ALMOST EVERYWHERE CONVERGENCE OF
RIESZ–RAIKOV SERIES

BY

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Let $T$ be a $d \times d$ matrix with integer entries and with eigenvalues $> 1$ in modulus. Let $f$ be a lipschitzian function of positive order. We prove that the series $\sum_{n=1}^{\infty} c_n f(T^n x)$ converges almost everywhere with respect to Lebesgue measure provided that $\sum_{n=1}^{\infty} |c_n|^2 \log^2 n < \infty$.

1. Introduction. Given an arbitrary nonatomic dynamical system $(X, T, \mu)$. Suppose

$$0 \leq c_n \downarrow, \quad c_n = O(n^{-1}), \quad \sum_{n=1}^{\infty} c_n = \infty.$$ 

Then there exists a function $f \in L^\infty(X)$ with $\int f \, d\mu = 0$ such that

$$(1) \quad \sum_{n=1}^{\infty} c_n f(T^n x)$$

diverges $\mu$-a.e. ([3], [4], [8]). On the other hand, it is easy to exhibit some specific functions like $f = g - T g$ with $g \in L^\infty(X)$ for which the series (1) converges $\mu$-a.e. It is then natural to ask whether there are other classes of functions such that the series (1) converges $\mu$-a.e.

In this paper, we consider a special system $(\mathbb{T}^d, T, dx)$ where $T$ is an endomorphism of $\mathbb{T}^d$ and $dx$ is Haar measure on $\mathbb{T}^d$. We prove that (1) converges a.e. for any lipschitzian continuous function.

In reality, more can be proved. For $f \in C(\mathbb{T}^d)$, we denote by $\omega_f(\cdot)$ the modulus of continuity of $f$. For $T \in M_d(\mathbb{Z})$, we denote by $\|T\|$ the operator norm of $T$ corresponding to a given norm on $\mathbb{R}^d$. Our main result is

**Theorem 1.** Let $T_n \in M_d(\mathbb{Z})$ with $\det T_n \neq 0$ and $f_n \in C(\mathbb{T}^d)$ with zero
mean value and \( \int |f_n| \, dx = O(1) \) \((n \geq 1)\). Suppose
\[
\sup_n \omega_{f_n}(\tau_{n,p}) = O(p^{-\sigma}) \quad (\sigma > 0),
\]
where
\[
\tau_{n,p} = \sum_{k=0}^{\infty} \| T_{n+1}^{-1} \cdots T_{n+p+k}^{-1} \| \quad (n \geq 1, \ p \geq 1).
\]
Then the series
\[
\sum_{n=1}^{\infty} c_n f_n(T_n T_{n-1} \cdots T_1 x)
\]
converges a.e. if one of the following conditions is satisfied:
\(\sigma > 1\),
\[
\sum_{n=1}^{\infty} |c_n|^2 \log^2 n < \infty,
\]
\(\sigma = 1\),
\[
\sum_{n=1}^{\infty} |c_n|^2 n^\varepsilon \log^2 n < \infty \quad (\text{for some } \varepsilon > 0),
\]
\(\sigma < 1\),
\[
\sum_{n=1}^{\infty} |c_n|^2 n^{1-\sigma} \log^2 n < \infty.
\]

**Corollary 1.** Let \( T \in M_d(\mathbb{Z}) \) with all eigenvalues > 1 in modulus. For any continuous function \( f \) such that \( \omega_f(r) = O((\log(1/r))^{-\sigma}) \) \((\text{for some } \sigma > 0)\), the series (1) converges a.e. provided one of the conditions (4)–(6) is satisfied.

The proof of Theorem 1 is based on the following Theorem 2. For \( n \geq 1 \), let \( X_n \) be a finite group equipped with the discrete topology and let \( \mu_n \) be a probability measure on \( X_n \). Consider then the infinite space \( X = \prod X_n \) and the infinite product measure \( \mu = \bigotimes \mu_n \). The topology of \( X \) can be defined by the usual ultrametric. We denote by \( I_n(x) \) the \( n \)-cylinder containing \( x \).
For \( f \in C(X) \), define
\[
\omega_n(f) = \sup_{I_n(x) \neq I_n(y)} |f(x) - f(y)|.
\]

**Theorem 2.** Let \( \{f_n\}_{n \geq 1} \) be a sequence of continuous functions. Suppose that \( f_n \) does not depend upon the \( n-1 \) first coordinates and \( E_{\mu} f_n = 0 \). Then
\[
|E_{\mu} f_n f_{n+p}| \leq \omega_{n+p-1}(f_n) E_{\mu} |f_{n+p}|
\]
for \( n \geq 1 \) and \( p \geq 1 \).

