redefine \( HA \) spaces \( A \) homogeneous Herz spaces \( A \) which is constant on the cosets of \( f \). That is, \( f \in \mathcal{S}'(G) \) (see [11]). For \( f \in \mathcal{S}'(G) \), we define its maximal function \( f^*(x) \) by

\[
f^*(x) = \sup_{n \in \mathbb{Z}} |f \ast \Delta_n(x)| = \sup_{n \in \mathbb{Z}} \left| \mu(G_n)^{-1} \int_{x + G_n} f(y) d\mu(y) \right|.
\]

Now we define new Hardy spaces \( HA^q_{p,\alpha}(G) \) associated with the non-homogeneous Herz spaces \( A^q_{p,\alpha}(G) \).

Definition 2.1. Let \( 0 < q \leq p < \infty \) and \( -1 < \alpha \leq 0 \). The Hardy spaces \( HA^q_{p,\alpha}(G) \) are defined by

\[
HA^q_{p,\alpha}(G) := \{ f \in \mathcal{S}'(G) : f^* \in A^q_{p,\alpha}(G) \}
\]

and

\[
\|f\|_{HA^q_{p,\alpha}(G)} := \|f^*\|_{A^q_{p,\alpha}(G)}.
\]

Evidently, \( HA^q_{p,\alpha}(G) \) is not a subset of \( H_{K^1_{p,\alpha}}^q(G) \).

Remark 2.1. Consider \( q = 1 < p < \infty \) and \( -1 < \alpha \leq 0 \). Since \( A^1_{p,\alpha}(G) \subseteq K^1_{p,\alpha}(G) \subseteq L^p(G) \), if \( f^* \in K^1_{p,\alpha}(G) \) then \( f \in L^p(G) \) by Lemma 3.5 of Kitada [3]. Therefore we can redefine \( HA^1_{p,\alpha}(G) \) by

\[
HA^1_{p,\alpha}(G) = \{ f \in L^1_{\alpha}(G) : f^* \in A^1_{p,\alpha}(G) \}.
\]

From this definition, we immediately deduce that \( HA^1_{p,\alpha}(G) = H_{K^1_{p,\alpha}}^1(G) \cap L^p_{\alpha}(G) \), and

\[
\|f\|_{HA^1_{p,\alpha}(G)} \sim \|f\|_{L^p_{\alpha}(G)} + \|f\|_{H_{K^1_{p,\alpha}}^1(G)}.
\]

In fact, let \( f \in HA^1_{p,\alpha}(G) \). Then \( f^* \in A^1_{p,\alpha}(G) = K^1_{p,\alpha}(G) \cap L^p_{\alpha}(G) \) by Proposition 1.1. Since \( f \in L^p_{\alpha}(G) \), it is easy to verify that \( f \in L^1_{\log}(G) \), and therefore \( \|f\|_{L^p_{\alpha}(G)} \leq \|f^*\|_{L^p_{\alpha}(G)} < \infty \), that is, \( f \in L^p_{\alpha}(G) \). In addition, since \( f^* \in K^1_{p,\alpha}(G) \) we know that \( f \in H_{K^1_{p,\alpha}}^1(G) \). Thus, \( f \in H_{K^1_{p,\alpha}}^1(G) \cap L^p_{\alpha}(G) \), and

\[
\|f\|_{L^p_{\alpha}(G)} + \|f\|_{H_{K^1_{p,\alpha}}^1(G)} \leq \|f^*\|_{L^p_{\alpha}(G)} + \|f^*\|_{K^1_{p,\alpha}(G)} \leq C\|f^*\|_{A^1_{p,\alpha}(G)} = C\|f\|_{HA^1_{p,\alpha}(G)} < \infty.
\]
Conversely, we have the general fact

\[ (2.2) \quad HK^q_{p,\alpha}(G) \cap L^p_\alpha(G) \subset HA^q_{p,\alpha}(G) \]

for \( 0 < p \leq 1 < q < \infty \) and \(-1 < \alpha \leq 0\).

In fact, if \( f \in HK^q_{p,\alpha}(G) \cap L^p_\alpha(G) \), then \( f^* \in K^q_{p,\alpha}(G) \cap L^p_\alpha(G) = A^q_{p,\alpha}(G) \).

That is, \( f \in HA^q_{p,\alpha}(G) \) and

\[
\|f\|_{HA^q_{p,\alpha}(G)} = \|f^*\|_{A^q_{p,\alpha}(G)} \leq C(\|f^*\|_{L^p_\alpha(G)} + \|f^*\|_{K^q_{p,\alpha}(G)}) \\
\leq C(\|f\|_{L^p(G)} + \|f\|_{HK^q_{p,\alpha}(G)}) < \infty,
\]

where we use Proposition 1.1 and the fact that \( f^* \) is bounded on \( L^p_\alpha(G) \) for \( 1 < p < \infty \) and \(-1 < \alpha \leq 0\) (see Kitada [4]).

In order to establish the characterization of the space \( HA^q_{p,\alpha}(G) \) in terms of decompositions, we need to introduce the following concept of a central atom of restricted type.

