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MINIMAX THEOREMS WITH APPLICATIONS TO CONVEX METRIC SPACES

BY

JÜRGEN KINDLER (DARMSTADT)

A minimax theorem is proved which contains a recent result of Pinelis and a version of the classical minimax theorem of Ky Fan as special cases. Some applications to the theory of convex metric spaces (farthest points, rendez-vous value) are presented.

1. Preliminaries. Throughout this paper let two nonvoid sets X and Y, a nonvoid convex subset C of $\mathbb{R}\cup\{-\infty\}$ and a function $a: X\times Y\to C$ with

$$\sup_{x\in X} a(x,y)\in C \quad \forall y\in Y$$

be given. The following notation will be used:

• We set

$$\begin{aligned}
\sup a &:= \sup\{a(x,y): x\in X, y\in Y\},\\
a^* &:= \inf_{y\in Y}\sup_{x\in X} a(x,y),\\
Y^* &:= \{\sup_{x\in X} a(x,\cdot)=a^*\} = \{y\in Y: \sup_{x\in X} a(x,y)=a^*\},\\
\widehat{X} &:= \bigcap_{y\in Y}\{a(\cdot,y)=\sup a\},\\
X(y) &:= \{a(\cdot,y)=\sup_{x\in X} a(x,y)\}, \quad y\in Y,\\
X(B) &:= \bigcap_{y\in B} X(y), \quad B\subset Y, \quad \text{with } X(\emptyset)=X,\\
\mathcal{R} &:= \{\{a(\cdot,y)\ge\lambda\}: y\in Y,\ \lambda\in\mathbb{R}\cup\{-\infty\}\},\\
\mathcal{B} &:= \text{smallest } \sigma\text{-algebra on } X \text{ containing } \mathcal{R} \text{ and the singletons } \{x\},\\
&\quad\ x\in X, \text{ and}\\
\mathcal{H} &:= \{S\subset X: \text{every function } a(\cdot,y), y\in Y, \text{ is constant on } S\}.
\end{aligned}$$

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• We denote by $\Psi(C)$ the set of all functions $\psi : C \times C \to C$ with the following properties:

(1) ψ is concave,
(2) ψ is nondecreasing in both variables,
(3) $\alpha, \beta \in C \cap \mathbb{R}$, $\alpha \neq \beta \Rightarrow \psi(\alpha, \beta) < \alpha \vee \beta$, and
(4) $-\infty, \alpha \in C \Rightarrow \psi(\alpha, -\infty) = \psi(-\infty, \alpha) = -\infty$.

• A nonvoid system of subsets of some set is called (*countably*) *compact* iff every (countable) subsystem with the finite intersection property has nonvoid intersection.

Finally, the following reformulation of a recent "minimax theorem with one-sided randomization" [12] will be used in the sequel:

THEOREM A. *Let $\mathcal{R}$ be countably compact, and suppose that for some $\psi \in \Psi(C)$,*

(5) $$\forall y_1, y_2 \in Y\ \exists y_0 \in Y\ \forall x \in X : a(x, y_0) \le \psi(a(x, y_1), a(x, y_2)).$$

Then there exists a probability measure p^* on $\mathcal{B}$ with

(6) $$\inf_{y \in Y} \int a(\cdot, y)\, dp^* = \inf_{y \in Y} \sup_{x \in X} a(x, y).$$

Proof. Apply Theorem 2 in [12] to $F = \{-a(\cdot, y) : y \in Y\}$, $\eta(\alpha) = \alpha$, $\xi(\alpha, \beta) = -\psi(-\alpha, -\beta)$, $\alpha, \beta \in D := -C$.

• In the following we say that a is ψ-*convex* (w.r.t. some $\psi \in \Psi(C)$) iff condition (5) is satisfied.

