C O L L O Q U I U M   M A T H E M A T I C U M

VOL. LXVIII | 1995 | FASC. 2

EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR A MODEL OF GRAVITATIONAL INTERACTION OF PARTICLES, III

BY

PIOTR BILER (WROCŁAW)

This is the third part of the study [7, 9]. We consider here local-in-time solutions of the parabolic-elliptic system from [9, (1)–(5)], which cannot be extended for all $t \ge 0$. We recall the system of partial differential equations defined in a bounded domain Ω of $\mathbb{R}^n$:

$$u_t = \Delta u + \nabla \cdot (u \nabla \varphi), \tag{1}$$

$$\Delta \varphi = u \tag{2}$$

with the nonlinear no-flux condition

$$\frac{\partial u}{\partial \nu} + u \frac{\partial \varphi}{\partial \nu} = 0, \tag{3}$$

where ν denotes the outward unit normal vector to $\partial\Omega$. For the potential φ we assume either

$$\varphi = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

or instead of the Dirichlet boundary condition above

$$\varphi = E_n * u \tag{4.2}$$

with E_n the fundamental solution of the n-dimensional Laplacian. For the initial-boundary problem we add the condition

$$u(x,0) = u_0(x) \ge 0. \tag{5}$$

For the interpretation of (1)–(5) we refer the reader to the introduction of [9].

In the one-dimensional case solutions are global in time (see the reasoning in [7, Sec. 3] based on an idea from [14], and [18]).

The first proof of nonexistence of solutions to (1)–(4) defined globally in time has been discovered for the radial problem in an n-dimensional ball, $n \ge 2$, in [7, Theorem 3]. The critical value of mass of the initial

1991 *Mathematics Subject Classification*: 35B40, 35K60, 35Q99, 82C21.

Key words and phrases: parabolic-elliptic system, blowing up solutions.

The preparation of this paper was partially supported by a KBN grant.

condition u_0 given in this theorem is $2n\sigma_n R^{n-2}$ for the ball of radius R, i.e. if $M = \int_\Omega u_0 > 2n\sigma_n R^{n-2}$ then any weak solution (in the sense of definitions in [4], [6], [8] or [9] in the radial case and the integrated densities formulation (6)–(7) in [7]) cannot exist globally in time. A more general result for star-shaped domains Ω has been proved later (but appeared in the first part of our study of the system (1)–(4.2)) in [9, Theorem 2(v)]. There a sufficient condition for nonexistence in the large of solutions is also expressed as a large mass condition $M > 2n\sigma_n(\operatorname{diam}\Omega)^{n-2}$. The idea of the proof is basically the same as in [7, Theorem 3]: we consider the evolution of the moment $w(t) = \int_\Omega u(x,t)|x|^2\,dx$ of a solution with $\int_\Omega u_0 = M$ sufficiently large.

We prove in this note a refinement of the above result. Namely, we show that if the concentration of u_0 is large enough, then no solution with the initial condition u_0 can be extended globally in time. The *concentration* is meant in a related but slightly different sense than in [9] where stationary solutions of moderate concentration appeared to be regular (and they do exist!). The approach used here is again the virial method, but now the moment w is compared with the energy integral I. Note that if $n \ge 3$, then this phenomenon of blow-up can occur for arbitrarily small mass $M = \int_\Omega u_0$.

A related result to that in [9, Theorem 2(v)], [7, Theorem 3] for the problem (1)–(4.1) in balls of $\mathbb{R}^n$ is proved here for arbitrary (not necessarily radial) large initial data. Replacing the condition (4.2) by the boundary condition (4.1) leads to a serious difficulty in the proof of blow-up of solutions.

We stress the fact that conditions ensuring blow-up of solutions (e.g. (6)) are nonlocal (even if they express a large concentration of mass in the vicinity of a point) according to the fact that the system (1)–(4) is nonlocal.

The phenomenon of nonexistence of global solutions to (1)–(4) has a different character than for semilinear parabolic equations with sources studied e.g. in [11–12], where the blow-up is asymptotically self-similar and local criteria for excluding blow-up can be given. For our system there are threshold values of the concentration of initial densities leading to a finite time blow-up (compare also [1] dealing with some quasilinear parabolic equations with transport terms). These size conditions on u_0 appear as a result of the competition between the diffusion modeled in (1) by the linear term Δu and the nonlinear source term $u\Delta\varphi = u^2$ with spreading effects described by $\nabla u \cdot \nabla \varphi$.