**Definition 2.2.** Let \( 0 < q \leq 1 < p < \infty \) and \(-1 < \alpha \leq 0\). A function \( a(x) \) on \( G \) is said to be a central \((q,p)_{\alpha}\)-atom of restricted type if

1. \( \text{supp } a \subset G_n \) for some \( n \in \mathbb{Z} \setminus \mathbb{N} \),
2. \( \left( \int_{G_n} |a(x)|^p \, d\mu_\alpha(x) \right)^{1/p} \leq \mu_\alpha(G_n)^{1/p - 1/q} \),
3. \( \int a(x) \, dx = 0 \).

**Remark 2.2.** The definition of a central atom of restricted type is a modification of the definition of a central atom in [5], where (1) reads: \( \text{supp } a \subset G_n \) for some \( n \in \mathbb{Z} \).

**Theorem 2.1.** Suppose \( 0 < q \leq 1 < p < \infty \) and \(-1 < \alpha \leq 0\). A distribution \( f \) on \( G \) is in \( HA^q_{p,\alpha}(G) \) if and only if \( f = \sum_j \lambda_j a_j \) in \( \mathcal{S}'(G) \), where \( a_j \)'s are central \((q,p)_{\alpha}\)-atoms of restricted type and \( \sum |\lambda_j|^q < \infty \). Then

\[
\|f\|_{HA^q_{p,\alpha}(G)} \sim \left\{ \sum |\lambda_j|^q \right\}^{1/q}.
\]

Moreover, for \( q = 1 \), the equality \( f(x) = \sum \lambda_j a_j(x) \) holds pointwise.

**Proof.** We can prove this theorem by a procedure similar to the proof of Theorem 2.5 in [5]. However, for \( q = 1 \), using (2.1), we can give a simple proof. In fact, let \( f \in HA^1_{p,\alpha}(G) \). Then \( f \in HK^1_{p,\alpha}(G) \cap L^1_\alpha(G) \). Therefore, by Theorem 2.5 of [5], we know that \( f(x) = \sum_j \lambda_j \pi_j(x) \), where each \( \pi_j \) is a central \((1,p)_{\alpha}\)-atom with support \( G_{n_j} \) for some \( n_j \in \mathbb{Z} \), and \( \sum |\lambda_j| < \infty \). Set \( I_1 = \{ j \in \mathbb{Z} : n_j \in \mathbb{N} \cup \{0\} \} \) and \( I_2 = \mathbb{Z} \setminus I_1 \). Write

\[
f(x) = \sum_{j \in I_1} \lambda_j \pi_j(x) + \sum_{j \in I_2} \lambda_j \pi_j(x) =: f_1(x) + f_2(x),
\]
where supp $f_1 \subset G_0$. In addition,
\[
\|f_2\|_{L^q(G)} \leq \sum_{j \in I_2} |\lambda_j| \|\pi_j(x)\|_{L^q(G)} \leq \sum_{j \in I_2} |\lambda_j| \mu_\alpha(G_{\lambda_j})^{1/p-1/q} \\
\leq C \sum_{j \in I_2} |\lambda_j| (m_{\lambda_j})^{-(\alpha+1)/(1/p-1/q)} \\
\leq C \sum_{j \in I_2} |\lambda_j| \leq C \sum_j |\lambda_j| < \infty.
\]
That is, $f_2 \in L^q(G)$, and $f_1 = f - f_2 \in L^q(G)$. Thus, $b(x) = \|f_1\|_{L^q(G)}^{-1} \times \mu_\alpha(G_0)^{1/p-1/q} f_1(x)$ is a central $(1,p)_\alpha$-atom of restricted type with support $G_0$, and
\[
f(x) = f_2(x) + \mu_\alpha(G_0)^{-1}(1/p-1/q)\|f_1\|_{L^q(G)} b(x).
\]
Set $\lambda_0 = \mu_\alpha(G_0)^{-1}(1/p-1/q)\|f_1\|_{L^q(G)}$, $a_0(x) = b(x)$, $\lambda_j = \lambda_j$ and $a_j(x) = \pi_j(x)$ for $j \in I_2$. Then $a_j(x)$’s ($j \in I_2 \cup \{0\}$) are central $(1,p)_\alpha$-atoms of restricted type and
\[
\sum_{j \in I_2 \cup \{0\}} |\lambda_j| = \mu_\alpha(G_0)^{-1}(1/p-1/q)\|f_1\|_{L^q(G)} + \sum_{j \in I_2} |\lambda_j| \\
\leq C \|f\|_{L^q(G)} + C \|f_2\|_{L^q(G)} + \sum_{j \in I_2} |\lambda_j| \\
\leq C \left(\|f\|_{L^q(G)} + \sum_j |\lambda_j|\right) \\
\leq C(\|f\|_{L^q(G)} + \|f\|_{HK^1_{p,\alpha}(G)}) \leq C\|f\|_{HA^1_{p,\alpha}(G)} < \infty,
\]
where we have used (2.1).

