MINIMAX THEOREMS WITH APPLICATIONS TO CONVEX METRIC SPACES

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A minimax theorem is proved which contains a recent result of Pinelis and a version of the classical minimax theorem of Ky Fan as special cases. Some applications to the theory of convex metric spaces (farthest points, rendez-vous value) are presented.

1. Preliminaries. Throughout this paper let two nonvoid sets $X$ and $Y$, a nonvoid convex subset $C$ of $\mathbb{R} \cup \{-\infty\}$ and a function $a : X \times Y \to C$ with

$$\sup_{x \in X} a(x, y) \in C \quad \forall y \in Y$$

be given. The following notation will be used:

- We set

$$\sup a := \sup \{a(x, y) : x \in X, y \in Y\},$$

$$a^* := \inf_{y \in Y} \sup_{x \in X} a(x, y),$$

$$Y^* := \{\sup_{x \in X} a(x, \cdot) = a^*\} = \{y \in Y : \sup_{x \in X} a(x, y) = a^*\},$$

$$\hat{X} := \bigcap_{y \in Y} \{a(\cdot, y) = \sup a\},$$

$$X(y) := \{a(\cdot, y) = \sup_{x \in X} a(x, y)\}, \quad y \in Y,$$

$$X(B) := \bigcap_{y \in B} X(y), \quad B \subset Y, \quad \text{with } X(\emptyset) = X,$$

$$\mathcal{R} := \{\{a(\cdot, y) \geq \lambda\} : y \in Y, \ \lambda \in \mathbb{R} \cup \{-\infty\}\},$$

$$\mathcal{B} := \text{smallest } \sigma\text{-algebra on } X \text{ containing } \mathcal{R} \text{ and the singletons } \{x\},$$

$$x \in X, \text{ and}$$

$$\mathcal{H} := \{S \subset X : \text{every function } a(\cdot, y), y \in Y, \text{ is constant on } S\}.$$

1991 Mathematics Subject Classification: 49J35, 54E35.
We denote by $\Psi(C)$ the set of all functions $\psi : C \times C \to C$ with the following properties:

1. $\psi$ is concave,
2. $\psi$ is nondecreasing in both variables,
3. $\alpha, \beta \in C \cap \mathbb{R}, \alpha \neq \beta \Rightarrow \psi(\alpha, \beta) < \alpha \lor \beta$, and
4. $-\infty, \alpha \in C \Rightarrow \psi(\alpha, -\infty) = \psi(-\infty, \alpha) = -\infty$.

A nonvoid system of subsets of some set is called (countably) compact iff every (countable) subsystem with the finite intersection property has nonvoid intersection.

Finally, the following reformulation of a recent “minimax theorem with one-sided randomization” [12] will be used in the sequel:

**Theorem A.** Let $\mathcal{R}$ be countably compact, and suppose that for some $\psi \in \Psi(C)$,

$$\forall y_1, y_2 \in Y \exists y_0 \in Y \forall x \in X : a(x, y_0) \leq \psi(a(x, y_1), a(x, y_2)).$$

Then there exists a probability measure $p^*$ on $\mathcal{B}$ with

$$\inf_{y \in Y} \int a(\cdot, y) \, dp^* = \inf_{y \in Y} \sup_{x \in X} a(x, y).$$

**Proof.** Apply Theorem 2 in [12] to $F = \{-a(\cdot, y) : y \in Y\}$, $\eta(\alpha) = \alpha$, $\xi(\alpha, \beta) = -\psi(-\alpha, -\beta)$, $\alpha, \beta \in D := -C$.

In the following we say that $a$ is $\psi$-convex (w.r.t. some $\psi \in \Psi(C)$) iff condition (5) is satisfied.

**2. Main results.** The following lemma summarizes some useful facts:

**Lemma 1.** (a) $a^* = -\infty \Rightarrow X(y^*) = X \forall y^* \in Y^*$.

(b) $X(y) \in \mathcal{R} \forall y \in Y$.

(c) If $\mathcal{R}$ is countably compact, then $X(y)$ is nonvoid for every $y \in Y$.

(d) Condition (6) implies $p^*(X(y^*)) = 1$ for every $y^* \in Y^*$. In particular, $X(Z)$ is nonvoid for every countable $Z \subset Y^*$.