We expect that understanding the mechanism of formation of large concentration clusters leading to gravitational collapse phenomena would help in determining sharp conditions for the existence of local solutions for initial data less regular than $u_0 \in L^p(\Omega)$ with $p > n/2$. We recall that the construction of weak solutions with such u_0's applies for the system (1)–(4)

like for the Debye system in [6]. In [9] we conjectured that existence of even weaker solutions (e.g. in the spirit of [15]) is possible only for u_0 not far from $L^p(\Omega)$, $p > n/2$. In particular, singularities of u_0 exceeding $|x - x_0|^{-2}$ should be excluded when we expect to obtain not too wild solutions of the evolution problem (i.e. with locally bounded potential φ), similarly to the case of stationary solutions studied in [8, Theorem 2(i)–(iii)] and [9, Theorem 1(ii)].

This conjecture was based on an analogous result in [2–3] for the semilinear parabolic equation $u_t = \Delta u + u^2$ that resembles (1) written as $u_t = \Delta u + u^2 + \nabla u \cdot \nabla\varphi$. The conditions for the existence of local solutions involved quantities which can be identified as the norm in the Morrey space $M^{n/2}(\Omega)$ more vast than $L^{n/2}(\Omega)$. However, methods pertinent to the semilinear case in [2–3] seem to be inapplicable to (1)–(2). We discuss questions of the minimal regularity of initial data for the Cauchy problem in $\mathbb{R}^n$ for (1), (2), (4.2) (converted into an integral equation) in the paper [5].

THEOREM 1. *If $\Omega \subset \mathbb{R}^n$, $n \ge 3$, is a star-shaped domain (with respect to the origin), then for u_0 such that $|u_0|_1 = M$ and*

$$\Big(M^{-1} \int_\Omega u_0(x)|x|^2\,dx\Big)^{n/2-1} < (2^{n/2} n\sigma_n)^{-1} M \tag{6}$$

there is no global solution to (1)–(3), (4.2), (5).

Proof. Define an auxiliary function

$$w(t) = \int_\Omega u(x,t)|x|^2\,dx, \tag{7}$$

the same as that considered in the proof of Theorem 2(v) in [9]. The function w is the second moment of the density u, so it measures the concentration of u at the origin. For any weak solution u of (1)–(4.2) we have

$$\begin{aligned} \frac{dw}{dt} &= -\int_\Omega (\nabla u + u\nabla\varphi)\cdot\nabla(|x|^2) = -2\int_\Omega \nabla u\cdot x - 2\int_\Omega u\nabla\varphi\cdot x \\ &= -2\int_{\partial\Omega} ux\cdot\nu + 2n\int_\Omega u \\ &\quad - 2\iint_{\Omega\times\Omega} u(x,t)(\nabla_x E(x-y))\cdot x)u(y,t)\,dy\,dx. \end{aligned} \tag{8}$$

Since Ω is star-shaped with respect to $x_0 = 0$ we have $x\cdot\nu \ge 0$ on $\partial\Omega$ and therefore

$$\frac{dw}{dt} \le 2nM - \frac{2}{\sigma_n}\iint_{\Omega\times\Omega} u(x,t)u(y,t)|x-y|^{-n}(|x|^2 - y\cdot x)\,dy\,dx \tag{9}$$

$$= 2nM - \frac{1}{\sigma_n} \iint_{\Omega\times\Omega} u(x,t)u(y,t) \times |x-y|^{-n}(|x|^2 - y\cdot x - x\cdot y + |y|^2)\,dy\,dx$$
$$= 2nM - \frac{1}{\sigma_n} \iint_{\Omega\times\Omega} u(x,t)u(y,t)|x-y|^{-n+2}\,dy\,dx$$
$$\equiv 2nM - \sigma_n^{-1} I,$$

by symmetry properties of the integral

$$I = \iint_{\Omega\times\Omega} u(x,t)u(y,t)|x-y|^{-n+2}\,dx\,dy.$$

Now we estimate I in a more careful manner than in [9]. Namely, using the Hölder inequality we can write

