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STOCHASTIC VIABILITY AND A COMPARISON THEOREM

BY

ANNA MILIAN (KRAKÓW)

We give explicit necessary and sufficient conditions for the viability of polyhedrons with respect to Itô equations. Using the viability criterion we obtain a comparison theorem for multi-dimensional Itô processes.

1. Introduction. The notion of *viable trajectories*, used in the theory of deterministic differential equations, refers to those trajectories which remain at any time in a fixed subset of the state space. The *viability problem* consists in characterizing a fixed subset and an equation such that the equation has viable trajectories in the subset for any initial state from the subset. Characterizing a subset and an equation in order for each solution to the equation starting from the subset to be viable in the subset is another problem, called the *invariance problem*.

In the theory of viable solutions the concept of the contingent cone plays a fundamental role. In fact, the pioneering theorem, proved in 1942 by Nagumo, gives a criterion for the existence of viable trajectories in terms of contingent cones. Namely, the Nagumo theorem states that if f is a bounded, continuous map from a closed subset K of $\mathbb{R}^m$ to $\mathbb{R}^m$, then a necessary and sufficient condition for the differential equation

$$x'(t) = f(x(t)), \qquad x(0) = x_0,$$

to have viable trajectories in K, for all initial states $x_0 \in K$, is that

$$\forall x \in K, \qquad f(x) \text{ belongs to the contingent cone to } K \text{ at } x.$$

Various generalizations of the Nagumo theorem provide viability conditions in terms of contingent cones (see for instance [2, Th. 1, p. 191]).

Viability and invariance with respect to Itô equations have been investigated first by J.-P. Aubin and G. Da Prato in [3]. The stochastic contingent set defined in that paper is an adaptation of the deterministic concept. Criterions for the viability and invariance of closed and *convex* subset of $\mathbb{R}^m$, given in [3], are expressed in terms of stochastic contingent sets. Their results were

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generalized to *arbitrary* subsets (which can also be time-dependent and random) by the present author in [12]. Independently in [8] similar, but not identical, results were obtained by S. Gautier and L. Thibault and according to [8] in a preprint [4] by J.-P. Aubin and G. Da Prato.

Conditions expressed by stochastic contingent sets are general but unfortunately not easy to check and the aim of the present paper is to give checkable conditions for general equations and sets which are polyhedrons.

Necessary and sufficient conditions for viability are given in Theorem 1. As an important corollary of Theorem 1 we obtain a comparison result for multidimensional Itô equations in the form of Theorem 2 generalizing [10, Th. 1.1, p. 352]. The proof of Theorem 1 is divided into several steps. First we give necessary conditions for the viability of half-spaces. Then we prove that the necessary conditions are also sufficient. Finally, we extend the result to general polyhedrons. The basic tool in this step is Theorem A1 about polyhedrons, of independent interest, postponed with its proof to the Appendix. The proof of Theorem 2 is an easy consequence of Theorem 1.

2. Formulation of the main theorems. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a right-continuous increasing family $F = (\mathcal{F}_t)_{t\ge 0}$ of sub-σ-fields of $\mathcal{F}$ each containing P-null sets, K be a closed subset of $\mathbb{R}^m$ and $J \subset \mathbb{R}$ an interval.

A stochastic process $X(t)$, $t \in J$, is said to be *viable in K on the interval J* if $P\{X(t) \in K,\ t \in J\} = 1$.

Given mappings $f = [f_i] : [0,\infty) \times \mathbb{R}^m \to \mathbb{R}^m$, $g = [g_{ij}] : [0,\infty) \times \mathbb{R}^m \to \mathbb{R}^{m\times r}$ and an r-dimensional F-Wiener process $W : [0,\infty) \times \Omega \to \mathbb{R}^r$, we consider the stochastic differential Itô equation

$$X(t) = x^0 + \int_{t_0}^{t} f(s, X(s))\, ds + \int_{t_0}^{t} g(s, X(s))\, dW(s), \qquad t \in [t_0, \infty). \tag{1}$$

A set K is said to be *stochastically invariant* for the pair (f,g) (or for the equation (1)) if for any $x^0 \in K$ and any $t_0 \ge 0$ every solution X to (1) is viable in K on the interval $[t_0,\infty)$.

We say that K has the *stochastic viability property* with respect to (f,g) (or with respect to (1)) if for any $x^0 \in K$ and any $t_0 \ge 0$ there exists a solution X to (1) which is viable in K on $[t_0,\infty)$.

The notions of stochastic viability and stochastic invariance, introduced by J.-P. Aubin and G. Da Prato, are more general because x^0 is assumed to be a random variable taking values in K almost surely. In this paper we restrict our investigation to the case of deterministic initial states. Let $P(a, \mathbf{n})$ denote the half-space in $\mathbb{R}^m$ determined by a point $a \in \mathbb{R}^m$ and a vector $\mathbf{n} \in \mathbb{R}^m$. Obviously

$$P(a, \mathbf{n}) = \{x \in \mathbb{R}^m : \langle x - a, \mathbf{n} \rangle \geq 0\},$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^m$. The term *polyhedron* refers to any set of the form

$$\bigcap_{\alpha \in I} P(a_\alpha, \mathbf{n}_\alpha),$$

where $I = \{1, \ldots, N\}$ is a finite subset of $\mathbb{N}$.

The aim of the paper is to prove the following theorem:

THEOREM 1. *Let $K = \bigcap_{\alpha \in I} P(a_\alpha, \mathbf{n}_\alpha)$ be a polyhedron in $\mathbb{R}^m$. Suppose that the coefficients $f(t, x)$ and $g(t, x)$ of* (1), *defined for $t \geq 0$ and $x \in \mathbb{R}^m$, satisfy the following conditions*:

(i) *For each $T > 0$ there exists $K_T > 0$ such that for all $x \in K$ and $t \in [0, T]$,*

$$\|f(t, x)\|^2 + \|g(t, x)\|^2 \leq K_T(1 + \|x\|^2).$$

(ii) *For all $T > 0$, $x \in K$, $y \in K$ and $t \in [0, T]$,*

$$\|f(t, x) - f(t, y)\| + \|g(t, x) - g(t, y)\| \leq K_T \|x - y\|.$$

(iii) *For each $x \in K$ the functions $f(\cdot, x)$ and $g(\cdot, x)$, defined for $t \geq 0$, are continuous.*

Then K has the viability property with respect to (f, g) if and only if the following condition holds:

(a) *For all $\alpha \in I$ and $x \in K$ such that $\langle x - a_\alpha, \mathbf{n}_\alpha \rangle = 0$, we have*

$$\langle f(t, x), \mathbf{n}_\alpha \rangle \geq 0, \quad \langle g_j(t, x), \mathbf{n}_\alpha \rangle = 0, \quad \text{for } t \geq 0,\ j = 1, \ldots, r,$$

where g_j is the jth column of the matrix $g = [g_{ij}]$.

From the above theorem we shall derive the following comparison theorem:

THEOREM 2. *Let I be a nonempty subset of $\{1, \ldots, m\}$. Assume that for each $T > 0$ there exists a constant $K_T > 0$ such that*

(i) *For all $x, y \in \mathbb{R}^m$ and $t \in [0, T]$,*

$$\|f(t, x) - f(t, y)\| + \|g(t, x) - g(t, y)\| \leq K_T \|x - y\|.$$

(ii) *For all $x \in \mathbb{R}^m$ and $t \in [0, T]$,*

$$\|f(t, x)\|^2 + \|g(t, x)\|^2 \leq K_T(1 + \|x\|^2).$$

(iii) *For each $x \in \mathbb{R}^m$ the functions $f(\cdot, x)$ and $g(\cdot, x)$, defined for $t \geq 0$, are continuous.*

Assume further that $\bar{f}$ and $\overline{g}$ also satisfy the above conditions. Let X and $\overline{X}$ be solutions to equation (1) *with coefficients* (f,g) *and* $(\bar{f},\overline{g})$, *respectively. Then the following conditions are equivalent*:

(a) *For all* $t_0 \ge 0$, $x^0 = (x_1^0,\dots,x_m^0) \in \mathbb{R}^m$ *and* $\overline{x}^0 = (\overline{x}_1^0,\dots,\overline{x}_m^0) \in \mathbb{R}^m$, *if* $x_i^0 \le \overline{x}_i^0$, $i \in I$, *then*

$$P\{X_i(t) \le \overline{X}_i(t),\ i \in I,\ t \ge t_0\} = 1.$$

(b) *For all* $p \in I$, $x = (x_1,\dots,x_m) \in \mathbb{R}^m$ *and* $\overline{x} = (\overline{x}_1,\dots,\overline{x}_m) \in \mathbb{R}^m$ *such that* $x_i \le \overline{x}_i$, $i \in I$, *if* $x_p = \overline{x}_p$, *then for all* $t \ge 0$, $j = 1,\dots,r$,

$$f_p(t,x) \le \bar{f}_p(t,\overline{x}) \quad \text{and} \quad g_{pj}(t,x) = \overline{g}_{pj}(t,\overline{x}).$$

3. Proof of Theorem 1. We begin with the following lemma.

LEMMA 1. *Let X be a solution to* (1) *such that* $x^0 = (x_1^0,\dots,x_m^0)$ *and* $x_q^0 = 0$ *for some* $q \in \{1,\dots,m\}$. *Suppose that conditions* (i)–(iii) *of Theorem* 2 *hold. Suppose further that*

(iv) $P\{X_q(t_0+t) \ge 0\ (\le 0),\ t \ge 0\} = 1$.