2. Main results. The following lemma summarizes some useful facts:

LEMMA 1. (a) $a^* = -\infty \Rightarrow X(y^*) = X\ \forall y^* \in Y^*$.
(b) $X(y) \in \mathcal{R}\ \forall y \in Y$.
(c) *If $\mathcal{R}$ is countably compact, then $X(y)$ is nonvoid for every $y \in Y$.*
(d) *Condition* (6) *implies $p^*(X(y^*)) = 1$ for every $y^* \in Y^*$. In particular, $X(Z)$ is nonvoid for every countable $Z \subset Y^*$.*

Proof. (a) and (b) are obvious, and (c) follows from the equality $X(y) = \bigcap_{n \in \mathbb{N}} \{a(\cdot, y) \ge \sup_{x \in X} a(x, y) - 1/n\}$, $y \in Y$.
(d) By (a) we may assume $a^* \in \mathbb{R}$. For $y^* \in Y^*$ we have $a(\cdot, y^*) - a^* \le 0$, but (6) implies $\int [a(\cdot, y^*) - a^*]\, dp^* \ge 0$, hence $p^*(X(y^*)) = 1$.

Now we can present our main results:

THEOREM 1. *Suppose that $\mathcal{R}$ is countably compact and a is ψ-convex w.r.t. some $\psi \in \Psi(C)$. Let $x^* \in X$ and $y^* \in Y^*$ satisfy*

(7) $$a(x^*, y) \ge a(x, y) \quad \forall x \in X(y^*),\ y \in Y.$$

Then (x^*, y^*) *is a saddle point of* a, *i.e.*,

$$a(x, y^*) \le a(x^*, y^*) \le a(x^*, y) \qquad \forall x \in X,\ y \in Y.$$

Proof. By Lemma 1(c) we have $X(y^*) \neq \emptyset$, hence $x^* \in X(y^*)$. Choose p^* according to Theorem A. Then, by Lemma 1(d), we obtain for arbitrary $x \in X$ and $y \in Y$,

$$\begin{aligned} a(x, y^*) &\le a^* \le \int a(\cdot, y)\, dp^* \\ &= \int_{X(y^*)} a(\cdot, y)\, dp^* \le \int_{X(y^*)} a(x^*, y)\, dp^* = a(x^*, y). \end{aligned}$$

Theorem 2. *Suppose that* $\mathcal{R}$ *is compact and* a *is* ψ*-convex w.r.t. some* $\psi \in \Psi(C)$. *Then*

(a) $X(Y^*)$ *is nonvoid*, *and*

(b) $\widehat{X}$ *is nonvoid iff* $Y^* = Y$.

Proof. (a) Apply Theorem A and Lemma 1(b) and (d).

(b) $Y^* = Y$ implies $\widehat{X} = X(Y^*)$ ($\neq \emptyset$ by (a)). Conversely, for $\widehat{x} \in \widehat{X}$ we have $\sup_{x \in X} a(x, z) \le \inf_{y \in Y} a(\widehat{x}, y) \le a^*$ for all $z \in Y$, hence $Y = Y^*$.

3. Standard situations. As our formulation of Theorems 1 and 2 is fairly abstract, it seems worthwhile to mention the standard situations:

Remark 1. For $\lambda \in (0, 1)$ we have $\mu_\lambda \in \Psi(\mathbb{R} \cup \{-\infty\})$ for the *weighted arithmetic means* $\mu_\lambda(\alpha, \beta) = \lambda\alpha + (1 - \lambda)\beta$.

If Y is a convex subset of some linear space, and if every $a(x, \cdot)$, $x \in X$, is convex, then a is μ_λ-convex for every $\lambda \in (0, 1)$.

Remark 2 (cf. [10]). Let X be a topological space.

(a) If X is compact and every function $a(\cdot, y)$, $y \in Y$, is upper semicontinuous, then $\mathcal{R}$ is compact.

(b) If X is countably compact and every function $a(\cdot, y)$, $y \in Y$, is upper semicontinuous, then $\mathcal{R}$ is countably compact.

(c) If X is pseudocompact (i.e., every continuous $f : X \to \mathbb{R}$ is bounded) and every function $a(\cdot, y)$, $y \in Y$, is continuous, then $\mathcal{R}$ is countably compact.

Proof. (a) and (b) are obvious.

(c) Let $\{\{a(\cdot, y_n) \ge \lambda_n\} : n \in \mathbb{N}\} \subset \mathcal{R}$ have the finite intersection property. Then $f := \sum_{n \in M} 2^{-n}(a(\cdot, y_n) - \lambda_n) \wedge 0 \vee (-1)$ with $M := \{n \in \mathbb{N} : \lambda_n \neq -\infty\}$ is continuous with $\sup_{x \in X} f(x) = 0$. Hence there exists an $x_0 \in X$ with $f(x_0) = 0$, for otherwise $1/f$ would be unbounded. Of course, $a(x_0, y_n) \ge \lambda_n$, $n \in \mathbb{N}$.