$$M^2 = \iint_{\Omega\times\Omega} u(x,t)u(y,t)\,dx\,dy$$
$$\le \Big(\iint_{\Omega\times\Omega} u(x,t)u(y,t)|x-y|^2\,dx\,dy\Big)^{1-2/n} \times \Big(\iint_{\Omega\times\Omega} u(x,t)u(y,t)|x-y|^{-n+2}\,dx\,dy\Big)^{2/n}$$
$$\le \Big(\iint_{\Omega\times\Omega} u(x,t)u(y,t)(|x|^2+|y|^2-2x\cdot y)\,dx\,dy\Big)^{1-2/n} I^{2/n}$$
$$= \Big(2Mw - 2\Big|\int_\Omega xu(x,t)\,dx\Big|^2\Big)^{1-2/n} I^{2/n} \le (2Mw(t))^{1-2/n} I^{2/n}.$$

This leads to

$$2^{1-n/2} M^{n/2+1} w(t)^{1-n/2} \le I, \tag{10}$$

and together with (9) we obtain

$$\frac{dw}{dt} \le 2nM - 2^{1-n/2}\sigma_n^{-1} M^{1+n/2} w^{1-n/2}.$$

The assumption (6) on the concentration of u_0 in Theorem 1 and monotonicity of the function on the right hand side of the above inequality allow us to write

$$\frac{2}{n}\frac{d(w^{n/2})}{dt} \le 2nMw^{n/2-1} - 2^{1-n/2}\sigma_n^{-1} M^{1+n/2}$$
$$= 2nM^{n/2}((w/M)^{n/2-1} - (2^{n/2} n\sigma_n)^{-1} M$$
$$\le 2nM^{n/2}((w(0)/M)^{n/2-1} - (2^{n/2} n\sigma_n)^{-1} M) < 0$$

(observe that $\frac{dw}{dt}(0)$ is strictly negative). After an integration this gives

$$w(t)^{n/2} \le w(0)^{n/2} - n^2 M^{n/2}((2^{n/2} n\sigma_n)^{-1} M - (w(0)/M)^{n/2-1})t,$$

which implies that

$$t \le T = w(0)^{n/2}(n^2 M^{n/2}((2^{n/2} n\sigma_n)^{-1} M - (w(0)/M)^{n/2-1}))^{-1},$$

unless $w(t) < 0$ which is absurd.

Note that the condition (6) sufficient for the finite time blow-up is satisfied for many initial densities u_0. For instance, if $u_0(x) = Mn\sigma_n^{-1} R^{-n}$ when $|x| \le R$ and $u_0(x) = 0$ elsewhere, then $\int_\Omega u_0 = M$ and (6) becomes

$$\left(\frac{n}{n+2}\right)^{n/2-1} R^{n-2} < (2^{n/2} n\sigma_n)^{-1} M, \tag{6'}$$

which is true for any fixed $M > 0$ and sufficiently small $R > 0$. The above condition (6′) is a quantitative expression of the wording "u_0 is of large concentration". Using the notion of the Morrey spaces considered in [9] for a description of regular stationary solutions to (1)–(4), we may interpret (6′) by saying that the $M^{n/2}(\Omega)$ norm of u_0 is large.

Similarly, for the Gaussian initial densities $u_0(x) = Mp_\varepsilon(x)(\int_\Omega p_\varepsilon)^{-1}$ in Ω, where $p_\varepsilon(x) = (2\pi\varepsilon)^{-n/2} \exp(-|x|^2/(2\varepsilon))$, $\varepsilon > 0$, and $M > 0$ is fixed, $w(0)$ is of order ε and $I(0) \sim \varepsilon^{2-n}$. Hence the blow-up occurs before $T \sim \varepsilon^{n/2} \sim |u_0|_\infty^{-1}$.

Given arbitrary $u_0 \not\equiv 0$ the sufficient blow-up condition (6) can be easily satisfied for the rescaled u_0, i.e. by taking $u_{0,\lambda}(x) = \lambda^n u_0(\lambda x)$ with $\lambda \ge 1$ large enough. Indeed, under this scaling the mass M is conserved, the moments are $w_\lambda = \lambda^{-2} w$, so (6) becomes $\lambda^{2-n} < c_n M$ with some constant $c_n > 0$.

Let us remark that this reasoning applies even to less regular solutions than those considered in [8], [9], e.g. to (suitably modified) L^1 solutions considered in [15].

We recover qualitatively the condition from [9, Theorem 2(v)]: $M > 2n\sigma_n d^{n-2}$, $d = \operatorname{diam} \Omega$, by estimating $(w/M)^{n/2-1}$ crudely by d^{n-2}.