We shall follow the idea of [2], showing quasi-orthogonality. But our techniques are different from those of [2]. In our case, we observe that the general term of (3) is invariant under the action of some finite group which becomes more and more dense when \( n \) increases and the group \( T^d \) can be
riesz–raikov series

represented by a suitable infinite product of finite groups (see §3). The problem then becomes one on an infinite product space which is treated in §2. The deduction of Theorem 1 from Theorem 2 is given in §4.

We call (1) and (3) Riesz–Raikov series because of the first works of D. A. Raikov ([5]) and of F. Riesz ([6]) in the case of one dimension. A similar one-dimensional result is contained in [7].

2. Proof of Theorem 2. We will consider the infinite product measure \( \mu \) as a \( G \)-measure in the sense of [1]. Here is the description.

Let \( n \geq 1 \). Define, for \( x = (x_1, x_2, \ldots) \in X \),

\[
F_n(x) = \prod_{j=1}^{n} \mu_j(x_j).
\]

We then have

\[
\sum_{\gamma \in \Gamma_n} F_n(\gamma) = 1 \quad (\forall x \in X),
\]

where \( \Gamma_n = \prod_{j=1}^{n} X_n \). \( \Gamma_n \) will be viewed as a subgroup of \( X \). So, for \( x \in X \) and \( \gamma \in \Gamma_n \), the group product \( \gamma x \) will mean \( (x_1 + \gamma_1, \ldots, x_n + \gamma_n, x_{n+1}, \ldots) \).

Denote by \( \mathcal{F}^n \) the \( \sigma \)-field generated by all but the first \( n \) coordinates of \( X \).

We have the following three facts:

**Fact 1.** The measure \( \mu \) is actually the unique measure such that for any \( n \geq 1 \),

\[
\frac{d\mu}{d\mu_n} = F_n \quad \mu\text{-a.e. where } \mu_n = \sum_{\gamma \in \Gamma_n} \mu \circ \gamma,
\]

\( \mu \circ \gamma \) being the image of \( \mu \) under the action of \( \gamma \).

**Fact 2.** For \( f \in L^1(\mu) \) we have

\[
\mathbb{E}_\mu(f \mid \mathcal{F}^n) = \sum_{\gamma \in \Gamma_n} f(\gamma x)F_n(\gamma x).
\]

**Fact 3.** For \( f \in C(X) \), the reverse martingale \( \mathbb{E}_\mu(f \mid \mathcal{F}^n) \) converges everywhere (even uniformly) to \( \mathbb{E}_\mu f \).

Facts 1 and 2 are easily verified and Fact 3 is a consequence of Fact 1 ([1]).

Let us now prove the estimate in Theorem 2. By Facts 3 and 2, we have

\[
\mathbb{E}_\mu f_n f_{n+p} = \lim_{N \to \infty} \mathbb{E}_\mu(f_n f_{n+p} \mid \mathcal{F}^N) = \lim_{N \to \infty} \sum_{\gamma \in \Gamma_N} f_n(\gamma x) f_{n+p}(\gamma x) F_N(\gamma x).
\]

Let \( \tilde{f}_n(x) = f_n(x_1, \ldots, x_{n+p-1}, 0, \ldots) \). As \( f_n = f_n - \tilde{f}_n + \tilde{f}_n \), the sum under the limit is bounded by

\[
\omega_{n+p-1}(f_n) \sum_{\gamma \in \Gamma_n} |f_{n+p}(\gamma x)| F_N(\gamma x) + \left| \sum_{\gamma \in \Gamma_N} \tilde{f}_n(\gamma x) f_{n+p}(\gamma x) F_N(\gamma x) \right|.
\]
Again by Facts 2 and 3, the first sum in the preceding expression has the limit
\[
\lim_{N \to \infty} \sum_{\gamma \in \Gamma_N} |f_{n+p}(\gamma x)| F_N(\gamma x) = E_{\mu} |f_{n+p}|
\]
Since the function \( \tilde{f}_n(x) \) depends only upon the first \( n + p - 1 \) coordinates and the function \( f_{n+p} \) does not depend upon the first \( n + p - 1 \) coordinates, the second sum can be written as
\[
\left( \sum_{\gamma' \in \Gamma_{n+p-1}} \tilde{f}_n(\gamma' x) F_{n+p-1}(\gamma' x) \right) \left( \sum_{\gamma'' \in X_{n+p} \times \cdots \times X_N} f_{n+p}(\gamma'' x) \prod_{j=n+p}^{N} \mu_j(\gamma'' x) \right).
\]
The first factor in the preceding product is independent of \( N \) and the second one equals \( E_{\mu} (f_{n+p} | F_N) \) and thus tends to \( E_{\mu} f_{n+p} = 0 \) as \( N \to \infty \). This completes the proof of Theorem 2.