Now suppose $f(x) = \sum_j \lambda_j b_j(x)$ satisfies the hypothesis of the theorem and supp $b_j \subset G_{n_j}$ for some $n_j \in \mathbb{Z} \setminus \mathbb{N}$. By Theorem 2.5 of [5], we know that $f \in HK^1_{p,\alpha}(G)$ and $\|f\|_{HK^1_{p,\alpha}(G)} \leq C(\sum_j |\lambda_j|)$. On the other hand, we have
\[
\|f\|_{L^q(G)} \leq \sum_j |\lambda_j| \|b_j\|_{L^q(G)} \leq C \sum_j |\lambda_j|(m_{\lambda_j})^{-(\alpha+1)/(1/p-1/q)} \\
\leq C \sum_j |\lambda_j| < \infty.
\]
That is, $f \in L^q(G)$. By (2.1), we know $f \in HK^1_{p,\alpha}(G) \cap L^q(G) = HA^1_{p,\alpha}(G)$ and
\[
\|f\|_{HA^1_{p,\alpha}(G)} \leq C(\|f\|_{L^q(G)} + \|f\|_{HK^1_{p,\alpha}(G)}) \leq C \sum_j |\lambda_j|.
\]
This finishes the proof of Theorem 2.1.
Similarly to the spaces $HK^q_{p,\alpha}(G)$, we have the following molecular decomposition characterization of the spaces $HA^q_{p,\alpha}(G)$ (see [5]).

**Definition 2.3.** Suppose $0 < q \leq 1 < p < \infty$, $-1 < \alpha \leq 0$ and $b > \max\{(1 + \alpha)(1/q - 1/p), 1 - (1 + \alpha)/p\}$. A function $M(x)$ on $G$ is said to be a central $(q, p, b)\alpha$-molecule of restricted type if

1. $\|M\|_{L^q(G)} \leq 1$,
2. $\mathfrak{M}_{p,\alpha}(M) := \|M\|_{L^q(G)}^{1-\theta} \|x\|^b \|M\|_{L^q(G)}^{\theta} < \infty$,
3. $\int M(x) \, dx = 0$.

where $\theta = (1/q - 1/p)(1 + \alpha)/b$.

**Remark 2.3.** The definition of a central molecule of restricted type is a modification of the definition of a central molecule in [5], where condition (1) is absent.

**Theorem 2.2.** Let $0 < q \leq 1 < p < \infty$, $-1 < \alpha < 0$ and suppose that $b > \max\{(1 + \alpha)(1/q - 1/p), 1 - (1 + \alpha)/p\}$. A distribution $f$ on $G$ is in $HA^q_{p,\alpha}(G)$ if and only if $f = \sum_k \lambda_k M_k$, both in $S'(G)$ and pointwise, where each $M_k$ is a central $(q, p, b)\alpha$-molecule of restricted type, $\mathfrak{M}_{p,\alpha}(M_k) \leq C < \infty$, $C$ is independent of $M_k$ and $\sum_k |\lambda_k|^q < \infty$. Moreover,

$$\|f\|_{HA^q_{p,\alpha}(G)} \sim \left( \sum_k |\lambda_k|^q \right)^{1/q}.$$

**Proof.** We can show this theorem using the same procedure as in the proof of Theorem 3.3 of [5]. For $q = 1$ we can once again give a simple proof using (2.1). In fact, let $f \in HA^1_{p,\alpha}(G)$. Then by (2.1), $f \in HK^1_{p,\alpha}(G) \cap L^p(G)$. Thus, from Theorem 3.3 of [5], we deduce that $f(x) = \sum_{k=1}^{\infty} \lambda_k \overline{M}_k(x)$, where each $\overline{M}_k(x)$ is a central $(1, p, b)\alpha$-molecule, $\mathfrak{M}_{p,\alpha}(\overline{M}_k) \leq C_0 < \infty$ with $C_0$ independent of $\overline{M}_k$ and $\sum_{k=1}^{\infty} |\lambda_k|^q \leq C_0$ independent of $\overline{M}_k$. Let $I_1 = \{k \in \mathbb{N} : \|\overline{M}_k\|_{L^q(G)} \leq 1\}$ and $I_2 = \mathbb{N} \setminus I_1$. Write

$$f(x) = \sum_{k=1}^{\infty} \lambda_k \overline{M}_k(x) = \sum_{k \in I_1} \lambda_k \overline{M}_k(x) + \sum_{k \in I_2} \lambda_k \overline{M}_k(x) =: f_1(x) + f_2(x).$$

We have

$$\|f_1\|_{L^p(G)} \leq \sum_{k \in I_1} |\lambda_k| \|\overline{M}_k\|_{L^q(G)} \leq \sum_{k \in I_1} |\lambda_k| < \infty,$$

that is, $f_1 \in L^p(G)$. Therefore, $f_2 = f - f_1 \in L^p(G)$ and

$$\|f_2\|_{L^p(G)} \leq \|f\|_{L^p(G)} + \|f_1\|_{L^p(G)} \leq \|f\|_{L^p(G)} + \sum_{k=1}^{\infty} |\lambda_k| < \infty.$$
Thus, $M_0(x) := (\|f\|_{L^p(G)} + \sum_{k=1}^{\infty} |\overline{\lambda}_k|)^{-1}f_2$ is a $(1,p,b)_{\alpha}$-molecule of restricted type, and

$$\mathcal{R}_{p,\alpha}(M_0) = \left(\|f\|_{L^p(G)} + \sum_{k=1}^{\infty} |\overline{\lambda}_k| \right)^{-1} f_2 \|f_2\|_{L^p(G)} \|x|^b f_2\|_{L^{\infty}(G)}$$

$$\leq \left(\|f\|_{L^p(G)} + \sum_{k=1}^{\infty} |\overline{\lambda}_k| \right)^{-\theta} \left( \sum_{k \in f_2} \|x|^b M_k \|f\|_{L^{\infty}(G)} \right)^{\theta} \leq C_0 < \infty.$$ 