Evidently, for M so large there are no (regular) stationary solutions to (1)–(3), (4.2)—otherwise they could be considered as global solutions to the evolution problem.

It is clear that if (6) holds then its modification with $\int_\Omega u_0(x)|x - \overline{x}|^2\, dx$ and $\overline{x} = M^{-1} \int_\Omega x u_0(x)\, dx$ (the center of mass of u_0) is also true. Moreover, if Ω is star-shaped with respect to $\overline{x}$, then the above proof is valid for the modified moment $\overline{w}(t) = \int_\Omega u(x,t)|x - \overline{x}|^2\, dx$.

Remark. The nonexistence result in Theorem 1 can be generalized to a larger class of domains including some non-star-shaped dumbbell-like ones.

The idea (reminiscent of [13, Prop. 1.7]) is to consider domains $\Omega \subset \mathbb{R}^n$ such that $\widetilde{x}\cdot\widetilde{\nu} \ge 0$ on $\partial\Omega$, where $\sim: x \mapsto \widetilde{x}$ is a nonisotropic dilation defined by $\widetilde{x} = (a_1x_1, \ldots, a_nx_n)$ for $x = (x_1, \ldots, x_n)$ and some fixed $a_1, \ldots, a_n > 0$. In the proof a modified moment function $\widetilde{w}(t) = \int_\Omega u(x,t)|\widetilde{x}|^2\,dx \ge (\min_{1\le k\le n} a_k^2)w(t)$ is considered. The counterpart of the integral I is

$$\widetilde{I} = \iint_{\Omega\times\Omega} u(x,t)u(y,t)|x-y|^{-n}|\widetilde{x}-\widetilde{y}|^2\,dx\,dy \ge (\min_{1\le k\le n} a_k^2)I,$$

and the sufficient blow-up condition (6) becomes

$$(\widetilde{w}(0)/M)^{n/2-1} < (\min_{1\le k\le n} a_k)^n\Big(2^{n/2}\Big(\sum_{k=1}^n a_k^2\Big)\sigma_n\Big)^{-1}M.$$

Recall that the system (1)–(3), (4.2) has an approximate Lyapunov function (see [9, (23), (24.2)]), which (for $n \ge 3$) can be written in the form

$$W(t) = -h(u) - (2\sigma_n)^{-1}I. \tag{11}$$

Here $h(u) = -\int_\Omega u\log u$ is the (differential) entropy of the density u, and the energy integral I is defined in (9). More precisely, if for some u_0, $h(u_0)$ and I are finite, then W provides us with a kind of a priori estimate

$$W(t) \le W(0) + C(\Omega, M), \qquad M = \int_\Omega u_0. \tag{12}$$

The proof of (12) is similar to that of (24.2) for $n = 2$ given in [9]. In general, the information contained in (12) is meaningless since both terms $-h$ and I may grow in time, but for $n = 2$ and small M the function W was useful in proving the global existence of solutions (cf. [9, Th. 2(iv)]).

We give below another proof of the nonexistence of global solutions to (1)–(3), (4.2) satisfying (12), still based on virial calculations, but this time a sufficient condition for blow-up is expressed in terms of $W(0)$ and its ingredients h, I.

First we formulate a version of the Shannon inequality for the entropy h and the second moment w of a finite measure with density $u \ge 0$.

LEMMA 1. *If Ω is a subset of $\mathbb{R}^n$, $u \ge 0$, $\int_\Omega u = M$, $h(u) = -\int_\Omega u\log u$, $\overline{x} = M^{-1}\int_\Omega xu(x)\,dx$ is the center of mass of u and $\overline{w} = \int_\Omega u(x)|x-\overline{x}|^2\,dx$, then*

$$-h(u) \ge c_1 - c_2\log\overline{w}$$

with some constants $c_j = c_j(\Omega, M)$, $j = 1, 2$, $c_2 > 0$, independent of u.