Then

(a) $f_q(t_0,x^0) \ge 0$ (≤ 0),
(b) $g_{qj}(t_0,x^0) = 0$ *for* $j = 1,\dots,r$.

P r o o f. It is enough to consider the case $t_0 = 0$. By (iv),

$$\frac{1}{t}\mathbb{E}X_q(t) = \frac{1}{t}\mathbb{E}\int_0^t f_q(s,X(s))\,ds \ge 0 \quad (\le 0).$$

From (i) and (iii) it follows that f is continuous. Therefore

$$\lim_{t\to 0^+}\frac{1}{t}\mathbb{E}\int_0^t f_q(s,X(s))\,ds = f_q(0,x^0) \ge 0 \quad (\le 0),$$

and (a) holds.

Suppose that (b) fails. Then $g_{qj}(0,x^0) \ne 0$ for some $j \in \{1,\dots,r\}$. Repeating the previous argument shows the continuity of g. Hence the process $g(t,X(t))_{t\ge 0}$ is continuous and by [11] or [15] it follows that $t_n \to 0$ as $n \to \infty$ implies that

$$\frac{1}{W_{t_n}}\int_0^{t_n} g_{qj}(s,X(s))\,dW_j(s) \to g_{qj}(0,x_0).$$

On the other hand, from Blumenthal's zero-one law (see [6, pp. 14–15]), if $t_n \to 0$ as $n \to \infty$ then

$$\liminf_{n\to\infty}\frac{W_j(t_n)}{\sqrt{t_n}} = -\infty \quad \text{a.s.}$$

Hence there is a sequence $t_n \to 0$ as $n \to \infty$ such that

$$\liminf_{n\to\infty} \frac{1}{\sqrt{t_n}} \int_0^{t_n} g_{qj}(s, X(s))\, dW_j(s) = -\infty \qquad \text{a.s.}$$

Applying the above arguments again, we conclude that

$$P\left\{ \liminf_{t\to 0^+} \frac{1}{\sqrt{t}} \sum_{j=1}^{r} \int_0^t g_{qj}(s, X(s))\, dW_j(s) = -\infty \right\} = 1.$$

But

$$\lim_{t\to 0^+} \frac{1}{\sqrt{t}} \left| \int_0^t f_q(s, X(s))\, ds \right| = 0 \qquad \text{a.s.}$$

Since $(1/\sqrt{t})X_q(t) \geq 0$ with probability 1, we have

$$P\left\{ \liminf_{t\to 0^+} \frac{1}{\sqrt{t}} X_q(t) \geq 0 \right\} = 1,$$

which is a contradiction. ∎

The next lemma states that conditions (a) and (b) of Lemma 1 are also sufficient for the viability of half-spaces.

LEMMA 2. *Let $f = (f_1, \ldots, f_m)$ and $g = [g_{i,j}]$, $i = 1, \ldots, m$, $j = 1, \ldots, r$ be defined in $P_q = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_q \geq 0\}$, where $q \in \{1, \ldots, m\}$. Suppose that*

(i) *For each $T > 0$ and $N > 0$ there exists a constant $K_{T,N} > 0$ such that for $0 \leq t \leq T$ and $x, y \in P_q$ with $|x| \leq N$ and $|y| \leq N$,*

$$\|g(t, x) - g(t, y)\| + \|f(t, x) - f(t, y)\| \leq K_{T,N}\|x - y\|.$$

(ii) *For each $T > 0$ there exists $K_T > 0$ such that for $0 \leq t \leq T$ and $x \in P_q$,*

$$\|g(t, x)\|^2 \leq K_T(1 + \|x\|^2).$$

(iii) *For each $T > 0$ there exists $K_T > 0$ such that for $0 \leq t \leq T$ and $x \in P_q$,*

$$\sum_{j=1}^{m} x_j f_j(t, x) \leq K_T(1 + \|x\|^2),$$

and moreover, f is locally bounded.

(iv) *For all $t \geq 0$, $x_i \in \mathbb{R}$, $i \in \{1, \ldots, m\}$, $i \neq q$, and $j \in \{1, \ldots, r\}$,*

$$g_{q,j}(t, x_1, \ldots, x_{q-1}, 0, x_{q+1}, \ldots, x_m) = 0.$$

(v) *For all $t \geq 0$ and $x_i \in \mathbb{R}$, $i \in \{1, \ldots, m\}$, $i \neq q$,*

$$f_q(t, x_1, \ldots, x_{q-1}, 0, x_{q+1}, \ldots, x_m) \geq 0.$$

Then

(a) *For each* $t_0 \geq 0$ *and* $x^0 = (x_1^0, \ldots, x_m^0)$ *such that* $x_q^0 \geq 0$ *there exists a solution* $X = (X_1, \ldots, X_m)$ *to* (1) *such that*

$$P\{X(t_0) = x^0\} = 1 \quad \textit{and} \quad P\{X_q(t) \geq 0,\ t \geq t_0\} = 1.$$

(b) *If* $x_q^0 > 0$ *then* $P\{X_q(t) > 0,\ t \geq t_0\} = 1$.

Proof. It is sufficient to prove the lemma with (i) replaced by

(i*) For each $T > 0$ there exists a constant $K_T > 0$ such that for $0 \leq t \leq T$ and $x, y \in P_q$,

$$\|g(t,x) - g(t,y)\| + \|f(t,x) - f(t,y)\| \leq K_T \|x - y\|.$$

Indeed, take $N > 0$ and consider $B_N = \{x \in \mathbb{R}^m : \|x - x^0\| \leq N\}$. Let π_N denote the projector onto B_N, associating with any $x \in \mathbb{R}^m$ the unique element $\pi_N(x) \in B_N$ such that

$$\|x - \pi_N(x)\| = \inf\{\|x - z\| : z \in B_N\}.$$

We consider equation (1) with coefficients $f_N(t,x) = f(t, \pi_N(x))$ and $g_N(t,x) = g(t, \pi_N(x))$. Since f_N and g_N satisfy (i*) and (ii)–(v), we conclude that the solution X_N of (1) with coefficients f_N and g_N satisfies (a) and (b). This gives the lemma.

Thus we are reduced to showing (a) and (b) under assumptions (i*) and (ii)–(v). First we prove (b) by means of a suitable change of variables. Set

$$\varphi(x) = p^{-1}(\ln x), \quad \text{where} \quad p(x) = \int_0^x \frac{ds}{\sqrt{1+s^2}}. \tag{2}$$

It is easily seen that φ is a one-to-one, increasing function from $(0,\infty)$ to $\mathbb{R}$. Let $\psi = \varphi^{-1}$. Then $\Phi(x_1, \ldots, x_m) = (x_1, \ldots, x_{q-1}, \varphi(x_q), x_{q+1}, \ldots, x_m)$ is a diffeomorphism from the interior of K to $\mathbb{R}^m$. Write $\Psi = \Phi^{-1}$ and let J_Φ be the Jacobi matrix of Φ. We consider equation (1) with coefficients

$$\bar{f}(t,x) = (\bar{f}_1(t,x), \ldots, \bar{f}_m(t,x)), \qquad \overline{g}(t,x) = J_\Phi(\Psi(x)) g(t, \Psi(x)),$$

where

$$\begin{aligned}\bar{f}_p(t,x) = \sum_{i=1}^m \frac{\partial}{\partial x_i} &\Phi^{(p)}(\Psi(x))) f_i(t, \Psi(x)) \\ &+ \frac{1}{2} \sum_{i,j=1}^m \sum_{l=1}^r \frac{\partial^2 \Phi^{(p)}}{\partial x_i \partial x_j} (\Psi(x)) g_{il}(t, \Psi(x))) g_{jl}(t, \Psi(x)),\end{aligned}$$

$\Phi = (\Phi^{(1)}, \ldots, \Phi^{(m)})$, $\Psi = (\Psi^{(1)}, \ldots, \Psi^{(m)})$ and $p = 1, \ldots, m$.