**Proof.** (a) and (b) are obvious, and (c) follows from the equality $X(y) = \bigcap_{n \in \mathbb{N}} \{a(\cdot, y) \geq \sup_{x \in X} a(x, y) - 1/n\}, y \in Y$.

(d) By (a) we may assume $a^* \in \mathbb{R}$. For $y^* \in Y^*$ we have $a(\cdot, y^*) - a^* \leq 0$, but (6) implies $\int [a(\cdot, y^*) - a^*] \, dp^* \geq 0$, hence $p^*(X(y^*)) = 1$.

Now we can present our main results:

**Theorem 1.** Suppose that $\mathcal{R}$ is countably compact and $a$ is $\psi$-convex w.r.t. some $\psi \in \Psi(C)$. Let $x^* \in X$ and $y^* \in Y^*$ satisfy

$$a(x^*, y) \geq a(x, y) \quad \forall x \in X(y^*), \: y \in Y.$$
Then \((x^*, y^*)\) is a saddle point of \(a\), i.e.,
\[ a(x, y^*) \leq a(x^*, y^*) \leq a(x^*, y) \quad \forall x \in X, y \in Y. \]

\textbf{Proof.} By Lemma 1(c) we have \(X(y^*) \neq \emptyset\), hence \(x^* \in X(y^*)\). Choose \(p^*\) according to Theorem A. Then, by Lemma 1(d), we obtain for arbitrary \(x \in X\) and \(y \in Y\),
\[
a(x, y^*) \leq a^* \leq \int_{X(y^*)} a(\cdot, y) \, dp^* \leq \int_{X(y^*)} a(x^*, y) \, dp^* = a(x^*, y). \]

\textbf{Theorem 2.} Suppose that \(\mathcal{R}\) is compact and \(a\) is \(\psi\)-convex w.r.t. some \(\psi \in \Psi(C)\). Then
(a) \(X(Y^*)\) is nonvoid, and
(b) \(\hat{X}\) is nonvoid iff \(Y^* = Y\).

\textbf{Proof.} (a) Apply Theorem A and Lemma 1(b) and (d).
(b) \(Y^* = Y\) implies \(\hat{X} = X(Y^*) \neq \emptyset\) by (a)). Conversely, for \(\hat{x} \in \hat{X}\) we have \(\sup_{x \in X} a(x, z) \leq \inf_{y \in Y} a(\hat{x}, y) \leq a^*\) for all \(z \in Y\), hence \(Y = Y^*\).

\section{Standard situations.} As our formulation of Theorems 1 and 2 is fairly abstract, it seems worthwhile to mention the standard situations:

\textbf{Remark 1.} For \(\lambda \in (0, 1)\) we have \(\mu_\lambda \in \Psi(\mathbb{R} \cup \{-\infty\})\) for the \textit{weighted arithmetic means} \(\mu_\lambda(\alpha, \beta) = \lambda \alpha + (1 - \lambda)\beta\).

If \(Y\) is a convex subset of some linear space, and if every \(a(\cdot, y), y \in Y\), is convex, then \(a\) is \(\mu_\lambda\)-convex for every \(\lambda \in (0, 1)\).

\textbf{Remark 2 (cf. [10]).} Let \(X\) be a topological space.
(a) If \(X\) is compact and every function \(a(\cdot, y), y \in Y\), is upper semicontinuous, then \(\mathcal{R}\) is compact.
(b) If \(X\) is countably compact and every function \(a(\cdot, y), y \in Y\), is upper semicontinuous, then \(\mathcal{R}\) is countably compact.
(c) If \(X\) is pseudocompact (i.e., every continuous \(f : X \to \mathbb{R}\) is bounded) and every function \(a(\cdot, y), y \in Y\), is continuous, then \(\mathcal{R}\) is countably compact.