Proof. The classical Shannon inequality for $\Omega = \mathbb{R}^n$ and $M = 1$ reads

$$h(u) \le \frac{n}{2}(1 + \log(2\pi\overline{w})),$$

with equality if and only if u is a Gaussian density (see [10, p. 249], [16, Ch. 9.1] and [17, Th. 1.11, Prob. 1.34]). Its demonstration follows by maximizing the entropy under the constraint that $\overline{w}$ is fixed. Rescaling u we obtain Lemma 1 with $C_1 = M((n/2+1)\log M-(n/2)(1+\log 2\pi))$, $C_2 = Mn/2$, which are optimal constants in the case $\Omega = \mathbb{R}^n$. Explicit values of the optimal $c_j(\Omega, M)$ are known only for particular Ω's. Since $w \geq \overline{w}$, the inequality

$$h(u) + c_1 \leq c_2 \log w \tag{13}$$

is a consequence of Lemma 1.

PROPOSITION 1. *If* $\Omega \subset \mathbb{R}^n$, $n \geq 3$, *is a star-shaped domain*,

$$W(0) = \int_\Omega u_0 \log u_0 - (2\sigma_n)^{-1} \iint_{\Omega\times\Omega} u_0(x)u_0(y)|x-y|^{-n+2}\,dx\,dy$$

is (*negative and*) *small enough*, *then there is no global solution to* (1)–(3), (4.2), (5).

Proof. From (9), (11) and (12) we get

$$\frac{dw}{dt} \leq 2nM - \sigma_n^{-1} I = 2nM + 2(W + h(u)) \tag{14}$$
$$\leq 2nM + 2W(0) + c_3(\Omega, M) + 2c_2 \log w.$$

If the right hand side of this inequality is strictly negative for $t = 0$, then we will arrive at a contradiction in the same way as in the proof of Theorem 1. Of course, there exist u_0's which satisfy the above condition (i.e. I large compared with $-h(u_0)$), e.g. the Gaussian ones (for small $\varepsilon > 0$) considered for an illustration of Theorem 1.

For $n = 2$, (14) is much simpler: $dw/dt \leq M(8\pi - M)/(2\pi)$. This shows that for $M > 8\pi$ solutions cease to exist because $w(t)$ tends to 0, so by the Shannon inequality $h(u)$ tends to $-\infty$.

Remark. The proof of Theorem 1, although indirectly, sheds some light on the question: How do the nonglobal solutions explode? (cf. the last remark in [9]).

The relation $w(t) \to 0$ as $t \to T$ implies $h(u) \to -\infty$ for the entropy of $u(t)$, hence $I = -2\sigma_n(W + h) \to \infty$ for solutions that admit the Lyapunov function (11).

Moreover, if

$$\|u; M^p(\Omega)\| \equiv \sup_{x\in\Omega, R>0} R^{n(1/p-1)} \int_{B_R(x)\cap\Omega} |u|$$

is the Morrey norm with exponent $1 < p < \infty$, then $\|u; M^p(\Omega)\|$ also tends to ∞ as $t \to T$. Indeed, for $u \geq 0$ with $\int_\Omega u = M$, the inequality

$$\|u; M^p(\Omega)\| \geq C_{n,p} M(M/w)^{(1-1/p)n/2} \tag{15}$$

holds with some constant $C_{n,p} > 0$. This follows from an obvious inequality $w \ge R^2 \int_{\Omega\setminus B_R} u$ so

$$R^{n(1/p-1)} \int\limits_{B_R} u \ge R^{-n}(R^{n/p}M - R^{n/p-2}w)$$

after the optimization (i.e. when $R = (n+2-n/p)(n-n/p)^{-1}wM^{-1}$).

Of course, all this does not give full information on when the solutions cease to exist, because the true existence time is strictly less than the blow-up time estimated by the indirect argument in the proof of Theorem 1. Nevertheless, together with [9, Th. 2(iv)] and some partial results concerning the continuation of solutions for $n \ge 3$ (e.g. a locally uniform estimate of type $\sup_{[0,T]}(|h(u)| + \|u; M^{n/2}(\Omega)\|) < \infty$ is sufficient for the continuation beyond $t = T$), this helps in the understanding of high concentration phenomena leading to a collapse of solutions.

Now we turn to the problem (1)–(3), (4.1) in a ball with (not necessarily radial) data (5).

THEOREM 2. *If Ω is a ball in $\mathbb{R}^n$, $n \ge 2$, and $|u_0|_1 = M$ is sufficiently large, then there is no global solution to the problem* (1)–(3), (4.1), (5).