We will prove that $\bar{f}$ and $\overline{g}$ satisfy all assumptions of the existence theorem for (1) ([7, p. 300, Th. 3.11]). To this end fix $T > 0$; we will estimate $\bar{f}(t,x)$ and $\overline{g}(t,x)$ for $t \in [0,T]$ and $x \in \mathbb{R}^m$. There exists $L > 0$ such that

$$|\psi(x)| \leq L(1 + |x|) \quad \text{ for } x \in \mathbb{R}.$$

Hence for $i \neq q$ we have $L_T > 0$ such that for $t \in [0,T]$ and $x \in \mathbb{R}^m$,

$$|\overline{g}_{ij}(t,x)|^2 = |g_{ij}(t,\Psi(x))|^2 \le L_T(1+\|\Psi(x)\|^2) \le L_T(1+\|x\|^2).$$

By (iv) we have

$$\begin{aligned}|\overline{g}_{qj}(t,x)| &= |\varphi'(\psi(x_q))g_{qj}(t,\Psi(x))|\\ &= \sqrt{1+x_q^2}|\psi(x_q)|^{-1}|g_{qj}(t,x_1,\ldots,x_{q-1},\psi(x_q),x_{q+1},\ldots,x_m)\\ &\qquad - g_{qj}(t,x_1,\ldots,x_{q-1},0,x_{q+1},\ldots,x_m)| \le L_T\sqrt{1+x_q^2}.\end{aligned}$$

Thus $|\overline{g}(t,x)|^2 \le L_T(1+\|x\|^2)$ for some constant $L_T > 0$ and all $t \in [0,T]$ and $x \in \mathbb{R}^m$. By assumption (ii),

$$\begin{aligned}\sum_{j=1}^r g_{qj}^2(t,\Psi(x)) = \sum_{j=1}^r &(g_{qj}(t,x_1,\ldots,x_{q-1},\psi(x_q),x_{q+1},\ldots,x_m)\\ &- g_{qj}(t,x_1,\ldots,x_{q-1},0,x_{q+1},\ldots,x_m))^2 \le L_T(\psi(x_q))^2.\end{aligned}$$

After some standard calculations, using assumptions (iii) and (iv) we find that

$$\begin{aligned}x_q\varphi''(\psi(x_q))\sum_{j=1}^r g_{qj}^2(t,\Psi(x)) &\le L_T(1+\|x\|^2),\\ \psi(x_q)f_q(t,\Psi(x)) + \sum_{j\neq q} x_j f_j(t,\Psi(x)) &\le L_T(1+\|x\|^2),\\ x_q\varphi'(\psi(x_q))f_q(t,\Psi(x)) - \psi(x_q)f_q(t,\Psi(x)) &\le L_T(1+\|x\|^2).\end{aligned}$$

Thus

$$\begin{aligned}\sum_{j=1}^m x_j\overline{f}_j(t,x) &= \frac{1}{2}x_q\varphi''(\psi(x_q))\sum_{j=1}^r g_{qj}^2(t,\Psi(x))\\ &\quad + x_q\varphi'(\psi(x_q))f_q(t,\Psi(x)) + \sum_{j\neq q} x_j f_j(t,\Psi(x)) \le L_T(1+\|x\|^2)\end{aligned}$$

for $x \in \mathbb{R}^m$ and $0 \le t \le T$. Hence there exists a process $\overline{X}(t)$, $t \ge t_0$, such that $\overline{X}(t_0) = \Phi(x^0)$ and

$$d\overline{X}(t) = \overline{f}(t,\overline{X}(t))dt + \overline{g}(t,\overline{X}(t))\,dW(t).$$

Using the Itô formula ([7, Th. 2.9, p. 287]) one can verify that the process $X(t) = \Psi(t,\overline{X}(t))$, $t \ge t_0$, is a solution to (1) with coefficients $f(t,x)$ and $g(t,x)$, starting at t_0 from x^0. Moreover, $P\{X_q(t) > 0,\ t \ge t_0\} = 1$.

Let now $x^0 = (x_1^0,\ldots,x_m^0)$ be such that $x_q^0 \ge 0$ and let $t_0 \ge 0$. Then there exists a sequence $x^n = (x_1^n,\ldots,x_m^n)$, $n \in \mathbb{N}$, converging to x^0 as $n \to \infty$ and such that $x_q^n > 0$. Denote by $X^0 = (X_1^0,\ldots,X_m^0)$ and $X^n = (X_1^n,\ldots,X_m^n)$ the solutions to equation (1) starting at t_0 from x^0 and x^n respectively. By the second part of the assertion which we have proved, $P\{X_q^n(t) > 0,\ t \geq t_0\} = 1$ for each $n \in \mathbb{N}$. Using the theorem on continuous dependence of solutions to (1) upon initial conditions, we obtain $\mathbb{E}|X^n(t) - X^0(t)|^2 \to 0$ as $n \to \infty$, for $t \geq t_0$. Hence for each $t \geq t_0$ we have $P\{X_q^0(t) \geq 0\} = 1$ and consequently $P\{X_q^0(t) \geq 0,\ t \geq t_0\} = 1$. This completes the proof of the lemma. ■

Proof of Theorem 1. Assume that K has the viability property with respect to (f, g). Fix $\alpha \in I$, $t_0 \geq 0$ and $x^0 \in K$ such that

$$\langle x^0 - a_\alpha, \mathbf{n}_\alpha \rangle = 0.$$

The viability implies that there exists a solution X to (1) such that $X(t_0) = x^0$ and $P\{X(t) \in K, t \geq t_0\} = 1$. Therefore $P\{\langle X(t) - a_\alpha, \mathbf{n}_\alpha\rangle \geq 0,\ t \geq t_0\} = 1$. Since $\mathbf{n}_\alpha \neq 0$, there exists $q \in \{1, \ldots, m\}$ such that $n_q^\alpha \neq 0$, where $\mathbf{n}_\alpha = (n_1^\alpha, \ldots, n_m^\alpha)$.

Let the one-to-one mapping

$$U : P(a_\alpha, \mathbf{n}_\alpha) \to P_q = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_q \geq 0\},$$

be given by the formula

$$\begin{aligned} &U(x_1, \ldots, x_m) \\ &= \Big(x_1 - a_1^\alpha, \ldots, x_{q-1} - a_{q-1}^\alpha, \sum_{i=1}^m (x_i - a_i^\alpha) n_i^\alpha, x_{q+1} - a_{q+1}^\alpha, \ldots, x_m - a_m^\alpha\Big), \end{aligned} \tag{3}$$

where $a_\alpha = (a_1^\alpha, \ldots, a_m^\alpha)$. By the Itô formula, the process $Z(t) = U(X(t))$, $t \geq t_0$, satisfies (1) with coefficients

$$\bar{f} = (\bar{f}_1, \ldots, \bar{f}_m), \quad \bar{g} = [\bar{g}_{ij}], \tag{4}$$

where

$$\bar{f}_i(t, x) = \begin{cases} f_i(t, U^{-1}(x)) & \text{for } i \neq q,\ i \in \{1, \ldots, m\}, \\ \sum_{j=1}^m n_j^\alpha f_j(t, U^{-1}(x)) & \text{for } i = q, \end{cases}$$

and

$$\bar{g}_{ij}(t, x) = \begin{cases} g_{ij}(t, U^{-1}(x)) & \text{for } i \neq q, i \in \{1, \ldots, m\}, j \in \{1, \ldots, r\}, \\ \sum_{p=1}^m n_p^\alpha g_{pj}(t, U^{-1}(x)) & \text{for } i = q,\ j \in \{1, \ldots, r\}. \end{cases}$$

Moreover, $Z(t_0) = z^0 = (z_1^0, \ldots, z_m^0)$, where $z_q^0 = 0$, and $P\{Z(t) \in P_q,\ t \geq t_0\} = 1$. Since $\bar{f}$ and $\bar{g}$ satisfy all assumptions of Lemma 1 we conclude that $\bar{f}_q(t_0, z^0) \geq 0$ and $\bar{g}_{qj}(t_0, z^0) = 0$ for $j \in \{1, \ldots, r\}$ and consequently we obtain condition (a) from the statement of our theorem.

The proof of sufficiency of (a) will be divided into several steps.