3. Some lemmas. Suppose the conditions of Theorem 1 are satisfied. Before giving the proof of Theorem 1 in the next section, we give here some lemmas.

Recall that \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \) is a quotient space. For simplicity, we introduce the following notation. Let \( \pi \) be the natural projection from \( \mathbb{R}^d \) onto \( \mathbb{R}^d / \mathbb{Z}^d \).

For \( x \in \mathbb{R}^d \), we write \( \dot{x} = \pi(x) \). By extension, if \( F \) is a map with values in \( \mathbb{R}^d \), we write \( \dot{F} = \pi \circ F \). Similarly, if \( t \) is a point of \( \mathbb{T}^d \) and \( G \) is a subgroup of \( \mathbb{T}^d \), we define \([t]_G = t + G\) which is the natural projection from \( \mathbb{T}^d \) into \( \mathbb{T}^d / G \).

Let \( \Phi \) be an endomorphism of \( \mathbb{R}^d \) defined by a nonsingular matrix with integer entries and \( \Psi \) be its inverse. We denote by \( \varphi \) the induced homomorphism of \( \Phi \) on \( \mathbb{T}^d \). Then the relation between \( \varphi \) and \( \Phi \) is \( \pi \circ \Phi = \varphi \circ \pi \), i.e.
\[
\hat{\Phi}(x) = \varphi(\dot{x}).
\]
The first lemma gives a correspondence between \( \mathbb{T}^d / \text{Ker} \varphi \) and \( \dot{\Psi}(D) \) where \( D \) is the hypercube \([0,1]^d\).

**Lemma 1.** The map \( \pi_\varphi : \dot{\Psi}(D) \to \mathbb{T}^d / \text{Ker} \varphi \) defined by \( \pi_\varphi(t) = [t]_{\text{Ker} \varphi} \) is one-to-one.

**Proof.** As \( D + \mathbb{Z}^d = \mathbb{R}^d \) and \( \Psi \) is nonsingular, we have the equality
\[
\Psi(D) + \mathbb{Z}^d = \mathbb{R}^d.
\]
Notice that \( \text{Ker} \varphi = \Psi(\mathbb{Z}^d) / \mathbb{Z}^d \). Thus the preceding equality implies that
\[
\dot{\Psi}(D) + \text{Ker} \varphi = \mathbb{T}^d.
\]
This equality implies the surjectivity of \( \pi_\varphi \). Suppose now we are given two
points $s$ and $t$ in $\hat{\Psi}(D)$. Suppose that $[s]_{\text{Ker } \varphi} = [t]_{\text{Ker } \varphi}$. We then have
\[ \varphi(s) = \varphi(t). \]
But $s = \hat{\Psi}(x)$ for some $x \in D$ and $t = \hat{\Psi}(y)$ for some $y \in D$. These facts, together with (7) and the last equality, imply $\hat{\Psi}(x) = \hat{\Psi}(y)$, which means $x = y \pmod{\mathbb{Z}^d}$. Thus $s = t$, so we have proved the injectivity. \[ \Box \]

For $n \geq 1$, we denote by $\Phi_n$ the endomorphism $T_n T_{n-1} \ldots T_1$ and by $\varphi_n$ the induced homomorphism on $\mathbb{T}^d$. Let
\[ G_n = \text{Ker } \varphi_n, \quad G^n = \mathbb{T}^d / G_n. \]
Obviously, $\{G_n\}_{n \geq 1}$ is an increasing sequence of finite subgroups of $\mathbb{T}^d$. By Lemma 1, $G^n$ is identified with $\hat{\Psi}_n(D)$. Now we introduce
\[ H_n = G_n / G_{n-1} \quad (N \geq 1) \]
($G_0 = \{0\}$).