Proof. We begin with some general computation valid for arbitrary star-shaped domains Ω in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$. Given a function ψ belonging to the Sobolev space $H^2(\Omega)$ define the moment functional

$$v(t) = \int\limits_{\Omega} u(x,t)\psi(x)\,dx. \tag{16}$$

Similarly to the proof of Theorem 1 we obtain a counterpart of (8),

$$\begin{aligned}\frac{dv}{dt} &= -\int\limits_{\Omega} (\nabla u + u\nabla\varphi)\cdot\nabla\psi \\ &= -\int\limits_{\partial\Omega} u\frac{\partial\psi}{\partial\nu} + \int\limits_{\Omega} u\Delta\psi - \int\limits_{\Omega} u\nabla\varphi\cdot\nabla\psi.\end{aligned} \tag{17}$$

Suppose that $0 \not\equiv \psi \ge 0$, $\partial\psi/\partial\nu \le 0$ and $-\Delta\psi \le C$ for some $C = C(\overline{\Omega}) > 0$ (the idea is that $\psi(x) \sim \mathrm{dist}(x, \partial\Omega)$, and the assumption $\partial\Omega \in C^2$ is used here). From (17) by symmetrization we arrive at an analogue of (9),

$$\begin{aligned}\frac{dv}{dt} &\ge -C\int\limits_{\Omega} u - \iint\limits_{\Omega\times\Omega} u(x,t)u(y,t)\nabla_x G(x,y)\cdot\nabla\psi(x)\,dy\,dx \\ &= -CM - \frac{1}{2}\iint\limits_{\Omega\times\Omega} u(x,t)u(y,t) \\ &\quad\times(\nabla_x G(x,y)\cdot\nabla\psi(x) + \nabla_y G(y,x)\cdot\nabla\psi(y))\,dy\,dx,\end{aligned} \tag{18}$$

where $G = G_\Omega$ is the Green function of the domain Ω. Now if the condition

$$\varrho(x,y) \equiv -(\nabla_x G(x,y)\cdot\nabla\psi(x) + \nabla_y G(y,x)\cdot\nabla\psi(y)) \\ \left(= -\frac{d}{ds}G(x+s\nabla\psi(x), y+s\nabla\psi(y))|_{s=0}\right) \geq \varepsilon\psi(x)\psi(y) \tag{19}$$

is satisfied for some $\varepsilon > 0$ and all $x, y \in \Omega$, then

$$\frac{dv}{dt} \geq -CM + \frac{\varepsilon}{2}\iint_{\Omega\times\Omega} u(x,t)\psi(x)\,u(y,t)\psi(y)\,dy\,dx \geq -CM + \frac{\varepsilon}{2}v(t)^2. \tag{20}$$

Integrating the differential inequality (20) we obtain

$$v(t) \geq \beta\frac{1+k\exp(\varepsilon\beta t)}{1-k\exp(\varepsilon\beta t)} \tag{21}$$

with $k = (v(0)-\beta)/(v(0)+\beta) \in (0,1)$ and $\beta = (2CM/\varepsilon)^{1/2}$ whenever $v(0) > \beta$. The inequality (21) means that $v(t)$ blows up in a finite time, contradicting the global existence of solutions.

Now we check the condition (19) for the ball $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 2$. Formally the computation below is valid for $n > 2$ but the case $n = 2$ is analogous. Take $\psi(x) = R^2 - |x|^2$ (so $\nabla\psi(x) = -2x$, $\Delta\psi = -2n \equiv -C$), and consider

$$G_{B_R}(x,y) = ((n-2)\sigma_n)^{-1}(-|x-y|^{2-n} + R^{n-2}|R^2x/|x| - y|x||^{2-n}),$$

so

$$\frac{1}{2}\varrho(x,y) = \frac{d}{ds}G_{B_R}(sx,sy)|_{s=1} \\ = \sigma_n^{-1}(|x-y|^{2-n} - 2R^{n-2}(|x|^2|y|^2 - R^2x\cdot y)|R^2x/|x| - y|x||^{-n}).$$