Step 1. We assume additionally that K is a half-space:

$$K = P(a, \mathbf{n}) = \{x \in \mathbb{R}^m : \langle x - a, \mathbf{n}\rangle \geq 0\},$$

and that f and g satisfy the assumptions of Theorem 1 with (ii) replaced by (ii′) $\forall T > 0\ \forall N > 0\ \exists K_{T,N} > 0\ \forall t \in [0,T]\ \forall x, y \in K,\ \|x\|, \|y\| \le N,$

$$\|f(t,x) - f(t,y)\| + \|g(t,x) - g(t,y)\| \le K_{T,N}\|x - y\|.$$

Fix $t_0 \ge 0$ and $x^0 \in K$. Our purpose is to find a process $X(t)$, $t \ge t_0$, satisfying (1) such that $P\{X(t) \in K,\ t \ge t_0\} = 1$. To this end, take $q \in \{1, \ldots, m\}$ such that $n_q \ne 0$, where $\mathbf{n} = (n_1, \ldots, n_m)$. We consider equation (1) with coefficients $\bar{f}$ and $\bar{g}$ defined on P_q by (4). Since they satisfy all assumptions of Lemma 2, there is a solution $Z(t) = (Z_1(t), \ldots, Z_m(t))$, $t \ge t_0$, to (1) with coefficients $\bar{f}$ and $\bar{g}$ such that $Z(t_0) = U(x^0)$ and $P\{Z_q(t) \ge 0, t \ge t_0\} = 1$. By the Itô formula, the process $X(t) = U^{-1}(Z(t))$, $t \ge t_0$, satisfies our requirements.

Step 2. Our next goal is to construct some special extensions of the coefficients f and g of (1).

Let $K^\varepsilon = \{x \in \mathbb{R}^m : d(x,K) \le \varepsilon\}$, where $d(x,K) = \inf\{\|x - z\| : z \in K\}$ and $\varepsilon > 0$. By Theorem A1 (see Appendix) there exist $\varepsilon > 0$ and $\pi : K^\varepsilon \to K$ such that

(a) $\pi|_K = \mathrm{id}$,
(b) π is Lipschitz continuous,
(c) $\forall \alpha \in I,\ \forall x \in K^\varepsilon,\ \langle x - a_\alpha, \mathbf{n}_\alpha\rangle = 0 \Rightarrow \langle \pi(x) - a_\alpha, \mathbf{n}_\alpha\rangle = 0.$

Fix $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$ and let $\varphi : [0,\infty) \to [0,1]$ be a Lipschitz continuous function such that $\varphi|_{[0,\varepsilon_1]} \equiv 1$ and $\varphi|_{[\varepsilon_2,\infty)} \equiv 0$. Define

$$\widetilde{f}(t,x) = \begin{cases} \varphi(d(x,K))f(t,\pi(x)), & x \in K^\varepsilon,\ t \ge 0, \\ 0, & x \notin K^\varepsilon,\ t \ge 0, \end{cases}$$

and

$$\widetilde{g}(t,x) = \begin{cases} \varphi(d(x,K))g(t,\pi(x)), & x \in K^\varepsilon,\ t \ge 0, \\ 0, & x \notin K^\varepsilon,\ t \ge 0. \end{cases}$$

We will show that $\widetilde{f}$ and $\widetilde{g}$ satisfy the following conditions:

(i′) For all $T > 0$ there exists $L_T > 0$ such that for all $t \in [0,T]$ and $x \in \mathbb{R}^m$,

$$\|\widetilde{f}(t,x)\|^2 + \|\widetilde{g}(t,x)\|^2 \le L_T(1 + |x|^2).$$

(ii′) For all $T, N > 0$ there exists $L_{T,N} > 0$ such that for all $t \in [0,T]$ and $x, y \in \mathbb{R}^m$ with $\|x\|, \|y\| \le N$,

$$\|\widetilde{f}(t,x) - \widetilde{f}(t,y)\| + \|\widetilde{g}(t,x) - \widetilde{g}(t,y)\| \le L_{T,N}\|x - y\|.$$

(iii′) For all $t \ge 0$, $x \in \mathbb{R}^m$, $\alpha \in I$ and $j = 1, \ldots, r$, the equality $\langle x - a_\alpha, \mathbf{n}_\alpha\rangle$
$= 0$ implies $\langle \widetilde{f}(t,x), \mathbf{n}_\alpha\rangle \ge 0$ and $\langle \widetilde{g}_j(t,x), \mathbf{n}_\alpha\rangle = 0$.

Here $\widetilde{g}_j$ is the jth column of the matrix $\widetilde{g} = [\widetilde{g}_{ij}]$.

P r o o f o f (i′)–(iii′). Condition (i′) is true because $\|\widetilde{f}\| \le \|f\|$, $\|\widetilde{g}\| \le \|g\|$, π is Lipschitz continuous, and f and g satisfy assumption (i) of Theorem 1.

In order to prove (ii′) we consider two cases.

C a s e 1: $x \in K^\varepsilon$, $y \in K^\varepsilon$. Then (ii′) holds because π and φ are Lipschitz continuous, and f and g satisfy assumptions (i) and (ii) of Theorem 1.

C a s e 2: $x \in K^\varepsilon$, $y \notin K^\varepsilon$. We consider the only interesting case $d(x,K) \le \varepsilon_2$. There exists z on the line segment with end points x and y such that $d(z,K) > \varepsilon_2$ and $z \in K^\varepsilon$. Hence $\varphi(d(z,K)) = \varphi(d(y,K)) = 0$ and using the result of Case 1 we obtain

$$\begin{aligned}\|\widetilde{f}(t,x) - \widetilde{f}(t,y)\| &\le \|\widetilde{f}(t,x) - \widetilde{f}(t,z)\| + \|\widetilde{f}(t,z) - \widetilde{f}(t,y)\| \\ &\le L_{T,N}\|x-z\| \le L_{T,N}\|x-y\|.\end{aligned}$$

Since f and g satisfy condition (a) of Theorem 1 and π satisfies (c), we have (iii′).

S t e p 3. Finally, we show that condition (a) of Theorem 1 implies the viability property of our polyhedron K. Having f and g, defined for $t \ge 0$ and $x \in K$, we extend them to all of $\mathbb{R}^m$, using the construction presented in Step 2. We denote the extensions by $\widetilde{f}$ and $\widetilde{g}$.

Let $X(t)$, $t \ge t_0$, be a solution to (1) with coefficients $\widetilde{f}$ and $\widetilde{g}$ and with $X(t_0) = x^0 \in K$. For every $\alpha \in I$, by the uniqueness theorem for (1) and by Step 1 we conclude that

$$P\{\langle X(t) - a_\alpha, \mathbf{n}_\alpha\rangle \ge 0,\ t \ge t_0\} = 1,$$

and consequently $P\{X(t) \in K,\ t \ge t_0\} = 1$. This completes the proof of Theorem 1. ∎

4. Proof of Theorem 2.

Define

$$K = \{(x_1,\dots,x_m,\overline{x}_1,\dots,\overline{x}_m) \in \mathbb{R}^{2m} : x_p \le \overline{x}_p,\ p \in I\}.$$

Denote by $\mathbf{e}_s$, $s \in \{1,\dots,2m\}$, the canonical basis of $\mathbb{R}^{2m}$. It is easy to see that $K = \bigcap_{p\in I} H_p$, where $H_p = \{z \in \mathbb{R}^{2m} : \langle z, \mathbf{n}_p\rangle \ge 0\}$ and

$$\langle \mathbf{n}_p, \mathbf{e}_s\rangle = \begin{cases} -1 & \text{for } s = p, \\ +1 & \text{for } s = m+p, \\ 0 & \text{for } s \ne p \text{ and } s \ne m+p. \end{cases}$$

Consider two m-dimensional Itô processes:

$$X(t) = x^0 + \int_{t_0}^{t} f(s,X(s))\,ds + \int_{t_0}^{t} g(s,X(s))\,dW(s),$$

$$\overline{X}(t) = \overline{x}^0 + \int_{t_0}^{t} \bar{f}(s, \overline{X}(s))\, ds + \int_{t_0}^{t} \overline{g}(s, \overline{X}(s))\, dW(s).$$

Define $Z(t) = (X_1(t), \ldots, X_m(t), \overline{X}_1(t), \ldots, \overline{X}_m(t))$ for $t \ge t_0$. It is easily seen that

$$Z(t) = z^0 + \int_{t_0}^{t} F(s, Z(s))\, ds + \int_{t_0}^{t} G(s, Z(s))\, dW(s),$$

where

$$F(s,z) = \begin{bmatrix} f(s,\pi_1(z)) \\ \bar{f}(s,\pi_2(z)) \end{bmatrix}, \quad G(s,z) = \begin{bmatrix} g(s,\pi_1(z)) \\ \overline{g}(s,\pi_2(z)) \end{bmatrix}, \quad z^0 = (x^0, \overline{x}^0),$$

$$\pi_1(x_1, \ldots, x_m, \overline{x}_1, \ldots, \overline{x}_m) = (x_1, \ldots, x_m),$$

$$\pi_2(x_1, \ldots, x_m, \overline{x}_1, \ldots, \overline{x}_m) = (\overline{x}_1, \ldots, \overline{x}_m).$$