\textbf{Proof.} (a) and (b) are obvious.
(c) Let \(\{a(\cdot, y_n) \geq \lambda_n : n \in \mathbb{N}\} \subset \mathcal{R}\) have the finite intersection property. Then \(f := \sum_{n \in \mathbb{N}} 2^{-n}(a(\cdot, y_n) - \lambda_n) \wedge 0 \vee (-1)\) with \(M := \{n \in \mathbb{N} : \lambda_n \neq -\infty\}\) is continuous with \(\sup_{x \in X} f(x) = 0\). Hence there exists an \(x_0 \in X\) with \(f(x_0) = 0\), for otherwise \(1/f\) would be unbounded. Of course, \(a(x_0, y_n) \geq \lambda_n, n \in \mathbb{N}\).
Remark 3. For \( y^* \in Y^* \) we have the implications
\[
X(y^*) \text{ is a singleton} \Rightarrow \emptyset \neq X(y^*) \in \mathcal{H} \\
\Rightarrow \text{condition (7) holds for every } x^* \in X(y^*).
\]

Example 1. Let \( X \) be a topological space, \( Y \) a nonvoid set, and \( a : X \times Y \to \mathbb{R} \cup \{ -\infty \} \) such that
\[
(i) \forall y_1, y_2 \in Y \exists y_0 \in Y \forall x \in X : a(x, y_0) \leq \frac{1}{2} a(x, y_1) + \frac{1}{2} a(x, y_2).
\]
Assume, moreover, that either
(ii.1) \( X \) is countably compact, and every function \( a(\cdot, y) \), \( y \in Y \), is upper semicontinuous, or
(ii.2) \( X \) is pseudocompact, and every function \( a(\cdot, y) \), \( y \in Y \), is continuous.

Then for every \( y^* \in Y^* \) with \( X(y^*) \in \mathcal{H} \) the set \( X(y^*) \) is nonvoid, and for \( x^* \in X \) the pair \((x^*, y^*)\) is a saddle point of \( a \) iff \( x^* \in X(y^*) \).

Proof. By Remark 2(b), resp. (c), and Lemma 1(c) every set \( X(y) \), \( y \in Y \), is nonvoid, and by Theorem 1 and Remarks 1–3 every pair \((x^*, y^*)\) with \( y^* \in Y^* \) and \( x^* \in X(y^*) \in \mathcal{H} \) is a saddle point of \( a \). Conversely, if \((x^*, y^*)\) is a saddle point, then, of course, \( y^* \in Y^* \) and \( x^* \in X(y^*) \).

Example 1 generalizes a recent minimax theorem of Pinelis which has interesting applications in statistical decision theory [17], [18]. In contrast to Pinelis we do not require any linear structure on the set \( Y \). This makes it possible to subsume also a version of the Ky Fan–Kőnig–Neumann minimax theorem [6], [13], [15].

Example 2.1. Let \( X \) and \( Y \) be countably compact topological spaces and \( a : X \times Y \to \mathbb{R} \cup \{ -\infty \} \) be such that
\[
(i) \text{ every function } a(\cdot, y) \text{, } y \in Y \text{, is upper semicontinuous,}
(ii) \text{ every function } a(x, \cdot) \text{, } x \in X \text{, is lower semicontinuous,}
(iii) \forall x_1, x_2 \in X \text{, } x_1 \neq x_2 \exists x_0 \in X \forall y \in Y : a(x_0, y) > \frac{1}{2} a(x_1, y) + \frac{1}{2} a(x_2, y),
(iv) \forall y_1, y_2 \in Y \exists y_0 \in Y \forall x \in X : a(x, y_0) \leq \frac{1}{2} a(x, y_1) + \frac{1}{2} a(x, y_2).
\]

Then \( a \) has a saddle point.

Proof. Condition (ii) implies \( Y^* \neq \emptyset \), and (i) implies \( X(y^*) \neq \emptyset \), \( y^* \in Y^* \), because \( X \) and \( Y \) are countably compact. From (iii) we infer that every \( X(y^*) \), \( y^* \in Y^* \), is a singleton. Now the assertion follows from Example 1 and Remarks 1 and 3.

In connection with Example 2.1 the following result ought to be mentioned:
EXAMPLE 2.2. Let $X$ be a pseudocompact and $Y$ a countably compact topological space, and let $a : X \times Y \to \mathbb{R}$ be continuous in each variable. Suppose that

(i) $\forall x_1, x_2 \in X \exists x_0 \in X \forall y \in Y: a(x_0, y) \geq \frac{1}{2}a(x_1, y) + \frac{1}{2}a(x_2, y),$

(ii) $\forall y_1, y_2 \in Y \exists y_0 \in Y \forall x \in X: a(x, y_0) \leq \frac{1}{2}a(x, y_1) + \frac{1}{2}a(x, y_2).$

Then $a$ has a saddle point.

