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# EMBEDDING INVERSE LIMITS OF NEARLY MARKOV INTERVAL MAPS AS ATTRACTING SETS OF PLANAR DIFFEOMORPHISMS

BY

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In this paper we address the following question due to Marcy Barge: For what $f : I \to I$ is it the case that the inverse limit of $I$ with single bonding map $f$ can be embedded in the plane so that the shift homeomorphism $\widehat{f}$ extends to a diffeomorphism ([BB, Problem 1.5], [BK, Problem 3])? This question could also be phrased as follows: Given a map $f : I \to I$, find a diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ so that $F$ restricted to its full attracting set, $\bigcap_{k\ge 0} F^k(\mathbb{R}^2)$, is topologically conjugate to $\widehat{f} : \overline{(I, f)} \to \overline{(I, f)}$. In this situation, we say that the inverse limit space, $\overline{(I, f)}$, can be embedded as the full attracting set of $F$.

The problem of realizing inverse limits of one-dimensional manifolds as attracting sets of diffeomorphisms was first addressed by R. F. Williams [W] and L. Block [Bl]. Williams constructed diffeomorphisms of $S^4$ which, when restricted to indecomposable subsets of their non-wandering sets, are conjugate to shift maps on inverse limits of one-dimensional branched manifolds [W, Theorem C]. Block generalized this construction to self-maps of manifolds which satisfy the condition that all singularities lie in the stable manifolds of orbits of sinks [Bl, Theorem A]. In this paper, we consider maps of the interval which satisfy Block's condition, and use a method similar to the one used by Block in [Bl] to embed inverse limits of the interval with this type of bonding map as attracting sets of diffeomorphisms of $\mathbb{R}^2$. We obtain a conjugacy on the entire attracting set rather than just an indecomposable subset of the non-wandering set.

Other related work includes [BM], [Ba], [M], and [Sz]. In [BM], M. Barge and J. Martin show that any inverse limit of the interval with a single bonding map can be embedded as the full attracting set of a homeomorphism of the plane. In [Ba], M. Barge gives conditions which guarantee that an

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inverse limit space can be embedded as the full attracting set of a diffeomorphism of the plane. Examples given in [Ba] include the "tent" map of the unit interval. In [M], M. Misiurewicz shows that the inverse limit of $I$ with bonding map $4x(1-x)$ can be embedded as the full attracting set of a diffeomorphism of any manifold of dimension greater than two, and as the full attracting set of a homeomorphism of any manifold of dimension two. In [Sz], W. Szczechla shows that if $f$ is a piecewise monotonic transitive interval map such that the orbit of every critical point is finite and does not contain any critical points, then $(I, f)$ can be embedded as the attractor of a diffeomorphism of any two-dimensional manifold.

**1. Inverse limits.** Let $I = [a, b]$ be an interval and $\{f_n\}_{n=0}^{\infty}$ a sequence of maps, $f_n : I \to I$. The inverse limit of $I$ with bonding maps $\{f_n\}_{n=0}^{\infty}$ is defined by

$$(I, \{f_n\}_{n=0}^{\infty}) = \{(x_0, x_1, \ldots) : x_n \in I \text{ and } f_n(x_{n+1}) = x_n,\ n = 0, 1, \ldots\}$$

and has topology induced by the metric

$$d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

In this paper, we are interested in inverse limit spaces defined by a single bonding map $f$, i.e. $f_n = f$ for $n = 0, 1, \ldots$ Let $(I, f)$ denote such an inverse limit space. In this case we may define $\widehat{f} : (I, f) \to (I, f)$ by $\widehat{f}((x_0, x_1, \ldots)) = (f(x_0), f(x_1), \ldots) = (f(x_0), x_0, \ldots)$. The map $\widehat{f}$ is a homeomorphism and is often referred to as the shift homeomorphism on $(I, f)$.

We say that a continuous interval map $f : I \to I$ is *nearly Markov with respect to* $A_1, \ldots, A_m$ if $A_1, \ldots, A_m$ are disjoint nondegenerate subintervals of $I$ such that the following conditions hold:

(1) $a \in A_1$ and $b \in A_m$,
(2) $f(\bigcup_{i=1}^{m} \overline{A}_i) \subset \bigcup_{i=1}^{m} \operatorname{int}(A_i)$,
(3) $\operatorname{diam} f^k(A_i) \to 0$ as $k \to \infty$ for $i = 1, \ldots, m$,
(4) $f$ is one-to-one on each component of $I - \bigcup_{i=1}^{m} A_i$.