R e m a r k 3. For $y^* \in Y^*$ we have the implications

$$X(y^*) \text{ is a singleton } \Rightarrow \emptyset \neq X(y^*) \in \mathcal{H}$$
$$\Rightarrow \text{condition (7) holds for every } x^* \in X(y^*).$$

EXAMPLE 1. Let X be a topological space, Y a nonvoid set, and $a : X \times Y \to \mathbb{R} \cup \{-\infty\}$ such that

(i) $\forall y_1, y_2 \in Y\ \exists y_0 \in Y\ \forall x \in X : a(x, y_0) \leq \frac{1}{2}a(x, y_1) + \frac{1}{2}a(x, y_2)$.

Assume, moreover, that either

(ii.1) X is countably compact, and every function $a(\cdot, y)$, $y \in Y$, is upper semicontinuous, or

(ii.2) X is pseudocompact, and every function $a(\cdot, y)$, $y \in Y$, is continuous.

Then for every $y^* \in Y^*$ with $X(y^*) \in \mathcal{H}$ the set $X(y^*)$ is nonvoid, and for $x^* \in X$ the pair (x^*, y^*) is a saddle point of a iff $x^* \in X(y^*)$.

P r o o f. By Remark 2(b), resp. (c), and Lemma 1(c) every set $X(y)$, $y \in Y$, is nonvoid, and by Theorem 1 and Remarks 1–3 every pair (x^*, y^*) with $y^* \in Y^*$ and $x^* \in X(y^*) \in \mathcal{H}$ is a saddle point of a. Conversely, if (x^*, y^*) is a saddle point, then, of course, $y^* \in Y^*$ and $x^* \in X(y^*)$.

Example 1 generalizes a recent minimax theorem of Pinelis which has interesting applications in statistical decision theory [17], [18]. In contrast to Pinelis we do not require any linear structure on the set Y. This makes it possible to subsume also a version of the Ky Fan–König–Neumann minimax theorem [6], [13], [15]:

EXAMPLE 2.1. Let X and Y be countably compact topological spaces and $a : X \times Y \to \mathbb{R} \cup \{-\infty\}$ be such that

(i) every function $a(\cdot, y)$, $y \in Y$, is upper semicontinuous,

(ii) every function $a(x, \cdot)$, $x \in X$, is lower semicontinuous,

(iii) $\forall x_1, x_2 \in X,\ x_1 \neq x_2,\ \exists x_0 \in X\ \forall y \in Y\colon\ a(x_0, y) > \frac{1}{2}a(x_1, y) + \frac{1}{2}a(x_2, y)$,

(iv) $\forall y_1, y_2 \in Y\ \exists y_0 \in Y\ \forall x \in X\colon\ a(x, y_0) \leq \frac{1}{2}a(x, y_1) + \frac{1}{2}a(x, y_2)$.

Then a has a saddle point.

P r o o f. Condition (ii) implies $Y^* \neq \emptyset$, and (i) implies $X(y^*) \neq \emptyset$, $y^* \in Y^*$, because X and Y are countably compact. From (iii) we infer that every $X(y^*)$, $y^* \in Y^*$, is a singleton. Now the assertion follows from Example 1 and Remarks 1 and 3.

In connection with Example 2.1 the following result ought to be mentioned:

EXAMPLE 2.2. Let X be a pseudocompact and Y a countably compact topological space, and let $a : X \times Y \to \mathbb{R}$ be continuous in each variable. Suppose that

(i) $\forall x_1, x_2 \in X\ \exists x_0 \in X\ \forall y \in Y\colon\ a(x_0, y) \ge \frac{1}{2}a(x_1, y) + \frac{1}{2}a(x_2, y)$,
(ii) $\forall y_1, y_2 \in Y\ \exists y_0 \in Y\ \forall x \in X\colon\ a(x, y_0) \le \frac{1}{2}a(x, y_1) + \frac{1}{2}a(x, y_2)$.