If we set $\lambda_0 = \|f\|_{L^p(G)} + \sum_{k=1}^{\infty} |\overline{\lambda}_k|$, $\lambda_k = \overline{\lambda}_k$ and $M_k = \overline{M}_k$ for $k \in I_1$, then $f(x) = \sum_{k \in I_1 \cup \{0\}} \lambda_k M_k(x)$, where each $M_k$ is a $(1,p,b)_{\alpha}$-molecule of restricted type, $\mathcal{R}_{p,\alpha}(M_k) \leq C_0 < \infty$ with $C_0$ independent of $M_k$, and

$$\sum_{k \in I_1 \cup \{0\}} |\lambda_k| \leq C \left(\|f\|_{L^p(G)} + \sum_{k=1}^{\infty} |\overline{\lambda}_k| \right) \leq C \|f\|_{L^p(G)} + \|f\|_{HK^1_{p,\alpha}(G)} \leq C \|f\|_{HA^1_{p,\alpha}(G)}$$

by (2.1).

Conversely, suppose $f(x) = \sum_{k=1}^{\infty} \lambda_k M_k(x)$ satisfies the hypothesis of the theorem. Then, by Theorem 3.3 of [5], we know that $f \in HK^1_{p,\alpha}(G)$ and

$$\|f\|_{HK^1_{p,\alpha}(G)} \leq C \left(\sum_{k=1}^{\infty} |\lambda_k| \right) < \infty.$$ 

On the other hand, we have

$$\|f\|_{L^p(G)} \leq \sum_{k=1}^{\infty} |\lambda_k| \|M_k\|_{L^p(G)} \leq \sum_{k=1}^{\infty} |\lambda_k| < \infty.$$ 

That is, $f \in L^p(G)$. Thus, by (2.1), $f \in HK^1_{p,\alpha}(G) \cap L^p(G) = HA^1_{p,\alpha}(G)$ and

$$\|f\|_{HA^1_{p,\alpha}(G)} \leq C \left(\|f\|_{L^p(G)} + \|f\|_{HK^1_{p,\alpha}(G)} \right) \leq C \left(\sum_{k=1}^{\infty} |\lambda_k| \right).$$ 

This finishes the proof of the theorem.

Similarly to the case of the space $HK^q_p(G)$, when $\alpha = 0$, for $HA^q_p(G) := HA^q_{p,0}(G)$, we can also obtain the dual space $RCMO^q_{p}(G)$ consisting of functions of central mean oscillation of restricted type.

**Definition 2.4.** Let $0 < q \leq 1 < p < \infty$. A function $f \in L^p_{loc}(G)$ is said to belong to $RCMO^q_{p}(G)$ if and only if for every $n \in \mathbb{Z} \setminus \{0\}$, there exists a constant $C_n$ such that

$$\sup_{n \in \mathbb{Z} \setminus \{0\}} \left( \mu(G_n)^{-1/q} \int_{G_n} |f(x) - C_n|^p \, dx \right)^{1/p} < \infty.$$
It is easy to verify that, if such \( C_n \)'s exist, we can take \( C_n = m_{G_n}(f) = \mu(G_n)^{-1} \int_{G_n} f(x) \, dx \). Set

\[
\|f\|_{\text{RCMO}_q^p(G)} := \sup_{n \in \mathbb{Z} \setminus \mathbb{N}} \mu(G_n)^{1-1/q} \left( \mu(G_n)^{-1} \int_{G_n} |f(x) - m_{G_n}(f)|^p \, dx \right)^{1/p}.
\]

**Remark 2.4.** The definition of the space \( \text{RCMO}_q^p(G) \) is a modification of the definition of \( \text{CMO}_q^p(G) \), the space of functions of central mean oscillation, where the supremum is taken over \( \mathbb{Z} \) instead of \( \mathbb{Z} \setminus \mathbb{N} \).

Similarly to Theorem 2.9 of [5], we can prove the following duality theorem (see [5] for the details).

**Theorem 2.3.** Let \( 0 < q \leq 1 < p < \infty \) and \( 1/p + 1/p' = 1 \). Then

\[(HA_q^p(G))^* = \text{RCMO}_{q'}^p(G)\]

in the following sense: given \( g \in \text{RCMO}_{q'}^p(G) \), the functional \( A_g \) defined for finite combinations of atoms \( f = \sum_{j \in \text{finite}} \lambda_j a_j \in HA_q^p(G) \) by

\[
A_g(f) = \int_{G} f(x) g(x) \, dx
\]

extends uniquely to a continuous linear functional \( A_g \in (HA_q^p(G))^* \), whose \( (HA_q^p(G))^* \) norm satisfies

\[
\|A_g\| \leq C \|g\|_{\text{RCMO}_{q'}^p(G)}.
\]

Conversely, given \( \Lambda \in (HA_q^p(G))^* \), there exists a unique (up to a constant) \( g \in \text{RCMO}_{q'}^p(G) \) such that \( \Lambda = A_g \). Moreover,

\[
\|g\|_{\text{RCMO}_{q'}^p(G)} \leq C \|\Lambda\|.
\]

In addition, as an application of the theory of atomic-molecular decompositions of \( HA_{p,0}^{q,0}(G) \), we can establish certain interpolation theorems and prove boundedness theorems on multipliers (see Theorems 2.6–2.7 and Theorems 4.1–4.4 of [5]). We omit the details. In the following sections, we obtain some other applications of the atomic theory of the spaces \( HA_{p,0}^{q,0}(G) \).