Condition (a) of Theorem 2 is equivalent to the stochastic invariance of K for the pair (F, G). Since all assumptions of the existence and uniqueness theorems for (1) are satisfied by f and g and consequently by F and G, the stochastic invariance of K for (F, G) is equivalent to the stochastic viability property of K with respect to (F, G). By Theorem 1 we conclude that the latter is equivalent to the following condition: $\forall p \in I,\ \forall z \in K$, if $\langle z, \mathbf{n}_p \rangle = 0$ then for $t \ge 0$,

$$\langle F(t,z), \mathbf{n}_p \rangle \ge 0 \quad \text{and} \quad \langle G_j(t,z), \mathbf{n}_p \rangle = 0 \quad \text{for } j = 1, \ldots, r,$$

where G_j is the jth column of the matrix $G = [G_{ij}]$. It is easily seen that the last condition is exactly condition (b) of our theorem. ∎

Appendix. Let $d(x, K)$ denote the distance between $x \in \mathbb{R}^m$ and a closed subset $K \subset \mathbb{R}^m$, defined by $d(x, K) = \inf\{\|x - z\| : z \in K\}$. Let $K^\varepsilon = \{x \in \mathbb{R}^m : d(x, K) \le \varepsilon\}$, $\varepsilon > 0$. Suppose that K is a polyhedron in $\mathbb{R}^m$. Recall this means that $K = \bigcap_{\alpha=1}^{N} P_\alpha(a_\alpha, \mathbf{n}_\alpha)$, where

$$P_\alpha(a_\alpha, \mathbf{n}_\alpha) = \{x \in \mathbb{R}^m : \langle x - a_\alpha, \mathbf{n}_\alpha \rangle \ge 0\}, \qquad a_\alpha, \mathbf{n}_\alpha \in \mathbb{R}^m.$$

Put

$$H_\alpha = \{x \in \mathbb{R}^m : \langle x - a_\alpha, \mathbf{n}_\alpha \rangle = 0\}, \qquad \alpha \in I = \{1, \ldots, N\}.$$

This section will be devoted to the proof of the following theorem.

THEOREM A1. *Let K be a polyhedron in $\mathbb{R}^m$. Then there exist $\varepsilon > 0$ and $f : K^\varepsilon \to K$ such that*

(1) $f|K = \mathrm{id}$,
(2) *f is Lipschitz continuous*,
(3) $\forall \alpha \in I,\ \forall x \in K^\varepsilon$, *if $x \in H_\alpha$ then $f(x) \in H_\alpha$*.

Before we start the proof, we state the relevant theorems from [5], [13] and [14], thus making the Appendix selfcontained.

THEOREM A2 [5, Th. 32.1, p. 85]. *The intersection of two hyperplanes, H' of dimension k' and H'' of dimension k'', lying in $\mathbb{R}^m$ is either empty or a hyperplane of dimension at least $k' + k'' - m$.*

THEOREM A3 [13, Th. 4.15, p. 49]. *Let K be a polyhedron in $\mathbb{R}^m$. Then there exist points $a_i \in \mathbb{R}^m, i = 1, \ldots, r$, and vectors $\mathbf{v}_j, j = 1, \ldots, s$, such that*

$$K = \Big\{ x = \sum_{i=1}^{r} \alpha_i a_i + \sum_{j=1}^{s} \beta_j \mathbf{v}_j : \alpha_i \ge 0,\ \sum_{i=1}^{r} \alpha_i = 1,\ \beta_j \ge 0 \Big\}.$$

THEOREM A4 [13, Th. 2.3, p. 19]. *If M_1 and M_2 are convex and disjoint subsets of $\mathbb{R}^m$, then there exists a vector $\mathbf{n} \in \mathbb{R}^m$ such that*

$$\langle x - y, \mathbf{n} \rangle \le 0 \quad \text{for all } x \in M_1,\ y \in M_2.$$

THEOREM A5 [13, Th. 2.4, p. 19]. *If M_1 and M_2 are convex, disjoint subsets of $\mathbb{R}^m$ and one of them is compact, then there exist a vector $\mathbf{n}$ and $\varepsilon > 0$ such that*

$$\langle x - y, \mathbf{n} \rangle \le -\varepsilon \quad \text{for all } x \in M_1,\ y \in M_2.$$

THEOREM A6 [14, Th. 1.31, p. 21]. *Let H be a Hilbert space, S any subset of H, and $\Phi : S \to H$. Suppose $\|\Phi(x) - \Phi(y)\| < D\|x - y\|$ for all $x, y \in S$. Then Φ can be extended to all of H in such a way that the extension satisfies the same Lipschitz condition.*

The proof of Theorem A1 will be divided into several parts.

Remark A1. Let K be a polyhedron in $\mathbb{R}^m$. Then by Theorem A3 there exist points $a_i \in \mathbb{R}^m$, $i = 1, \ldots, r$, and vectors $\mathbf{v}_j$, $j = 1, \ldots, s$, such that

$$K = \Big\{ x = \sum_{i=1}^{r} \alpha_i a_i + \sum_{j=1}^{s} \beta_j \mathbf{v}_j : \alpha_i \ge 0,\ \sum_{i=1}^{r} \alpha_i = 1,\ \beta_j \ge 0 \Big\}.$$

We shall say that K *is determined by the points a_i and the vectors $\mathbf{v}_j$*.

Assume, moreover, that

$$\forall x \in K, \quad \langle x - a, \mathbf{n} \rangle \le 0$$

for a given vector $\mathbf{n} \in \mathbb{R}^m$ and a point $a \in \mathbb{R}^m$. Set

$$H = \{x : \langle x - a, \mathbf{n} \rangle = 0\}.$$

It is easy to see that either $K \cap H = \emptyset$ or $K \cap H$ is the polyhedron in H determined by all points a_i, $i \in \{1, \ldots, r\}$, such that $a_i \in H$ and by all vectors $\mathbf{v}_j$, $j \in \{1, \ldots, s\}$, such that $\langle \mathbf{v}_j, \mathbf{n} \rangle = 0$.

Lemma A1. *Suppose that K and L are disjoint polyhedrons in $\mathbb{R}^m$. Then there exists a half-space $P = \{x : \langle x - a, \mathbf{n}\rangle \geq 0\}$ such that $K \subset P$ and $L \cap P = \emptyset$.*

Proof. The proof is by induction on m. In fact, assuming the lemma to hold for $m \leq k$, we will construct the required half-space for $m = k + 1$, using our inductive assumption twice: for $m = 2$ and for $m = k$. Hence we begin with the cases $m = 1$ and $m = 2$. We give the proof only for the latter.

By Theorem A3, K is determined by some points $a_1, \dots, a_r$ and vectors $\mathbf{n}_1, \dots, \mathbf{n}_s$ and L is determined by $b_1, \dots, b_p$ and $\mathbf{w}_1, \dots, \mathbf{w}_q$. By Theorem A4, there exists a vector $\mathbf{n} \in \mathbb{R}^2$ such that

$$\forall x \in K,\ \forall y \in L, \qquad \langle x - y, \mathbf{n}\rangle \leq 0.$$

Define

$$x \leq y \quad \text{if and only if} \quad \langle x - y, \mathbf{n}\rangle \leq 0.$$

Let $a_i \leq a_p$ for all $i \in \{1, \dots, r\}$. Consider the line

$$l = \{x \in \mathbb{R}^2 : \langle x - a_p, \mathbf{n}\rangle = 0\}.$$

Of course, $\langle a_p - y, \mathbf{n}\rangle \leq 0$ for each $y \in L$ and it is easy to check that $\langle x - a_p, \mathbf{n}\rangle \leq 0$ for each $x \in K$. Hence, if $K \cap l = \emptyset$ or $L \cap l = \emptyset$ then the proof of the lemma in the case $m = 2$ is finished.