If $f$ is nearly Markov with respect to $A_1, \ldots, A_m$, let $I_1, \ldots, I_{m-1}$ be the components of $I - \bigcup_{i=1}^{m} A_i$ such that if $i < j$ and $x \in I_i$ and $y \in I_j$, then $x < y$. In this situation, we use the notation $I_1 < \ldots < I_{m-1}$.

Our first theorem describes a situation where two continuous maps of a metric space yield inverse limits of that metric space with topologically conjugate shift homeomorphisms. In the case when the metric space is an interval, and the maps are nearly Markov, we obtain a corollary which is useful in this work.

THEOREM 1.1. *Suppose that $f$ and $g$ are continuous maps of a metric space $X$ and $A_1, \ldots, A_m$ are closed disjoint subsets of $X$ such that*

(1) $f(x) = g(x)$ *for all* $x \in X - \bigcup_{i=1}^m A_i$,
(2) $\operatorname{diam}(f^k(A_i)) \to 0$ *and* $\operatorname{diam}(g^k(A_i)) \to 0$ *as* $k \to \infty$ *for* $i = 1, \ldots, m$,
(3) *for each* $i = 1, \ldots, m$, *there exists* $j$ *such that* $f(A_i) \cup g(A_i) \subset A_j$.

*Then* $\widehat{f} : (X, f) \to (X, f)$ *is topologically conjugate to* $\widehat{g} : (X, g) \to (X, g)$.

Proof. Let $P = \{p_1, \ldots, p_r\}$ be the periodic points of $f$ contained in $\bigcup_{i=1}^m A_i$ and $Q = \{q_1, \ldots, q_r\}$ be the corresponding periodic points of $g$ contained in $\bigcup_{i=1}^m A_i$. Conditions (2) and (3) guarantee a one-to-one correspondence between $P$ and $Q$. Let $(x_0, x_1, \ldots) \in (X, f)$. To define $\phi : (X, f) \to (X, g)$ we need to consider two cases. First, consider $(x_0, x_1, \ldots) \in (X, f)$ such that $x_n \in \bigcup_{i=1}^m A_i$ for all $n$. In this case, $(x_0, x_1, \ldots) = (p_{i_1}, p_{i_2}, \ldots, p_{i_s}, p_{i_1}, \ldots)$ where $\{p_{i_1}, p_{i_2}, \ldots, p_{i_s}\}$ is a subset of $P$. Let $\phi((x_0, x_1, \ldots)) = (q_{i_1}, q_{i_2}, \ldots, q_{i_s}, q_{i_1}, \ldots)$ where $\{q_{i_1}, q_{i_2}, \ldots, q_{i_s}\}$ is the corresponding periodic orbit for $g$. Note that

$$\begin{aligned}
\phi(\widehat{f}(p_{i_1}, p_{i_2}, \ldots, p_{i_s}, p_{i_1}, \ldots)) &= \phi((p_{i_s}, p_{i_1}, \ldots, p_{i_s}, p_{i_1}, \ldots)) \\
&= (q_{i_s}, q_{i_1}, \ldots, q_{i_s}, q_{i_1}, \ldots) \\
&= \widehat{g}((q_{i_1}, q_{i_2}, \ldots, q_{i_s}, q_{i_1}, \ldots)) \\
&= \widehat{g}(\phi((p_{i_1}, p_{i_2}, \ldots, p_{i_s}, p_{i_1}, \ldots))).
\end{aligned}$$

In the second case, let $x_n$ be the first coordinate of $(x_0, x_1, \ldots)$ such that $x_n \notin \bigcup_{i=1}^m A_i$. Let $\phi((x_0, x_1, \ldots)) = (g^n(x_n), g^{n-1}(x_n), \ldots, g(x_n), x_n, \ldots)$.
Note that

$$\begin{aligned}
\phi(\widehat{f}((x_0, x_1, \ldots))) &= \phi((f(x_0), x_0, \ldots)) \\
&= (g^{n+1}(x_n), g^n(x_n), \ldots, g(x_n), x_n, \ldots) \\
&= \widehat{g}((g^n(x_n), \ldots, g(x_n), x_n, \ldots)) = \widehat{g}(\phi((x_0, x_1, \ldots))).
\end{aligned}$$

It follows that $\phi \circ \widehat{f} = \widehat{g} \circ \phi$.

The map $\phi$ is one-to-one and onto since we may define $\phi^{-1}$ by interchanging the roles of $f$ and $g$ and $(I, f)$ and $(I, g)$ in the above proof. This completes the proof of Theorem 1.1.