Proof. Apply [5], Corollaire 1, and [11], Satz 3.12.

Remark 4. An inspection of our proof shows that condition (iii) in Example 2.1 can be replaced by the weaker assumption

(iii) $\forall x_1, x_2 \in X, x_1 \neq x_2, \forall y^* \in Y^* \exists x_0 \in X: a(x_0, y^*) > a(x_1, y^*) \wedge a(x_2, y^*).$

In view of Example 2.2 one might conjecture that Example 2.1 remains true also when in condition (iii) “>” is replaced by “≥”. This is disproved, however, by the following counter-example:

EXAMPLE 2.3. The space $X$ of all countable ordinal numbers, endowed with the order topology, is sequentially compact and therefore countably compact. Let $Y = X$ and $a(x, y) = 1(0)$ for $x > y (x \leq y)$. Then every function $a(\cdot, y), y \in Y$, is continuous and every function $a(x, \cdot), x \in X$, is lower semicontinuous. But, of course, $a$ has no saddle point.

Moreover, we have $Y^* = Y$ but $X(Y^*) = \emptyset$ and $\hat{X} = \emptyset$. This shows that Theorem 2 is false if $R$ is only assumed to be countably compact.

4. Convex metric spaces. In the following let $(X, d)$ be a compact metric space. Recall that $\delta(X) := \sup_{(x, y) \in X \times X} d(x, y)$ is the diameter, $R(X) := \inf_{y \in X} \sup_{x \in X} d(x, y)$ is the Chebyshev radius, and $Z(X) := \{y \in X : \sup_{x \in X} d(x, y) = R(X)\}$ is the Chebyshev center of $X$. Moreover, let $P(X)$ denote the set of all Baire probability measures on $X$. Then we call $G(X) := \sup_{\mu \in P(X)} \inf_{y \in X} \int d(\cdot, y) d\mu$ the Gross value of $X$.

Suppose that for some $\psi \in \Psi(C)$ with $[0, \delta(X)) \subset C$ we have

\[ (8) \quad \forall y_1, y_2 \in X \exists y_0 \in X \forall x \in X : d(x, y_0) \leq \psi(d(x, y_1), d(x, y_2)). \]

Then $(X, d)$ will be called $\Psi$-convex (w.r.t. $\psi$).

Remark 5. Let $\mu_\lambda$ be as in Remark 1.

(a) Takahashi [20] calls a metric space convex iff it is $\Psi$-convex w.r.t. every $\mu_\lambda, \lambda \in (0, 1)$. Of course, every convex subset of a normed space is of this type. (Compare also [2].)

(b) Kijima [9] considers "$\mu_1/\mu_2$-convex" metric spaces. Also the convex metric spaces studied by Yang Lu and Zhang Jingzong are of this type ([21], Lemma 1).
Example 3. Let \((X,d)\) be a compact \(\Psi\)-convex metric space. Then

(a) the Chebyshev radius \(R(X)\) coincides with the Gross value \(G(X)\);
(b) the points in the Chebyshev center \(Z(X)\) have a common farthest point in \(X\);
(c) the following are equivalent:
   (i) \(\delta(X) > 0\),
   (ii) \(Z(X) \neq X\),
   (iii) \(\delta(X) > R(X)\) (i.e., \(X\) contains a nondiametral point);
   (iv) every point in the Chebyshev center \(Z(X)\) has at least two different farthest points in \(X\);
(d) a continuous map \(T : X \rightarrow X\) is constant iff every \(y \in X\) has a unique farthest point in \(T(X)\).

Takahashi [20] proved the implication (i)\(\Rightarrow\)(iii) of (c) in the situation of Remark 5(a), and Astaneh [1] established a result of type (i)\(\Rightarrow\)(iv) of (c) in a Hilbert space setting.

Proof. For (a), (b), and (c), let \(Y = X\) and \(a = d\). Then, in the terminology of Section 1, \(Y^*\) coincides with the (nonvoid) Chebyshev center \(Z(X)\), \(X(y)\) is the (nonvoid) set of farthest points of \(y\) in \(X\), and \(B\) is the Baire \(\sigma\)-algebra.