It remains to consider the more difficult case $K \cap l \neq \emptyset$ and $L \cap l \neq \emptyset$. Since $K \cap l$ and $L \cap l$ are disjoint polyhedrons contained in l, there exists an ordering "$<$" in l such that $x < y$ for all $x \in K \cap l$ and $y \in L \cap l$. Let

$$d_0 = \sup\{x : x \in K \cap l\} \quad \text{and} \quad g_0 = \inf\{x : x \in L \cap l\}.$$

Of course, $d_0 < g_0$. Take points c_1 and c_2 such that $d_0 < c_1 < c_2 < g_0$. By Theorem A5 there exist vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ such that for all $x \in K$ and $y \in L$,

$$\langle x - c_1, \mathbf{n}_1\rangle > 0 \quad \text{and} \quad \langle y - c_2, \mathbf{n}_2\rangle > 0.$$

If $\mathbf{n}_1 \parallel \mathbf{n}_2$, then we can assume that $\mathbf{n}_1 = \mathbf{n}_2$ or $\mathbf{n}_1 = -\mathbf{n}_2$. Since we have $\langle c_2 - g_0, \mathbf{n}_2\rangle < 0$ and $\langle c_2 - g_0, \mathbf{n}_1\rangle > 0$ we conclude that only the case $\mathbf{n}_1 = -\mathbf{n}_2$ is possible. Clearly, $\langle c_2 - c_1, \mathbf{n}_1\rangle < 0$ and $\langle y - c_2, \mathbf{n}_1\rangle < 0$ for $y \in L$. Consequently, $\langle y - c_1, \mathbf{n}_1\rangle < 0$ for every $y \in L$ and the half-space

$$P_1 = \{x : \langle x - c_1, \mathbf{n}_1\rangle \geq 0\}$$

satisfies our requirements.

Now we consider the case when $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel. Set

$$l_1 = \{x : \langle x - c_1, \mathbf{n}_1\rangle = 0\}, \quad l_2 = \{x : \langle x - c_2, \mathbf{n}_2\rangle = 0\}, \quad l_1 \cap l_2 = \{x\}.$$

We have two possibilities : either $\langle x - c_1, \mathbf{n}\rangle > 0$ or $\langle x - c_1, \mathbf{n}\rangle < 0$. We consider the case $\langle x - c_1, \mathbf{n}\rangle > 0$, the other one is analogous. We will show that

$$P = \{x : \langle x - c_2, \mathbf{n}_2\rangle < 0\}$$

is the required half-space. Observe that $\langle y - c_2, \mathbf{n}_2\rangle > 0$ for $y \in L$. If $y \in K$ then there exist $\alpha \geq 0$ and $\beta \geq 0$ such that

$$y = c_1 + \alpha(d_0 - c_1) + \beta(c_1 - x).$$

Since

$$\operatorname{sgn}(d_0 - c_1, \mathbf{n}_2) = \operatorname{sgn}(c_2 - g_0, \mathbf{n}_2) = -1,$$
$$\operatorname{sgn}(c_1 - c_2, \mathbf{n}_2) = \operatorname{sgn}(c_2 - g_0, \mathbf{n}_2) = -1,$$

for each $y \in K$ we have

$$\begin{aligned}\langle y - c_2, \mathbf{n}_2\rangle &= \langle y - c_1, \mathbf{n}_2\rangle + \langle c_1 - c_2, \mathbf{n}_2\rangle \\ &= \alpha\langle d_0 - c_1, \mathbf{n}_2\rangle + \beta\langle c_1 - x, \mathbf{n}_2\rangle + \langle c_1 - c_2, \mathbf{n}_2\rangle < 0,\end{aligned}$$

which proves the lemma in the case $m = 2$.

Assume now that the lemma holds for $m \leq k$, where $k \geq 2$. We will prove it for $m = k+1$. Suppose that K and L are disjoint polyhedrons in $\mathbb{R}^{k+1}$. By Theorem A3, K is determined by some points $a_1, \ldots, a_t$ and vectors $\mathbf{w}_1, \ldots, \mathbf{w}_r$ and L is determined by $b_1, \ldots, b_l$ and $\mathbf{v}_1, \ldots, \mathbf{v}_\kappa$.

By Theorem A4, there exists a vector $\mathbf{n} \in \mathbb{R}^m$ such that $\langle y - x, \mathbf{n}\rangle \geq 0$ for all $y \in K$ and $x \in L$. Let $b_i \leq \overline{b}$ for $\overline{b} \in \{b_1, \ldots, b_l\}$ and $i \in \{1, \ldots, l\}$ (here $b_i \leq b_j$ if and only if $\langle b_i - b_j, \mathbf{n}\rangle \leq 0$). Define

$$P = \{x : \langle x - \overline{b}, \mathbf{n}\rangle = 0\}.$$

Of course, $\langle y - \overline{b}, \mathbf{n}\rangle \leq 0$ for all $y \in L$. If $\langle a_i - \overline{b}, \mathbf{n}\rangle > 0$ for each $i \in \{1, \ldots, t\}$, then it is easy to see that $\langle y - \overline{b}, \mathbf{n}\rangle > 0$ for all $y \in K$ and P is the required half-space.

It remains to consider the case when there exists a_i such that $\langle a_i - \overline{b}, \mathbf{n}\rangle = 0$. Since $K \cap P = K_1 \neq \emptyset$ and $L \cap P = L_1 \neq \emptyset$, K_1 and L_1 are disjoint polyhedrons in the k-dimensional space P.

Using our inductive assumption we have a vector $\overline{\mathbf{n}}$ parallel to P and such that $\langle y_1 - x_1, \overline{\mathbf{n}}\rangle > 0$ for all $x_1 \in L_1$ and $y_1 \in K_1$. By Remark A1, L_1 is determined by $b_{i_1}, \ldots, b_{i_p}$ and $\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_q}$, where $b_{i_s} \in P$ and $\mathbf{v}_{j_t}$ is parallel to P for $s = 1, \ldots, p$ and $t = 1, \ldots, q$. Considering the relation

$$b_i \leq b_j \quad \text{if and only if} \quad \langle b_i - b_j, \overline{\mathbf{n}}\rangle \leq 0$$

we have $\overline{\overline{b}} \in \{b_{i_1}, \ldots, b_{i_p}\}$ such that $b_{i_s} \leq \overline{\overline{b}}$ for $s = 1, \ldots, p$. We can assume that $\overline{\overline{b}} = 0$. We write H for the linear subspace of $\mathbb{R}^m$ generated by the vectors $\mathbf{n}$ and $\overline{\mathbf{n}}$. Let π_H denote the projector onto H, associating with any $x \in \mathbb{R}^m$ the unique element $\pi_H(x) \in H$ satisfying $|x - \pi_H(x)| = d(x, H)$. Define $K' = \pi_H(K)$ and $L' = \pi_H(L)$. Of course, K' and L' are polyhedrons in H. Moreover, H was chosen so that K' and L' are disjoint. Hence using our inductive assumption again, we have a vector $\widetilde{\mathbf{n}} \,\|\, H$ such that $\langle y' - x', \widetilde{\mathbf{n}}\rangle > 0$ for all $y' \in K'$ and $x' \in L'$.

Let $y \in K$, $x \in L$ and let $H^{\perp}$ be the orthogonal complement of H. Then $y = y' + \overline{y}$, where $y' \in H$ and $\overline{y} \in H^{\perp}$. Analogously, $x = x' + \overline{x}$, where $x' \in H$ and $\overline{x} \in H^{\perp}$. Hence we have $\langle y - x, \widetilde{\mathbf{n}}\rangle = \langle y' - x', \widetilde{\mathbf{n}}\rangle + \langle \overline{y} - \overline{x}, \widetilde{\mathbf{n}}\rangle = \langle y' - x', \widetilde{\mathbf{n}}\rangle > 0$. Choose $b \in \{b_1, \ldots, b_l\}$ such that $b_i \le b$ for $i = 1, \ldots, l$. Then

$$\{x : \langle x - b, \widetilde{\mathbf{n}}\rangle \ge 0\}$$

is the desired half-space and the proof is complete. ∎

LEMMA A2. *Assume that K and L are disjoint polyhedrons in $\mathbb{R}^m$. Then $d(K, L) = \inf\{|x - y| : x \in K,\ y \in L\} > 0$.*

Proof. By Lemma A1 we can suppose without any loss of generality that $L = \{x : \langle x - a, \mathbf{n}\rangle \ge 0\}$ for some $a \in \mathbb{R}^m$ and $\mathbf{n} \in \mathbb{R}^m$.

Moreover, we can assume that $a = 0$. Let

$$K = \Big\{ \sum_{i=1}^{r} \alpha_i x_i + \sum_{j=1}^{s} \beta_j \mathbf{v}_j : \alpha_i \ge 0,\ \sum_{i=1}^{r} \alpha_i = 1,\ \beta_j \ge 0 \Big\}.$$

Since $K \cap L = \emptyset$ we have $\langle \sum_{i=1}^{r} \alpha_i x_i + \sum_{j=1}^{s} \beta_j \mathbf{v}_j, \mathbf{n}\rangle < 0$ for α_i and β_j as above.