### 3. The relation between \( HA_{p,0}^{1,0}(G) \) and \( Y^*(G) \).

In order to establish certain \( H^p(G) \) multiplier results, Onneweer and Quek [10] introduce a subspace \( Y^*(G) \) of \( H^1(G) \) as follows:

\[
Y^*(G) = \left\{ f \in L^1(G) : \int G f(x) \, dx = 0 \quad \text{and} \quad \|f\|_{L^1(G)} + \left\| f \log^+ \left( \frac{|f|}{\|f\|_{L^1(G)}} \right) \right\|_{L^1(G)} + \| |f(x)| \log^+ |x| \|_{L^1(G)} < \infty \right\}.
\]
On the other hand, as we point out in §2, \( HA_{p,0}^1(G) \) is also a subspace of \( H^1(G) \). In this section, as another application of atomic theory for the space \( HA_{p,0}^1(G) \), we discuss the relation between \( HA_{p,0}^1(G) \) and \( Y^*(G) \). First, we have \( HA_{p,0}^1(G) \not\subset Y^*(G) \) for \( 1 < p < \infty \).

Let \( \Gamma' \) denote the dual group of \( G \) and for each \( n \in \mathbb{Z} \) let \( \Gamma_n = \{ \gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n \} \). Take \( \gamma_0 \in \Gamma \setminus \Gamma_0 \) and define

\[
f(x) = \begin{cases} \frac{m_k}{|k|^{3/2}} \gamma_0(x), & x \in G_k \setminus G_{k+1} \text{ and } k = -1, -2, -3, \ldots, \\ 0, & \text{otherwise.} \end{cases}
\]

For \( k \in \mathbb{Z} \setminus \{0 \cup \mathbb{N}\} \), let \( b_k(x) = m_k \gamma_0(x) \chi_{G_k \setminus G_{k+1}}(x) \). Then supp \( b_k \subset G_k \) and

\[
\int b_k(x) \, dx = m_k \left( \int_{G_k \setminus G_{k+1}} \gamma_0(x) \, dx - \int_{G_{k+1}} \gamma_0(x) \, dx \right) = 0.
\]

Moreover,

\[
\|b_k\|_{L^p(G)} = m_k \left( \int_{G_k \setminus G_{k+1}} |\gamma_0(x)|^p \, dx \right)^{1/p} \leq m_k (m_k)^{-1/p} \leq \mu(G_k)^{1/p-1}.
\]

Thus, \( b_k(x) \) is a central \((1, p)_0\)-atom of restricted type. Since \( \sum_{k=-\infty}^{-1} 1/|k|^{3/2} < \infty \) and \( f(x) = \sum_{k=-\infty}^{-1} b_k(x)/|k|^{3/2} \), by Theorem 2.1 we know that \( f \in HA_{p,0}^1(G) \) and

\[
\|f\|_{HA_{p,0}^1(G)} \leq C \sum_{k=-\infty}^{-1} \frac{1}{|k|^{3/2}} < \infty.
\]

But note that \( |y| = (m_k)^{-1} \geq 2^{-k} \) if \( y \in G_k \setminus G_{k+1} \), and so \( \log^+ |y| \geq C(-k) \).

Therefore,

\[
\int |f(y)| \log^+ |y| \, dy = \sum_{k=-\infty}^{-1} \int_{G_k \setminus G_{k+1}} |f(y)| \log^+ |y| \, dy \geq \sum_{k=-\infty}^{-1} \frac{m_k}{|k|^{3/2}} (-k) \mu(G_k \setminus G_{k+1}) = \infty.
\]

Thus, \( f \not\in Y^*(G) \).

However, we have the following theorem:
Theorem 3.1. Let $1 < p < \infty$. Then
\[
\left\{ f \in A_{p,0}^1(G) : \int f(x) \, dx = 0 \right\}
\] with
\[
\|f\|_{L^1(G)} + \|f(x) \log^+ |x| \|_{L^1(G)} < \infty \}
\] \subset H A_{p,0}^1(G).

From this theorem we can deduce that \((Y^*(G) \cap A_{p,0}^1(G)) \subset H A_{p,0}^1(G)\).

Proof of Theorem 3.1. Let \(f^*(x) = \sup_{n \in \mathbb{Z}} |f \ast \Delta_n(x)|\). Write
\[
\|f\|_{H A_{p,0}^1(G)} = \|f^*\|_{A_{p,0}^1(G)} = \sum_{l = -\infty}^{-1} (m_l)^{-1/(1-p)} \|f^* \chi_{G_l \setminus G_{l+1}}\|_{L^p(G)}
\] + \(\|f^* \chi_{G_0}\|_{L^p(G)} =: I_1 + I_2\).