(a) By Theorem A and Remark 2 we have \(G(X) \geq R(X)\); the converse inequality is obvious.
(b) By Theorem 2(a) and Remark 2 there exists a point \(\tilde{x} \in X(Y^*)\), i.e.,
\[d(\tilde{x}, y^*) = \sup_{x \in X} d(x, y^*) \forall y^* \in Z(X).\]
(c) (i)\(\Rightarrow\)(ii). Assume that (ii) is violated, i.e., \(Y^* = Y\). By Theorem 2(b) and Remark 2 there is an \(\hat{x} \in \hat{X}\). This implies \(\delta(X) = d(\hat{x}, \tilde{x}) = 0\) in contradiction to (i).
(i)\(\Rightarrow\)(iv). Suppose that (iv) is violated, i.e., \(X(y^*) = \{x^*\}\) is a singleton for some \(y^* \in Y^*\). By Theorem 1 and Remarks 2 and 3 we obtain \(d(x, y^*) \leq d(x^*, y)\) for all \(x \in X\), \(y \in X\); hence \(X = \{y^*\}\), a contradiction.

As the implications (ii)\(\Leftrightarrow\)(iii)\(\Rightarrow\)(i) and (iv)\(\Rightarrow\)(i) are obvious, everything is proved.

Finally, (d) follows by applying Theorem 1 and Remarks 2 and 3 to the restriction \(a = d(T(X) \times X\). (The farthest point property yields the existence of a pair \((x^*, y^*) \in X \times X\ with \(d(Tx, y^*) \leq d(Tx^*, y)\) for all \(x \in X, y \in X\), and we arrive at \(T(X) = \{y^*\}\).

Now we recall a result of Gross which is an easy consequence of Theorem A or one of its ancestors due to Glicksberg [7] or to Peck–Dulmage [16] (compare [8] and also [4], [14], [19]).
**Theorem B (Gross).** Let $(X, d)$ be a compact connected metric space. Then there exists a uniquely determined constant $A(X)$ such that

$$\forall x_1, \ldots, x_n \in X, \ n \in \mathbb{N}, \ \exists y \in X: \ \frac{1}{n} \sum_{i=1}^{n} d(x_i, y) = A(X).$$

This “rendez-vous value” $A(X)$ coincides with the Gross value $G(X)$.

We use this theorem to prove the following generalization of a result of Esther and George Szekeres ([4], Theorem 5) and of Yang Lu and Zhang Jingzong [21]:

**Example 4.** Let $(X, d)$ be a compact metric space. Suppose that $(X, d)$ is $\psi$-convex w.r.t. some $\psi \in \Psi(C)$, $C \supset [0, \delta(X)]$, satisfying

$$\psi(0, \alpha) + \psi(0, \alpha) \leq \alpha, \quad 0 < \alpha \leq \delta(X).$$

Then $(X, d)$ is arcwise connected, and its rendez-vous value $A(X)$ coincides with the Chebyshev radius $R(X)$.

**Proof.** By Theorem B and Example 3(a) it remains to show that $(X, d)$ is arcwise connected. By a well-known theorem of Menger ([3], Theorem 6.2) it is sufficient to prove that for $y_1, y_2 \in X$, $y_1 \neq y_2$, the “segment”

$$(y_1, y_2) := \{ y \in X : d(y_1, y) + d(y, y_2) = d(y_1, y_2) \} - \{y_1, y_2\}$$

is nonvoid. We choose $y_0 \in X$ according to (8) and show that $y_0 \in (y_1, y_2)$:

$$d(y_1, y_2) \leq d(y_1, y_0) + d(y_2, y_0) \leq \psi(d(y_1, y_1), d(y_1, y_2)) + \psi(d(y_2, y_1), d(y_2, y_2)) \leq d(y_1, y_2) \quad \text{(by (9))}$$

we infer that $d(y_1, y_2) = d(y_1, y_0) + d(y_0, y_2)$. Suppose that $y_0 = y_2$, say. Then $d(y_1, y_2) = d(y_1, y_0) \leq \psi(0, d(y_1, y_2))$ contradicts condition (3).

**Remark 6.** Condition (9) is satisfied for $\psi = \mu_\lambda$ as in Remark 1 and, more generally, for every positively homogeneous $\psi \in \Psi([0, \infty))$. It would be interesting to know whether condition (9) is dispensable in Example 4.

**References**