Let

$$K_0 = \Big\{ \sum_{i=1}^{r} \alpha_i x_i : \alpha_i \ge 0,\ \sum_{i=1}^{r} \alpha_i = 1 \Big\}.$$

For each $x \in K_0$ we have $\langle x, \mathbf{n}\rangle < 0$. Since K_0 is compact, there exists a positive constant γ such that $\langle x, \mathbf{n}\rangle \le -\gamma$ for each $x \in K_0$. It is easily seen that $\langle \mathbf{v}_j, \mathbf{n}\rangle \le 0$ for each $j = 1, \ldots, s$. Hence $\langle x, \mathbf{n}\rangle \le -\gamma$ for each $x \in K$ and so $d(K, L) > 0$. ∎

We shall use the next lemmas to prove that the projection f from the statement of Theorem A1, which we construct, satisfies the Lipschitz condition.

LEMMA A3. *Let H_1 and H_2 be subspaces of $\mathbb{R}^m$ such that $H_1 \cap H_2 \ne \emptyset$. Let $f : H_1 \cup H_2 \to \mathbb{R}^m$ be such that $f|_{H_1}$ and $f|_{H_2}$ are Lipschitz continuous. Then f is Lipschitz continuous.*

Proof. Set $E = H_1 \cap H_2$. Let $x \in H_1 \setminus H_2$, $y \in H_2 \setminus H_1$ and $a = \pi_E(x)$. We consider two cases:

1) The points x, y and a are collinear. Since H_1 and H_2 are convex, a belongs to the line segment with endpoints x and y. Then

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f(a)| + |f(a) - f(y)| \\ &\le L|x - a| + L|a - y| = L|x - y|. \end{aligned}$$

2) The points x, y and a are not collinear, so they determine a triangle xya. Let α, β, γ denote the angles axy, xya and xay respectively. Then

$$\frac{|x-a|}{\sin\beta} = \frac{|a-y|}{\sin\alpha} = \frac{|x-y|}{\sin\gamma} = c,$$

and consequently

$$|f(x)-f(y)| \le L(|x-a|+|a-y|) \le 2L|x-y|\frac{1}{\sin\gamma}.$$

It is sufficient to show that there exists $\delta>0$ such that $\sin\gamma\ge\delta$. Let $e\in E$ and $w\in E^\perp\cap H_2$ be such that $y-a=e+w$. Since $x-a\in E^\perp$, we have

$$|\cos\gamma| = \left|\left\langle \frac{y-a}{|y-a|}, \frac{x-a}{|x-a|}\right\rangle\right| = \left|\left\langle \frac{w}{|e+w|}, \frac{x-a}{|x-a|}\right\rangle\right| \le \left|\left\langle \frac{w}{|w|}, \frac{x-a}{|x-a|}\right\rangle\right|$$

($x-a\ne 0$, $y-a\ne 0$ and $w\ne 0$, because $x\notin E$ and $y\notin E$). Since $E^\perp\cap H_1\cap H_2=\{0\}$, it follows that for each $u\in E^\perp\cap H_1$ and $v\in E^\perp\cap H_2$ such that $|u|=|v|=1$ we have $|\langle u,v\rangle|<1$. As the sphere in a finite-dimensional space is compact, there exists $\delta>0$ such that $\sin\gamma\ge\delta$, which completes the proof. ∎

LEMMA A4. *Let* $K=\bigcap_{\alpha=1}^r P_\alpha$, *where*

$$P_\alpha = P(a_\alpha,\mathbf{n}_\alpha) = \{x\in\mathbb{R}^m : \langle x-a_\alpha,\mathbf{n}_\alpha\rangle = 0\},\ a_\alpha,\mathbf{n}_\alpha\in\mathbb{R}^m,\ \alpha=1,\dots,r,$$

and define

$$H_\alpha = H(a_\alpha,\mathbf{n}_\alpha) = \{x\in\mathbb{R}^m : \langle x-a_\alpha,\mathbf{n}_\alpha\rangle = 0\}.$$

Let $S=H_{i_1}\cap\ldots\cap H_{i_n}\ne\emptyset$, *where* $\{i_1,\dots,i_n\}\subset\{1,\dots,r\}$. *Then there exist* $C>0$ *and* $\delta>0$ (*depending on* S) *such that for all* $x\in S$, *and all* $\varepsilon\in(0,\delta)$,

$$d(x,K)\le\varepsilon \quad \textit{implies} \quad d(x,S\cap K)\le C\varepsilon.$$

Proof. The proof will be divided into two steps.

Step 1. Let $H_1,\dots,H_d$ be $(m-1)$-dimensional subspaces of $\mathbb{R}^m$ and $E=H_1\cap\ldots\cap H_d\ne\emptyset$. Let π and π_i denote the orthogonal projectors onto E and H_i respectively, $i=1,\dots,d$. Then there exists $C>0$ such that for each $x\in\mathbb{R}^m$ we have

$$|x-\pi(x)| \le C\max\{|x-\pi_i(x)| : i=1,\dots,d\}. \tag{1}$$

The proof is by induction on m. Of course, (1) is true for $m=1$. Assuming (1) to hold for $m-1$, we will prove it for m. Fix $x\in\mathbb{R}^m$. If $x\in E$ we can take $C=1$.

Hence we consider the case when $x\notin H_r$ for some $r\in\{1,\dots,d\}$. Since $E\subset H_r$ one can check that $\pi(\pi_r(x))=\pi(x)$. Namely,

$$|\pi(\pi_r(x))-\pi(x)|^2 = \langle \pi(\pi_r(x))-\pi_r(x), \pi(\pi_r(x))-\pi(x)\rangle + \langle \pi_r(x)-x, \pi(\pi_r(x))-\pi(x)\rangle + \langle x-\pi(x), \pi(\pi_r(x))-\pi(x)\rangle = 0.$$

Consequently, by the Pythagoras theorem we have

$$|x-\pi(x)| = (|x-\pi_r(x)|^2 + |\pi_r(x)-\pi(x)|^2)^{1/2}. \tag{2}$$

Of course,

$$|x-\pi_r(x)| \le \max\{|x-\pi_i(x)| : i=1,\ldots,d\}. \tag{3}$$

Since H_i and H_r are not parallel, we conclude that $\dim H_i \cap H_r = m-2$ by Theorem A2, and we can use our inductive assumption:

$$\begin{aligned}|\pi_r(x)-\pi(x)| &= d(\pi_r(x), (H_1\cap H_r)\cap\ldots\cap(H_{r-1}\cap H_r) \\ &\qquad \cap (H_{r+1}\cap H_r)\cap\ldots\cap(H_d\cap H_r)) \\ &\le C\max\{d(\pi_r(x), H_i\cap H_r) : i=1,\ldots,d,\ i\ne r\}.\end{aligned} \tag{4}$$

Let $\mathbf{n}_i$ and $\mathbf{n}_r$ be normal vectors to H_i and H_r respectively, and let H denote the plane parallel to $\mathbf{n}_i$ and $\mathbf{n}_r$ such that $x \in H$. It is obvious that $\pi_i(x), \pi_r(x) \in H$. Since $\mathbf{n}_i, \mathbf{n}_r \perp H_i \cap H_r$, we have $\dim(H_i\cap H_r)^\perp \ge 2$. But by Theorem A2 we know that $\dim H_i \cap H_r \ge m-2$, so $\dim(H_i\cap H_r)^\perp = 2$. Moreover, since $H \subset (H_i\cap H_r)^\perp$ we conclude that $H = (H_i\cap H_r)^\perp$. Let $z = \pi_{H_i\cap H_r}(\pi_r(x))$. As $z-\pi_r(x) \perp H_i\cap H_r$ we have $z-\pi_r(x) \parallel H$ and since $\pi_r(x) \in H$ we conclude that $z \in H$. Thus the quadrilateral with vertices $x, \pi_r(x), \pi_i(x)$ and z lies in the 2-dimensional plane H. The vectors $\pi_i(x)-x, z-\pi_i(x)$ are orthogonal, and so are $x-\pi_r(x)$, $\pi_r(x)-x$. Moreover, $|x-\pi_r(x)| > 0$. Considering all possible cases:

(i) $z = \pi_r(x)$,
(ii) $z \ne \pi_r(x)$, $\pi_i(x) = x$,
(iii) $z \ne \pi_r(x)$, $\pi_i(x) \ne x$,

we conclude that there exists $C > 0$ such that

$$|z-\pi_r(x)| \le C\max\{|x-\pi_r(x)|, |x-\pi_i(x)|\}.$$

Taking into account (4) and the last inequality, we obtain (1).