We first estimate \(I_1\). Suppose \(x \in G_l \setminus G_{l+1}\) for some \(l \in \mathbb{Z} \setminus \{N \cup \{0\}\}\). Let \(n \leq l\). Then \(x \in G_l \subset G_n\), and \(\Delta_n(x) = m_n\). In addition, since \(\int_{G_l} f(y) \, dy = 0\), we have
\[
|f \ast \Delta_n(x)| = \left| \int_{G_l} f(y)(\Delta_n(x-y) - \Delta_n(x)) \, dy \right|
\]
\[
\leq \int_{G_l \setminus G_i} |f(y)| |\Delta_n(x-y) - \Delta_n(x)| \, dy,
\]
where we have used the fact that if \(y \in G_l\), then \(x-y \in G_n\) and therefore \(\Delta_n(x-y) = \Delta_n(x) = m_n\). Further, we have
\[
|f \ast \Delta_n(x)| \leq 2m_n \int_{G_l \setminus G_i} |f(y)| \, dy \leq 2m_l \int_{G_l \setminus G_i} |f(y)| \, dy.
\]
Thus,
\[
I_1 = \sum_{l = -\infty}^{-1} (m_l)^{-1/(1-p)} \|f^* \chi_{G_l \setminus G_{l+1}}\|_{L^p(G)}
\]
\[
\leq C \sum_{l = -\infty}^{-1} \int_{G_l \setminus G_i} |f(y)| \, dy
\]
\[
+ \sum_{l = -\infty}^{-1} (m_l)^{-1/(1-p)} \|\sup_{n \geq l} |f \ast \Delta_n(x)|\|_{\chi_{G_l \setminus G_{l+1}}(x)}\|_{L^p(G)}
\]
\[
=: C I_1 + I_2.
\]

Note that if \(y \in G_j \setminus G_{j+1}\), we have \(|y| = (m_j)^{-1} \geq 2^{-j}\), so that
$-j \leq C \log |y| = C \log^+ |y|$ as long as $j \leq -1$. From this, we deduce that

$$II_1 = \sum_{l=-\infty}^{-1} \int_{G \setminus G_l} |f(y)| \, dy = \sum_{l=-\infty}^{-1} \sum_{j=-\infty}^{l-1} \int_{G_j \setminus G_{j+1}} |f(y)| \, dy$$

$$= \sum_{j=-\infty}^{-2} \int_{G_j \setminus G_{j+1}} |f(y)| \, dy \leq C \sum_{j=-\infty}^{-2} \int_{G_j \setminus G_{j+1}} |f(y)| \log |y| \, dy$$

$$\leq C \|f(x)\|_{L^1(G)} < \infty.$$  

Now we estimate $II_2$. Set $E_l(x) = (f \chi_{G_l \setminus G_{l+1}})(x)$. Note that

$$|f * \Delta_n(x)| \leq \int_G |f(y)| \Delta_n(x-y) \, dy.$$  

If $n > l$ and $x \in G_l \setminus G_{l+1}$, then $x \notin G_n$. Note that if $y \notin x + G_n$, then $\Delta_n(x-y) = 0$ and $x + G_n \subset x + G_{l+1} \subset (G_l \setminus G_{l+1})$. Thus, $\Delta_n(x-y) \neq 0$ only if $y \in G_l \setminus G_{l+1}$. From this it follows that for $x \in G_l \setminus G_{l+1}$ and $l \leq -1$,

$$\sup_{n>l} |f * \Delta_n(x)| \leq \sup_{n>l} \int_{G_l} |E_l(y)| \Delta_n(x-y) \, dy \leq (E_l)^* (x).$$

Therefore,

$$II_2 = \sum_{l=-\infty}^{-1} (m_l)^{-1/(1-p)} \| (\sup_{n>l} |f * \Delta_n(x)| \chi_{G_l \setminus G_{l+1}}(x)) \|_{L^p(G)}$$

$$\leq C \sum_{l=-\infty}^{-1} (m_l)^{-1/(1-p)} \| (E_l)^*(x) \|_{L^p(G)}$$

$$\leq C \sum_{l=-\infty}^{-1} (m_l)^{-1/(1-p)} \| E_l \|_{L^p(G)} \leq C \|f\|_{A^{1,1}_p(G)} < \infty,$$

where we have used the $L^p(G)$-boundedness of $(E_l)^*$ (see Kitada [4]).

