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SET MAPPINGS ON GENERALIZED LINEAR CONTINUA

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A major branch of combinatorial set theory concerns *set mappings*, i.e., functions $f : X \to P(X)$, where one seeks for a large *free subset*, that is, a subset $Y \subseteq X$ such that $x \notin f(y)$ for any two distinct $x, y \in Y$. P. Erdős [2] proved that if the ground set is the reals and $f(x)$ is nowhere dense for $x \in \mathbb{R}$ then there is an infinite free set. Bagemihl [1] extended this by showing that in fact an everywhere dense free set exists. Muthuvel became interested in these questions for $\mathbb{R}_\kappa$, the linear continuum of zero-one sequences of length κ for $\kappa > \omega$ [3, 4]. He showed that under GCH if κ is regular there always exists a free set of cardinal κ [4]. We improve this result by showing that an everywhere dense free set exists.

If κ is a regular cardinal then $\mathbb{R}_\kappa$ consists of all nonconstant functions f from κ into $2 = \{0, 1\}$ such that there is no last $\alpha < \kappa$ such that $f(\alpha) = 0$. We order $\mathbb{R}_\kappa$ by the lexicographic ordering. Subsets of $\mathbb{R}_\kappa$ of the form $I(g) = \{f \in \mathbb{R}_\kappa : f \supseteq g\}$ where $g : \gamma \to 2$ for some $\gamma < \kappa$ are the *intervals*. So what we call intervals are really the nonempty dyadic intervals. Let $\mathcal{I}$ be the set of intervals; notice that $|\mathcal{I}| = 2^{<\kappa} = \kappa$ under GCH.

A set $A \subseteq \mathbb{R}_\kappa$ is *everywhere dense* if $A \cap I \neq \emptyset$ for every interval I; A is *nowhere dense* if for every interval I there is a subinterval $I' \subseteq I$ such that $A \cap I' = \emptyset$; A is of *first category* if it is the union of κ nowhere dense sets; otherwise, it is of *second category*.

First we give a (probably well known) construction of a strong Luzin type set in $\mathbb{R}_\kappa$.

LEMMA. (GCH) *There is a set $A = \{r(\alpha) : \alpha < \kappa^+\} \subseteq \mathbb{R}_\kappa$ such that A is of second category in every interval and whenever $\{x^\alpha_\xi : \xi < \mu(\alpha)\}$ are disjoint subsets of κ^+ with $\mu(\alpha) < \kappa$ then there is an $S \subseteq \kappa^+$, $|S| = \kappa$, and there are intervals J_ξ $(\xi < \mu)$ such that $\mu(\alpha) = \mu$ $(\alpha \in S)$ and if $J'_\xi \subseteq J_\xi$ are subintervals $(\xi < \mu)$ then there is an $\alpha \in S$ such that $r(x^\alpha_\xi) \in J'_\xi$ for every $\xi < \mu$.*

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Proof. First we remark that the assumption on $|S|$ is inessential as it is obvious from the other assumptions that if there is a good set then there is one of cardinal κ.

By GCH there are κ^+ objects of the form $((d_\xi : \xi < \mu), F)$ where $\mu < \kappa$, d_ξ are distinct functions $\gamma \to 2$ for some $\gamma < \kappa$, and F is a function from $\mu \times \mathcal{I}$ such that $F(\tau, I)$ is always a subinterval of I. We can, therefore, enumerate them as $\{((d^\alpha_\xi : \xi < \mu(\alpha)), F_\alpha) : \alpha < \kappa^+\}$.

To construct $r(\alpha)$ for $\alpha < \kappa^+$ we re-order α as $\{\beta(\tau) : \tau < \kappa\}$ and determine successively the digits of $r(\alpha)$, or, in other words, define a descending sequence of intervals $\{I^\alpha_\tau : \tau < \kappa\}$ and get $r(\alpha)$ as the unique element of the intersection.

At step $\tau < \kappa$ shrink $\bigcap\{I^\alpha_{\tau'} : \tau' < \tau\}$ to an interval $\widehat{I}^\alpha_\tau$ such that there is at most one $d = d^{\beta(\tau)}_\xi$ for some $\xi < \mu(\beta(\tau))$ such that $\widehat{I}^\alpha_\tau \subseteq I(d)$. This is possible as those intervals are disjoint. Then add a final 0 and 1 (this will ensure that $r(\alpha)$ has cofinally many zeroes and ones, i.e., $r(\alpha) \in \mathbb{R}_\kappa$). For $\tau = 0$ let $\widehat{I}^\alpha_0$ be an arbitrary interval; we only require that every interval should occur κ^+ times as $\widehat{I}^\alpha_0$. Then let $I^\alpha_\tau = F_{\beta(\tau)}(\xi, \widehat{I}^\alpha_\tau)$, where ξ is the above index. Finally, as we have already said, $\{r(\alpha)\} = \bigcap\{I^\alpha_\tau : \tau < \kappa\}$.