We obtain the following useful corollary.

COROLLARY 1.2. *Suppose that $f$ and $g$ are nearly Markov interval maps with respect to $A_1, \ldots, A_m$, and $f(x) = g(x)$ for all $x \in I - \bigcup_{i=1}^m A_i$. Then $\widehat{f} : (I, f) \to (I, f)$ is topologically conjugate to $\widehat{g} : (I, g) \to (I, g)$.*

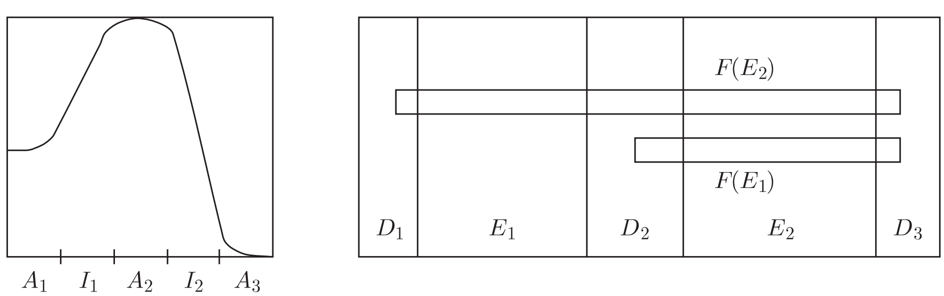
**2. Embedding inverse limits of nearly Markov maps.** In this section we state and prove the main result of the paper, which answers Barge's question for nearly Markov interval maps. We use the following definition and notation: If $F$ is a self-homeomorphism of a manifold $M$, we call the intersection of the forward images of $F$ the *full attracting set* of $F$ and write $\Lambda_F = \bigcap_{k\ge 0} F^k(M)$.

THEOREM 2.1. *If $f : I \to I$ is nearly Markov with respect to $A_1, \ldots, A_m$ and $f|I - \bigcup_{i=1}^m A_i$ is differentiable of class $C^k$ ($k = 1, \ldots, \infty$), then there exists a $C^k$-diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F|\Lambda_F$ is topologically conjugate to $\widehat{f} : (I, f) \to (I, f)$.*

Proof. Suppose that $f : I \to I$ is nearly Markov with respect to $A_1 = [a_1, b_1], \ldots, A_m = [a_m, b_m]$. Let $I_1 < \ldots < I_{m-1}$ denote the components of $I - \bigcup_{i=1}^m A_i$. For each $i = 1, \ldots, m$, choose $r_i$ and $s_i$ such that $[r_i, s_i] \subset \operatorname{int}(A_i)$, and $f(\bigcup_{i=1}^m A_i) \subset \bigcup_{i=1}^m [r_i, s_i]$. Without loss of generality, we may assume that $f(a_i) = f(b_i)$ for $i = 1, \ldots, m$, for if this is not the case, let $A_i' = [r_i, s_i]$, $i = 1, \ldots, m$, and define $f' : I \to I$ as follows: $f'(x) = f(r_i)$ for $x \in A_i'$, $i = 1, \ldots, m$, $f'(x) = f(x)$ for $x \in I - \bigcup_{i=1}^m A_i$, and extend $f'$ to $\bigcup_{i=1}^m (A_i - A_i')$ so that $f'$ is nearly Markov with respect to $A_1', \ldots, A_m'$. Note that $f$ is also nearly Markov with respect to $A_1', \ldots, A_m'$ and so it follows from Corollary 1.2 that $\widehat{f} : (I, f) \to (I, f)$ is topologically conjugate to $\widehat{f'} : (I, f') \to (I, f')$ and so we replace $f$ with $f'$ and $A_i$ with $A_i'$ for $i = 1, \ldots, m$. Then $f'$ satisfies the condition that it is constant on the endpoints of $A_i'$ for each $i = 1, \ldots, m$.