Then a has a saddle point.

Proof. Apply [5], Corollaire 1, and [11], Satz 3.12.

Remark 4. An inspection of our proof shows that condition (iii) in Example 2.1 can be replaced by the weaker assumption

(iii)* $\forall x_1, x_2 \in X,\ x_1 \ne x_2,\ \forall y^* \in Y^*\ \exists x_0 \in X\colon\ a(x_0, y^*) > a(x_1, y^*) \wedge a(x_2, y^*)$.

In view of Example 2.2 one might conjecture that Example 2.1 remains true also when in condition (iii) "$>$" is replaced by "$\ge$". This is disproved, however, by the following counter-example:

EXAMPLE 2.3. The space X of all countable ordinal numbers, endowed with the order topology, is sequentially compact and therefore countably compact. Let $Y = X$ and $a(x, y) = 1(0)$ for $x > y$ ($x \le y$). Then every function $a(\cdot, y)$, $y \in Y$, is continuous and every function $a(x, \cdot)$, $x \in X$, is lower semicontinuous. But, of course, a has no saddle point.

Moreover, we have $Y^* = Y$ but $X(Y^*) = \emptyset$ and $\widehat{X} = \emptyset$. This shows that Theorem 2 is false if $\mathcal{R}$ is only assumed to be countably compact.

4. Convex metric spaces. In the following let (X, d) be a compact metric space. Recall that $\delta(X) := \sup_{(x,y)\in X\times X} d(x, y)$ is the *diameter*, $R(X) := \inf_{y\in X} \sup_{x\in X} d(x, y)$ is the *Chebyshev radius*, and $Z(X) := \{y \in X : \sup_{x\in X} d(x, y) = R(X)\}$ is the *Chebyshev center* of X. Moreover, let $P(X)$ denote the set of all Baire probability measures on X. Then we call $G(X) := \sup_{p\in P(X)} \inf_{y\in X} \int d(\cdot, y)\, dp$ the *Gross value* of X.

Suppose that for some $\psi \in \Psi(C)$ with $[0, \delta(X)] \subset C$ we have

$$(8)\qquad \forall y_1, y_2 \in X\ \exists y_0 \in X\ \forall x \in X : d(x, y_0) \le \psi(d(x, y_1), d(x, y_2)).$$

Then (X, d) will be called *Ψ-convex* (w.r.t. ψ).

Remark 5. Let μ_λ be as in Remark 1.

(a) Takahashi [20] calls a metric space convex iff it is Ψ-convex w.r.t. every μ_λ, $\lambda \in (0, 1)$. Of course, every convex subset of a normed space is of this type. (Compare also [2].)

(b) Kijima [9] considers "$\mu_{1/2}$-convex" metric spaces. Also the convex metric spaces studied by Yang Lu and Zhang Jingzong are of this type ([21], Lemma 1).

EXAMPLE 3. Let (X,d) be a compact Ψ-convex metric space. Then

(a) the Chebyshev radius $R(X)$ coincides with the Gross value $G(X)$;

(b) the points in the Chebyshev center $Z(X)$ have a common farthest point in X;

(c) the following are equivalent:

(i) $\delta(X) > 0$,

(ii) $Z(X) \neq X$,

(iii) $\delta(X) > R(X)$ (i.e., X contains a nondiametral point);

(iv) every point in the Chebyshev center $Z(X)$ has at least two different farthest points in X;

(d) a continuous map $T : X \to X$ is constant iff every $y \in X$ has a unique farthest point in $T(X)$.

Takahashi [20] proved the implication (i)⇒(iii) of (c) in the situation of Remark 5(a), and Astaneh [1] established a result of type (i)⇒(iv) of (c) in a Hilbert space setting.

Proof. For (a), (b), and (c), let $Y = X$ and $a = d$. Then, in the terminology of Section 1, Y^* coincides with the (nonvoid) Chebyshev center $Z(X)$, $X(y)$ is the (nonvoid) set of farthest points of y in X, and $\mathcal{B}$ is the Baire σ-algebra.

(a) By Theorem A and Remark 2 we have $G(X) \geq R(X)$; the converse inequality is obvious.