For $I_2$, we first deduce that if $x \in G_0$ and $n \leq 0$, then

$$|f * \Delta_n(x)| \leq m_n \int_{x + G_n} |f(y)| \, dy \leq \|f\|_{L^1(G)}.$$
If \( n > 0 \) and \( x \in G_0 \), then \( \Delta_n(x - y) = 0 \) for \( y \notin G_0 \). Thus,
\[
|\Delta_n * f(x)| \leq \int_G |f(y)| \Delta_n(x - y) \, dy \leq \int_G |f(y)| \Delta_n(x - y) \, dy
\]
\[
= \int_{G_0} E_0(y) \Delta_n(x - y) \, dy,
\]
where we set \( E_0(y) := f(y) \chi_{G_0}(y) \). Therefore, if \( x \in G_0 \), we have
\[
f^*(x) = \sup_{n \in \mathbb{Z}} |f \ast \Delta_n(x)| \leq \sup_{n > 0} |f \ast \Delta_n(x)| + \sup_{n \leq 0} |f \ast \Delta_n(x)|
\]
\[
\leq \|f\|_{L^1(G)} + (E_0)^*(x).
\]
Thus,
\[
I_2 = \|f^*(x)\chi_{G_0}(x)\|_{L^p(G)} \leq \|f\|_{L^1(G)} + \|(E_0)^*\|_{L^p(G)}
\]
\[
\leq \|f\|_{L^1(G)} + C\|E_0\|_{L^p(G)} \leq \|f\|_{L^1(G)} + C\|f\|_{A^q_{p,0}(G)} < \infty.
\]
That is, \( f \in HA^q_{p,0}(G) \) and we finish the proof of Theorem 3.1.

4. The relation between \( HA^q_{p,0}(G) \) and the general non-homogeneous Herz spaces. In this section, we first introduce general non-homogeneous Herz spaces \( K(\beta, p, q; G) \) on locally compact Vilenkin groups. For the definitions of the general homogeneous Herz spaces on locally compact Vilenkin groups see [8] and [6].

**Definition 4.1.** Let \( \beta \in \mathbb{R} \) and \( 0 < q, p \leq \infty \). The non-homogeneous Herz space \( K(\beta, p, q; G) \) is defined by
\[
K(\beta, p, q; G) := \{ f : f \text{ is a measurable function on } G \text{ and } \|f\|_{K(\beta, p, q; G)} < \infty \},
\]
where
\[
\|f\|_{K(\beta, p, q; G)} := \left\{ \sum_{l=-\infty}^{-1} \mu(G_l)^{\beta q} \|f\chi_{G_l \setminus G_{l+1}}\|_{L^p(G)}^q + \|f\chi_{G_0}\|^q_{L^p(G)} \right\}^{1/q}.
\]
Obviously, \( K(1/q - 1/p, p, q; G) = A^q_{p,0}(G) \) for \( 0 < q \leq 1 \leq p < \infty \). Concerning the relation between the spaces \( K(\beta, p, q; G) \) and \( HA^q_{p,0}(G) \), we have the following fact.

**Theorem 4.1.** Let \( 0 < q \leq 1 < p < \infty \). If \( \beta > 1/q - 1/p \), then
\[
K(\beta, p, q; G) \cap \left\{ f \in L^1(G) : \int f(x) \, dx = 0 \right\} \subset HA^q_{p,0}(G).
\]

**Remark 4.1.** If \( q = 1 < p < \infty \), by Theorem 2.1 and the definition of
the space $H A^1_{p,0}(G)$, we obviously have
\[ H A^1_{p,0}(G) \subset K \left(1 - \frac{1}{p}, p, 1; G\right) \cap \left\{ f \in L^1(G) : \int f(x) \, dx = 0 \right\} \]
\[ = A^1_{p,0}(G) \cap \left\{ f \in L^1(G) : \int f(x) \, dx = 0 \right\}. \]
Thus, a natural question is whether the following equality holds:
\[ H A^1_{p,0}(G) = A^1_{p,0}(G) \cap \left\{ f \in L^1(G) : \int f(x) \, dx = 0 \right\}. \]

\textbf{Proof of Theorem 4.1.} Let $f \in K(\beta, p; q) \cap \left\{ f \in L^1(G) : \int f(x) \, dx = 0 \right\}$. Set $C_0 = G_0$, $C_i = G_{-i} \setminus G_{-i+1}$, $i = 1, 2, \ldots$. Write
\[ f(x) = \sum_{n=0}^{\infty} (f(x) - f_{C_n})\chi_{C_n}(x) + \sum_{n=0}^{\infty} f_{C_n}\chi_{C_n}(x) =: E(x) + F(x), \]
where $f_{C_n} = \mu(C_n)^{-1} \int_{C_n} f(t) \, dt$.
If we set
\[ M = \sup\{\text{order}(G_k/G_{k+1})\} < \infty, \]
then from $\int f(y) \, dy = 0$ we deduce that
\[ F(x) = \sum_{n=0}^{\infty} \left( \int_{C_n} f(t) \, dt \right) \frac{\chi_{C_n}(x)}{\mu(C_n)} \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{j=n+1}^{\infty} \int_{C_j} f(t) \, dt \right) \left( \frac{\chi_{C_{n+1}}(x)}{\mu(C_{n+1})} - \frac{\chi_{C_n}(x)}{\mu(C_n)} \right) \]
\[ = \sum_{n=0}^{\infty} \lambda_n^{(1)} a_n^{(1)}(x), \]
where
\[ a_n^{(1)}(x) = M^{1/p - 1} \left(1 + M^{-1/p}\right)^{-1} \mu(G_{n-1})^{1 - 1/q} \]
\[ \times \left( \frac{\chi_{C_{n+1}}(x)}{\mu(C_{n+1})} - \frac{\chi_{C_n}(x)}{\mu(C_n)} \right) \]
and
\[ \lambda_n^{(1)} = M^{1 - 1/p} \left(1 + M^{-1/p}\right) \mu(G_{n-1})^{1/q - 1} \left( \sum_{j=n+1}^{\infty} \int_{C_j} f(t) \, dt \right). \]
It is easy to see that \( \text{supp} a_n^{(1)} \subset G_{-n-1} \), \( \int a_n^{(1)}(x) \, dx = 0 \) and

\[
\|a_n^{(1)}\|_{L^p(G)} \leq \frac{1}{M^{1-1/p}(1 + M^{1-1/p})} \mu(G_{-n-1})^{1/q-1} \times \left( \frac{1}{\mu(G_{-n})^{1-1/p} + \mu(G_{-n+1})^{1-1/p}} \right) \\
\leq \frac{1}{\mu(G_{-n-1})^{1/q-1/p}}.
\]