Assume now that this construction fails to meet the requirements of the Lemma and $\{x^\alpha_\xi : \xi < \mu(\alpha)\}$ witness this. Then they are disjoint subsets of κ^+ for $\alpha < \kappa^+$, $\mu(\alpha) < \kappa$. Without loss of generality, we can assume that $\mu(\alpha) = \mu$ for $\alpha < \kappa^+$. We can also assume that there exists a $\gamma < \kappa$ such that $r(x^\alpha_\xi)|\gamma = d_\xi$ for $\xi < \mu$ and these μ functions from γ into 2 are different.

By our indirect assumption there is a function $F : \mu \times \mathcal{I} \to \mathcal{I}$ such that given I_ξ $(\xi < \mu)$ there is no $\alpha < \kappa^+$ such that $r(x^\alpha_\xi) \in F(\xi, I_\xi)$ for every $\xi < \mu$. There is an $\varepsilon < \kappa^+$ such that $\mu(\varepsilon) = \mu$ and $((d_\xi : \xi < \mu), F) = ((d^\varepsilon_\xi : \xi < \mu), F_\varepsilon)$.

Let now α be so large that $\varepsilon < x^\alpha_\xi$ for $\xi < \mu$. When $r(x^\alpha_\xi)$ was constructed ε occurred in the $\tau(\xi)$th step for a $\tau(\xi) < \kappa$. We know that $\widehat{I}^{x^\alpha_\xi}_{\tau(\xi)} \subseteq I(d_\xi)$ by assumption. Then we get the interval $I^{x^\alpha_\xi}_{\tau(\xi)} = F(\xi, I')$ for an $I' = \widehat{I}^{x^\alpha_\xi}_{\tau(\xi)}$. But then $\{x^\alpha_\xi : \xi < \mu\}$ contradict what was assumed about F.

We need to show that A is of second category in every interval. Assume not. Then there is an interval I and there are functions $F^\xi : \mathcal{I} \to \mathcal{I}$ (for $\xi < \kappa$) such that $I \cap A$ decomposes as $I \cap A = \bigcup\{A_\xi : \xi < \kappa\}$, $F^\xi(I') \subseteq I'$ for every $I' \subseteq I$ and $A_\xi \cap F^\xi(I') = \emptyset$ for $\xi < \kappa$. Put $I = I(d)$. Then (d, F^ξ) occurs in the above enumeration at the $\alpha(\xi)$th step (say). Select $\alpha > \alpha(\xi)$ such that $\widehat{I}^\alpha_0 = I$. Then the successive digits of $r(\alpha)$ are so chosen that $r(\alpha) \in I$, and for every $\xi < \kappa$, $r(\alpha) \in F^\xi(I')$ for some $I' \subseteq I$, i.e.,

$r(\alpha) \notin A_\xi$, that is, $r(\alpha) \notin A$, a contradiction. ∎

THEOREM. (GCH) *If* κ *is regular and* $f(x) \subseteq \mathbb{R}_\kappa$ *is nowhere dense for* $x \in \mathbb{R}_\kappa$ *then there is an everywhere dense free set.*

Let $A \subseteq \mathbb{R}_\kappa$ be as in the Lemma. Enumerate $\mathcal{I}$ as $\{I_\mu : \mu < \kappa\}$. For some $\mu < \kappa$ the sequence $\{x_\xi : \xi < \mu\}$ is called *bad* if $f(y) \cap \{r(x_\xi) : \xi < \mu\} \neq \emptyset$ for all but first category many y in $A \cap I_\mu$.

CLAIM. *There is a* $\delta < \kappa^+$ *such that no set* $\{x_\xi : \xi < \mu\}$ *with* $x_\xi > \delta$ $(\xi < \mu)$ *is bad.*

Proof. Otherwise for every $\delta < \kappa^+$ there is a bad set above it, so by transfinite recursion we can choose disjoint bad sets $\{x^\alpha_\xi : \xi < \mu(\alpha)\}$. The Lemma gives S, μ, and certain intervals J_ξ. As $\{x^\alpha_\xi : \xi < \mu\}$ is bad for $\alpha \in S$ and $|S| = \kappa$ all but first category many elements $r(y)$ of $A \cap I_\mu$ have $f(r(y)) \cap \{r(x^\alpha_\xi) : \xi < \mu(\alpha)\} \neq \emptyset$. Let $r(y)$ be such an element. As $f(r(y))$ is nowhere dense there exist subintervals $J'_\xi \subseteq J_\xi$ such that $J'_\xi \cap f(r(y)) = \emptyset$. But then, for some $\alpha \in S$, $r(x^\alpha_\xi) \in J'_\xi$ for $\xi < \mu$, and yet $f(r(y)) \cap \{r(x^\alpha_\xi) : \xi < \mu(\alpha)\} \neq \emptyset$, a contradiction. ∎

If we now have δ as in the Claim, we can select by transfinite induction the free $\{r(x_\xi) : \xi < \kappa\}$, $x_\xi > \delta$, $r(x_\xi) \in A \cap I_\xi$ at every step we have a second category set of good extensions. ∎

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