(b) By Theorem 2(a) and Remark 2 there exists a point $\widetilde{x} \in X(Y^*)$, i.e., $d(\widetilde{x}, y^*) = \sup_{x\in X} d(x,y^*)\ \forall y^* \in Z(X)$.

(c) (i)⇒(ii). Assume that (ii) is violated, i.e., $Y^* = Y$. By Theorem 2(b) and Remark 2 there is an $\widehat{x} \in \widehat{X}$. This implies $\delta(X) = d(\widehat{x}, \widehat{x}) = 0$ in contradiction to (i).

(i)⇒(iv). Suppose that (iv) is violated, i.e., $X(y^*) = \{x^*\}$ is a singleton for some $y^* \in Y^*$. By Theorem 1 and Remarks 2 and 3 we obtain $d(x,y^*) \leq d(x^*, y)$ for all $x \in X$, $y \in X$; hence $X = \{y^*\}$, a contradiction.

As the implications (ii)⇔(iii)⇒(i) and (iv)⇒(i) are obvious, everything is proved.

Finally, (d) follows by applying Theorem 1 and Remarks 2 and 3 to the restriction $a = d|T(X) \times X$. (The farthest point property yields the existence of a pair $(x^*, y^*) \in X \times X$ with $d(Tx, y^*) \leq d(Tx^*, y)$ for all $x \in X$, $y \in X$, and we arrive at $T(X) = \{y^*\}$.)

Now we recall a result of Gross which is an easy consequence of Theorem A or one of its ancestors due to Glicksberg [7] or to Peck–Dulmage [16] (compare [8] and also [4], [14], [19]).

THEOREM B (Gross). *Let (X,d) be a compact connected metric space. Then there exists a uniquely determined constant $A(X)$ such that*

$$\forall x_1,\dots,x_n \in X,\ n\in\mathbb{N},\ \exists y\in X: \quad \frac{1}{n}\sum_{i=1}^{n} d(x_i,y) = A(X).$$

This "rendez-vous value" $A(X)$ coincides with the Gross value $G(X)$.

We use this theorem to prove the following generalization of a result of Esther and George Szekeres ([4], Theorem 5) and of Yang Lu and Zhang Jingzong [21]:

EXAMPLE 4. Let (X,d) be a compact metric space. Suppose that (X,d) is ψ-convex w.r.t. some $\psi\in\Psi(C)$, $C\supset[0,\delta(X)]$, satisfying

$$\psi(0,\alpha)+\psi(\alpha,0)\le\alpha,\qquad 0<\alpha\le\delta(X). \tag{9}$$

Then (X,d) is arcwise connected, and its rendez-vous value $A(X)$ coincides with the Chebyshev radius $R(X)$.

Proof. By Theorem B and Example 3(a) it remains to show that (X,d) is arcwise connected. By a well-known theorem of Menger ([3], Theorem 6.2) it is sufficient to prove that for $y_1,y_2\in X$, $y_1\ne y_2$, the "segment"

$$(y_1,y_2):=\{y\in X: d(y_1,y)+d(y,y_2)=d(y_1,y_2)\}-\{y_1,y_2\}$$

is nonvoid. We choose $y_0\in X$ according to (8) and show that $y_0\in(y_1,y_2)$:

From $d(y_1,y_2)\le d(y_1,y_0)+d(y_2,y_0)\le\psi(d(y_1,y_1),\ d(y_1,y_2))+\psi(d(y_2,y_1),d(y_2,y_2))\le d(y_1,y_2)$ (by (9)) we infer that $d(y_1,y_2)=d(y_1,y_0)+d(y_0,y_2)$. Suppose that $y_0=y_2$, say. Then $d(y_1,y_2)=d(y_1,y_0)\le\psi(0,d(y_1,y_2))$ contradicts condition (3).

Remark 6. Condition (9) is satisfied for $\psi=\mu_\lambda$ as in Remark 1 and, more generally, for every positively homogeneous $\psi\in\Psi([0,\infty))$. It would be interesting to know whether condition (9) is dispensable in Example 4.

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FACHBEREICH MATHEMATIK
TECHNISCHE HOCHSCHULE DARMSTADT
SCHLOSSGARTENSTR. 7
D-64289 DARMSTADT, GERMANY

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