By homogeneity it suffices to consider $R = 1$. First observe that $\varrho(x,y) \geq 0$ and $\varrho(x,y) = 0$ if and only if $|x| = |y| = 1$. To see this we represent

$$\begin{aligned} &\tfrac{1}{2}\sigma_n\varrho(x,y)|x-y|^n|x/|x| - y|x||^n \\ &\quad= |x-y|^2|x/|x| - y|x||^n + (-2|x|^2|y|^2 + 2x\cdot y)|x-y|^n \\ &\quad= |x-y|^2(|x/|x| - y|x||^n - |x-y|^n) \\ &\qquad+ (|x|^2 + |y|^2 - 2|x|^2|y|^2)|x-y|^n, \end{aligned} \tag{22}$$

and note that for $|x|, |y| \leq 1$,

$$|x/|x| - y|x||^2 - |x-y|^2 = (1-|x|^2)(1-|y|^2) \geq 0$$

(therefore $|x/|x| - y|x|| \geq |x-y|$), and

$$|x|^2 + |y|^2 - 2|x|^2|y|^2 = (|x| - |y|)^2 + 2|x||y|(1-|x||y|) \geq 0.$$

Using (22) we can easily check that $\inf \varrho(x,y)(1-|x|^2)^{-1}(1-|y|^2)^{-1} \equiv \varepsilon > 0$.

For $n = 2$ recall that $G_{B_1}(x, y) = (2\pi)^{-1}(\log |x - y| - \log |x/|x| - y|x||)$, hence $\pi\varrho(x, y) = |x/|x| - y|x||^{-2}(1 - |x|^2|y|^2)$.

We do not know whether (19) holds for arbitrary star-shaped domains in $\mathbb{R}^n$.

Acknowledgements. The author would like to thank Jan Goncerzewicz, Andrzej Krzywicki, Tadeusz Nadzieja and Michel Pierre for interesting discussions, and Anzelm Iwanik for the reference [10]. Part of this research was done while the author was visiting the universities of Nancy-1 and Paris–Val de Marne in 1993.

REFERENCES

[1] J. Aguirre and M. Escobedo, *On the blow-up of solutions of a convective reaction-diffusion equation*, Proc. Roy. Soc. Edinburgh 123A (1993), 433–460.

[2] P. Baras et M. Pierre, *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. 18 (1984), 111–149.

[3] —, —, *Critère d'existence de solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré Anal. Non Linéaire 2 (1985), 185–212.

[4] P. Biler, *Existence and asymptotics of solutions for a parabolic-elliptic system with nonlinear no-flux boundary conditions*, Nonlinear Anal. 19 (1992), 1121–1136.

[5] —, *The Cauchy problem and self-similar solutions for a nonlinear parabolic equation*, preprint, 1994.

[6] P. Biler, W. Hebisch and T. Nadzieja, *The Debye system: existence and large time behavior of solutions*, Nonlinear Anal. 23 (1994), 1189–1209.

[7] P. Biler, D. Hilhorst and T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interaction of particles, II*, Colloq. Math. 67 (1994), 297–308.

[8] P. Biler and T. Nadzieja, *A class of nonlocal parabolic problems occurring in statistical mechanics*, ibid. 66 (1993), 131–145.

[9] —, —, *Existence and nonexistence of solutions for a model of gravitational interaction of particles, I*, ibid. 66 (1994), 319–334.

[10] Y. Derriennic, *Entropie, théorèmes limite et marches aléatoires*, in: Probability Measures on Groups VIII, H. Heyer (ed.), Lecture Notes in Math. 1210, Springer, Berlin, 1986, 241–284.

[11] Y. Giga and R. V. Kohn, *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. 38 (1985), 297–319.

[12] —, —, *Nondegeneracy of blowup for semilinear heat equations*, ibid. 42 (1989), 845–884.

[13] O. Kavian, *A remark on the blowing-up solutions to the Cauchy problem for nonlinear Schrödinger equations*, Trans. Amer. Math. Soc. 299 (1987), 193–203.

[14] A. Krzywicki and T. Nadzieja, *A nonstationary problem in the theory of electrolytes*, Quart. Appl. Math. 50 (1992), 105–107.

[15] A. A. Lacey and D. E. Tzanetis, *Global unbounded solutions to a parabolic equation*, J. Differential Equations 101 (1993), 80–102.

[16] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics*, Springer, New York, 1994.

[17] R. McEliece, *The Theory of Information and Coding*, Encyclopedia Math. Appl. 3, Addison-Wesley, Reading, 1977.

[18] T. Nadzieja, *A model of radially symmetric cloud of self-attracting particles*, Applicationes Math., to appear.

MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND
E-mail: BILER@MATH.UNI.WROC.PL

Reçu par la Rédaction le 15.6.1994