Step 2. Let K and S satisfy the assumptions of Lemma A4 and let $C > 0$ be the constant given by Step 1. Set

$$\mathcal{G} = \{H_{\alpha_1}\cap\ldots\cap H_{\alpha_k} : \{\alpha_1,\ldots,\alpha_k\} \subset \{1,\ldots,r\}\}$$

and $\varrho = \min\{d(D,K) : D\in\mathcal{G},\ D\cap K = \emptyset\}$, where $d(D,K) = \inf\{|x-y| : x\in D,\ y\in K\}$. Assume $x \in S$ and $d(x,K) \le \varepsilon$, where $\varepsilon \in (0, \varrho/(C+1)^r)$. We now construct a finite sequence of points $x_0, x_1, \ldots, x_l$ such that:

(i) $x_0 = x$,

(ii) $x_k \in S_k = S \cap H_{\alpha_1} \cap \ldots \cap H_{\alpha_k}$, where $H_{i_1}, \ldots, H_{i_n}, H_{\alpha_1}, \ldots, H_{\alpha_k}$ are all distinct and $S_k \cap K \neq \emptyset$,

(iii) $d(x_k, K) \leq (1 + C)^k \varepsilon$,

(iv) $|x_{k+1} - x_k| \leq C(1 + C)^k \varepsilon$,

(v) $x_k \notin K$ for $k < l$ and $x_l \in K$.

Note that x_0 satisfies conditions (i)–(iii).

Suppose that a sequence $x_0, \ldots, x_k$ satisfies conditions (i)–(iv) and $x_k \notin K$. Hence there exists $\alpha \in \{1, \ldots, r\}$ such that $\langle x_k - a_\alpha, \mathbf{n}_\alpha \rangle < 0$. Since $S_k \cap K \neq \emptyset$ there exists $y \in S_k$ such that $\langle y - a_\alpha, \mathbf{n}_\alpha \rangle \geq 0$ and consequently $S_k \cap H_\alpha \neq \emptyset$. We define S_{k+1} to be $S_k \cap H_\alpha$ and $x_{k+1} = \pi_{S_{k+1}}(x_k)$. By Step 1 we have for $j = 1, \ldots, n$, $s = 1, \ldots, k$,

$$|x_{k+1} - x_k| \leq C \max\{d(x_k, H_{i_j}), d(x_k, H_{\alpha_s}), d(x_k, H_\alpha)\}.$$

But $d(x_k, H_\alpha) \leq d(x_k, K)$ so by (iii), keeping in mind that $x_k \in S_k$, we obtain $|x_{k+1} - x_k| \leq C(1+C)^k \varepsilon$. Consequently, $d(x_{k+1}, K) \leq |x_{k+1} - x_k| + |x_k - \pi_K(x_k)| \leq (C + 1)^{k+1} \varepsilon$. We observe that $k + 1 \leq r - s$.

Since $d(S_{k+1}, K) \leq |x_{k+1} - \pi_K(x_k)| < \varrho$, we conclude that $S_{k+1} \cap K \neq \emptyset$. We have only a finite number of different H_α so we can continue our construction at most $l = r - n$ times and finally $x_l \in K$.

Having such a sequence, it is easy to finish the proof of the lemma:

$$d(x, S \cap K) \leq |x - x_l| \leq \sum_{k=0}^{l-1} |x_{k+1} - x_k| \leq \sum_{k=0}^{l-1} C(C + 1)^k \varepsilon \leq rC(C + 1)^r \varepsilon.$$

We are now in a position to prove Theorem A1. The proof consists in the construction of $f : K^\varepsilon \to K$, for ε sufficiently small. Recall that $K = \bigcap_{i \in I} P_i(a_i, \mathbf{n}_i)$, where $I = \{1, \ldots, N\}$. Define

$$\mathcal{G} = \{S = H_{i_1} \cap \ldots \cap H_{i_n} : i_1, \ldots, i_n \in I\},$$

$$\begin{aligned}\varrho = \min\{&\min\{d(S, K) : S \in \mathcal{G},\ S \neq \emptyset,\ S \cap K = \emptyset\},\\ &\min\{d(S, S') : S, S' \in \mathcal{G},\ S \cap S' = \emptyset\},\ \min\{\delta(S) : S \in \mathcal{G},\ S \neq \emptyset\}\}.\end{aligned}$$

By Lemmas A2 and A4, ϱ is positive. We will define f on K^ϱ. Let

$$C = \max\{C(S) : S \in \mathcal{G},\ S \neq \emptyset\}, \quad \text{where } C(S) \text{ is given by Lemma A4},$$

$$\mathcal{G}_d = \{S \in \mathcal{G} : S \cap K \neq \emptyset,\ S \text{ is not a subset of } K,\ \dim S = d\},$$

$$d_0 = \min\{d : \mathcal{G}_d \neq \emptyset\}.$$

I. First we extend f to $\bigcup\{S \cap K^\varrho : S \in \mathcal{G}_{d_0}\}$. For fixed $S \in \mathcal{G}_{d_0}$ we define $f|_{K^\varrho \cap S} = \pi_{K \cap S}$. If S_1 and S_2 belong to $\mathcal{G}_{d_0}$ and $S_1 \cap S_2 \neq \emptyset$ then either $S_1 \cap S_2$ is a subset of K or not. In the first case $f|_{S_1 \cap S_2} = \mathrm{id}$, in the second case we have two possibilities:

(i) $S_1 \cap S_2 = S_1$,

(ii) $S_1 \cap S_2 \neq S_1$.

If (i) holds, then $S_1 \subset S_2$ and $\dim S_1 = \dim S_2 = d_0$ so $S_1 = S_2$. In the case (ii) we have $\dim S_1 \cap S_2 < d_0$ and by the definition of d_0 we conclude that $S_1 \cap S_2 \cap K = \emptyset$ and consequently $S_1 \cap S_2 \cap K^\varrho = \emptyset$. So f is well defined on $\bigcup\{S \cap K^\varrho : S \in \mathcal{G}_{d_0}\}$.

II. It is clear that for each $S \in \mathcal{G}_{d_0}$, if $x \in S \cap K^\varrho$ then $f(x) \in S \cap K$.

III. We have to show that f is Lipschitz continuous on $\bigcup\{S \cap K^\varrho : S \in \mathcal{G}_{d_0}\}$. Let $x \in S_1 \in \mathcal{G}_{d_0}$ and $y \in S_2 \in \mathcal{G}_{d_0}$. If $S_1 \cap S_2 \neq \emptyset$ then $f|_{S_1 \cup S_2}$ satisfies the Lipschitz condition by Lemma A3. If $S_1 \cap S_2 = \emptyset$ then $|x-y| \geq d(S_1, S_2) \geq \varrho$ and

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|} &\leq \frac{|f(x)-x|}{|x-y|} + \frac{|x-y|}{|x-y|} + \frac{|y-f(y)|}{|x-y|}\\ &\leq 1 + \frac{|f(x)-x|}{\varrho} + \frac{|f(y)-y|}{\varrho}.\end{aligned}$$

Since $d(x, K) \leq \varrho$, by Lemma A4 we have $|x - f(x)| \leq C\varrho$ and so $|f(x) - x|/\varrho \leq C$. Analogously, $|f(y) - y|/\varrho \leq C$ and consequently f satisfies the Lipschitz condition on $\bigcup\{S \cap K^\varrho : S \in \mathcal{G}_{d_0}\}$.

IV. Assume that f is Lipschitz continuous on $\bigcup\{S : S \in \mathcal{G}_d,\ d \leq k\}$, and for all $d \leq k$ and $S \in \mathcal{G}_d$, if $x \in S \cap K^\varrho$ then $f(x) \in S \cap K$. Fix $S \in \mathcal{G}_{k+1}$.

By our inductive assumption, f is well defined on $\bigcup_i S_i \cap K^\varrho$, where $S_i \in \mathcal{G},\ \dim S_i \leq k,\ S_i \cap S \neq \emptyset$, and is Lipschitz continuous.

We extend f to $K \cap S$ by setting $f|_{K\cap S} = \mathrm{id}$; it is easy to see that f is still Lipschitz continuous. By Theorem A6, f has a Lipschitz continuous extension $f_1 : S \to S$. We define $f|_{K^\varrho \cap S} = \pi_{S\cap K} \circ f_1$.

By our construction f is well defined on $\bigcup\{S \cap K^\varrho : s \in \mathcal{G},\ \dim S \leq k+1\}$. The Lipschitz property of f is checked as in III. It follows from our construction that

$$\forall S \in \mathcal{G}_{k+1},\ \forall x \in S \cap K^\varrho, \qquad f(x) \in S \cap K.$$

We continue this procedure till $d = m - 1$ and finally we extend f to K by setting $f|_K = \mathrm{id}$. We extend this function to K^ϱ using Theorem A6. This is the desired function, which proves the theorem. ∎

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INSTITUTE OF MATHEMATICS
CRACOW UNIVERSITY OF TECHNOLOGY
WARSZAWSKA 24
31-155 KRAKÓW, POLAND
E-mail: MILIAN@OETO.PK.EDU.PL

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