Next let $D = I \times [0,1]$, $D_i = A_i \times [0,1]$ for $i = 1, \ldots, m$, and $E_i = I_i \times [0,1]$ for $i = 1, \ldots, m-1$. Define $F : D \to D$ as follows: Let $\delta = \min\{b_i - a_i\}_{i=1}^m$ and let $\{[p_1, q_i]\}_{i=1}^{m-1}$ be subintervals of $[0,1]$ such that $0 < p_1 < q_1 < p_2 < q_2 < \ldots < p_{m-1} < q_{m-1} < 1$ and $q_{i+1} - p_i < \delta$ for $i = 1, \ldots, m-2$. This last condition is necessary so that it will be possible to define $F$ so that it is a contraction on $\bigcup_{i=1}^m [r_i, s_i] \times [0,1]$. First we define $F$ on $\bigcup_{i=1}^{m-1} E_i$ (Figure 1):

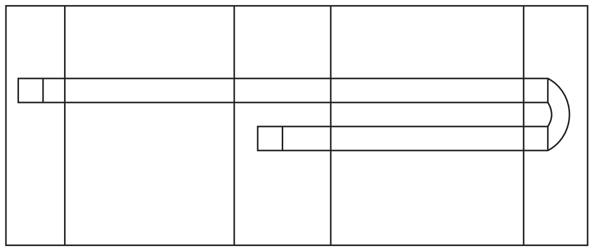
$$F(x,t) = \begin{cases} (f(x), (1-t)p_i + tq_i) & \text{for } x \in I_i \text{ and } f'(x) > 0, \\ (f(x), tp_i + (1-t)q_i) & \text{for } x \in I_i \text{ and } f'(x) < 0. \end{cases}$$




$f$ $\qquad$ $F|\bigcup_{i=1}^{m-1} E_i$

Fig. 1

Note that $F(\{x\} \times [0,1]) \subset \{f(x)\} \times [0,1]$ for all $x \in \bigcup_{i=1}^{m-1} I_i$ and $F : \bigcup_{i=1}^{m-1} E_i \to D$ is a $C^k$-diffeomorphism. Now extend $F$ to a $C^k$-diffeomorphism of all of $D$ so that $F(\bigcup_{i=1}^{m} D_i) \subset \bigcup_{i=1}^{m}(r_j, s_j) \times [0,1]$ and so that $F|\bigcup_{i=1}^{m}[r_i, s_i] \times [0,1]$ is a contraction (Figure 2).



$F$

Fig. 2

To prove that $F$ is the desired diffeomorphism, let $G : D \to D$ be any continuous map such that $G(x,y) = (f(x), \cdot)$ for all $(x,y) \in D$, $G(x,y) = F(x,y)$ for all $(x,y) \in D - \bigcup_{i=1}^{m} D_i$, and $G$ contracts in the $y$ direction. Note that $G$ is not necessarily one-to-one on $\bigcup_{i=1}^{m} D_i$. In fact, if $x$ is a critical point of $f$, then $G(\{x\} \times [0,1])$ is a single point. It is easy to check that $F$, $G$, and $D_1, \ldots, D_m$ satisfy the conditions of Theorem 1.1 and so $\widehat{F} : (D,F) \to (D,F)$ is topologically conjugate to $\widehat{G} : (D,G) \to (D,G)$. Furthermore, $\widehat{F} : (D,F) \to (D,F)$ is topologically conjugate to $F|\Lambda_F$ [Sc, Theorem 37]. Therefore, to complete the proof we need to check that $\widehat{G} : (D,G) \to (D,G)$ is topologically conjugate to $\widehat{f} : (I,f) \to (I,f)$. To this end, define $\phi : (D,G) \to (I,f)$ as follows: If $(z_0, z_1, \ldots) \in (D,G)$ let $\phi((z_0, z_1, \ldots)) = (x_0, x_1, \ldots)$ where $x_i$ is the first coordinate of $z_i$. Since $G(x,y) = (f(x), \cdot)$ for all $(x,y)$ in $D$, it follows that $\phi((z_0, z_1, \ldots)) \in (I,f)$. Furthermore, since $G$ contracts in the $y$ direction and $\operatorname{diam}(f^k(A_i)) \to 0$ as $k \to \infty$ for $i = 1, \ldots, m$, it follows that $\phi$ is one-to-one. To see that $\phi$ is onto, let $(x_0, x_1, \ldots) \in (I,f)$. Let $\pi_n : (D,G) \to D$ be the projection onto the $n$th coordinate of an element of $(D,G)$. Since $G(x,y) = (f(x), \cdot)$ for all $(x,y)$ in $D$, it follows that $\bigcap_{n \geq 0} \pi_i^{-1}(x_i)$ is a nonempty subset of $(D,G)$ and that $\phi(\bigcap_{n \geq 0} \pi_n^{-1}(x_n)) = (x_0, x_1, \ldots)$.

We have established that $\widehat{G} : (D,G) \to (D,G)$ is topologically conjugate to $\widehat{f} : (I,f) \to (I,f)$, which completes the proof of Theorem 2.1.

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