That is, \( a_n^{(1)}(x) \) is a central \((q,p)_0\)-atom. Also, by (1.1), we have

\[
\sum_{n=0}^{\infty} |\lambda_n^{(1)}|^q \leq C \sum_{n=0}^{\infty} \mu(G_{-n-1})^{1-q} \left| \sum_{j=n+1}^{\infty} \int_{C_j} f(t) \, dt \right|^q \\
\leq C \sum_{n=0}^{\infty} \sum_{j=n+1}^{\infty} \left( \int_{C_j} |f(t)| \, dt \right)^q \mu(G_{-n-1})^{1-q} \\
\leq C \sum_{n=0}^{\infty} \sum_{j=n+1}^{\infty} \left( \int_{C_j} |f(t)|^p \, dt \right)^{q/p} \mu(C_j)^{(1-1/p)q} \mu(G_{-n-1})^{1-q} \\
\leq C \sum_{n=0}^{\infty} \sum_{j=n+1}^{\infty} \left( \int_{C_j} |f(t)|^p |t|^\beta p \, dt \right)^{q/p} \\
\times (m_j)^{\beta q - (1-1/p)q} (m_{-n-1})^{q-1} \\
\leq C \sum_{j=1}^{\infty} \left( \int_{C_j} |f(t)|^p |t|^\beta p \, dt \right)^{q/p} (m_j)^{\beta q - (1-1/p)q} \\
\times \left( \sum_{n=0}^{\infty} (m_{-n-1})^{q-1} \right) \\
\leq C \sum_{j=1}^{\infty} \left( \int_{C_j} |f(t)|^p |t|^\beta p \, dt \right)^{q/p} (m_j)^{q(\beta - 1/q + 1/p)} \\
\leq C \left( \int |f(t)|^p |t|^\beta p \, dt \right)^{q/p} \\
\leq C \|f\|_{L^p(G)}^{q(\beta - 1/q + 1/p)} \\
\leq C \left\{ \sum_{l=-\infty}^{1} \mu(G_l)^{\beta q} \left\| f_{G_l \setminus G_{l+1}} \right\|_{L^p(G)}^{q} + \|f_{G_0}\|_{L^p(G)}^{q} \right\} \\
= C \left\| f \right\|_{K(\beta,p,q,G)}^{q} < \infty.
\]

Thus, by Theorem 2.1 we know that \( F \in HA_{p,0}^1(G) \).
Now, we turn to $E(x)$. Write
\[
E(x) = \sum_{n=0}^{\infty} (f(x) - f_{C_n}) \chi_{C_n}(x)
\]
\[
= \sum_{n=0}^{\infty} \mu(G_{-n})^{1/q-1/p} \|f - f_{C_n}\|_{L^p(G)}
\times \frac{\{f(x) - f_{C_n}(x)\} \chi_{C_n}(x)}{\|f - f_{C_n}\|_{L^p(G)} \mu(G_{-n})^{1/q-1/p}}
=:\sum_{n=0}^{\infty} \lambda_n^{(2)} a_n^{(2)}(x).
\]

It is easy to see that
\[
a_n^{(2)}(x) = \frac{\{f(x) - f_{C_n}\} \chi_{C_n}(x)}{\|f - f_{C_n}\|_{L^p(G)} \mu(G_{-n})^{1/q-1/p}}
\]
is a central $(q, p)_{0}$-atom, and
\[
\sum_{n=0}^{\infty} |\lambda_n^{(2)}|^q = \sum_{n=0}^{\infty} \mu(G_{-n})^{1-q/p} \|f - f_{C_n}\|_{L^p(G)}^q
\leq \sum_{n=0}^{\infty} \mu(G_{-n})^{1-q/p} \|f \chi_{C_n}\|_{L^p(G)} + |f_{C_n}| \mu(C_n)^{1/p} q
\leq C \sum_{n=0}^{\infty} \mu(G_{-n})^{1-q/p} \|f \chi_{C_n}\|_{L^p(G)}^q
= C \|f\|_{K(1/q-1/p, p, q; G)}^q \leq C \|f\|_{K(\beta, p, q; G)}^q < \infty.
\]

Thus, by Theorem 2.1 we know that $E \in HA_{1, 0}^p(G)$. Therefore, $f = E + F \in HA_{1, 0}^p(G)$. This finishes the proof of Theorem 4.1.

REFERENCES


DEPARTMENT OF MATHEMATICS
BEIJING NORMAL UNIVERSITY
BEIJING 100875
THE PEOPLE’S REPUBLIC OF CHINA

*Reçu par la Rédaction le 13.12.1993*