**Lemma 2.** Given a point $h \in H_n$, there is one and only one point $t \in G_n \cap \hat{\Psi}_{n-1}(D)$ such that $h = [t]_{G_{n-1}}$.

**Proof.** Let $t_0 \in G_n$ be a representative of $h$. As $\hat{\Psi}_{n-1}(D) + G_{n-1} = \mathbb{T}^d$, there exist a $g \in G_{n-1}$ and a $t \in \hat{\Psi}_{n-1}(D)$ such that $t_0 = t + g$. So $h = [t]_{G_{n-1}}$ and $t \in G_n \cap \hat{\Psi}_{n-1}(D)$ since $g \in G_{n-1} \subset G_n$. Such a $t$ is unique since each point of $\hat{\Psi}_{n-1}(D)$ corresponds to a unique coset of $G_n$. \[ \Box \]

Let $\| \cdot \|$ be a norm of $\mathbb{R}^d$. We introduce the associated quotient metric on $\mathbb{T}^d$ defined by
\[ d(x, y) = \inf_{z \in \mathbb{Z}^d} \| (x - y) - z \|. \]
This metric on $\mathbb{T}^d$ is invariant under translations. We sometimes write $d(x, y) = \| x - y \|_{\mathbb{T}^d}$. For two subsets $A$ and $B$ of $\mathbb{T}^d$, we denote by $d(A, B)$ the distance from $A$ to $B$. By the two preceding lemmas, $G^n$ and $H_n$ can be identified with subsets of $\mathbb{T}^d$. From now on $G^n$ and $H_n$ will denote their corresponding subsets on $\mathbb{T}^d$. The following fact is evident.

**Lemma 3.** $d(0, G^n) \leq \| \Psi_n \|$ and $d(0, H_n) \leq \| \Psi_{n-1} \|$.

We therefore construct the infinite product $X = \prod_{n=1}^{\infty} H_n$ equipped with the usual ultrametric, and the map $q : X \to \mathbb{T}^d$ defined by
\[ q(h_1, h_2, \ldots) = \sum_{n=1}^{\infty} h_n. \]

**Lemma 4.** The map $q : X \to \mathbb{T}^d$ is continuous and surjective.

**Proof.** We have the continuity because of (2) which implies
\[ \sum_{n} \| T_n^{-1} \ldots T_1^{-1} \| < \infty. \]
Let $\Gamma_n = \prod_{j=1}^n H_j$. Then $\Gamma_n$ can be regarded as a subset of $X$. Consider the restriction of $q$ to $\Gamma_n$. We claim that $q(\Gamma_n) = G_n$. In fact, first we observe that $q(\Gamma_n) \subset G_n$. Suppose $h_j' \in H_j (1 \leq j \leq n)$ and $h_1' + \cdots + h_n' = h_1'' + \cdots + h_n''$. Then $h_n'-h_n'' \in G_n-1$. According to Lemma 1, this is impossible unless $h_n' = h_n''$. By induction, it follows that $h_j' = h_j'' (1 \leq j \leq n)$. This proves the injectivity of the restriction of $q$ to $\Gamma_n$. However, the cardinality of $\Gamma_n$ is the same as that of $G_n$, so $q(\Gamma_n) = G_n$. By condition (2), the union of $G_n (n \geq 1)$ is dense in $G$. Thus the closure of the image of $q$ is $G$. But $X$ is compact and hence $q$ is surjective.

Let $\{\mu_n\}$ be the sequence of probability measures defined by

$$\mu_n(h) = |H_n|^{-1} \quad (h \in H_n).$$

Let $\mu = \otimes_{n=1}^\infty \mu_n$ and let $q \mu$ be the image by $q$ of $\mu$. That is to say, $q \mu$ is the measure on $T^d$ characterized by

$$(8) \quad \int_{T^d} f \, dq \mu = \int_X f \circ q \, d\mu \quad (f \in C(T^d)).$$

**Lemma 5.** If $\mu$ is defined as above, then $q \mu$ is Haar measure $\lambda$ on $T^d$.

**Proof.** Let $G_n^\infty \gamma \in G_n$ being a partition of $T^d$, we have

$$\lambda(\gamma + G_n^\infty) = \frac{1}{|G_n|} \quad (\gamma \in G_n),$$

because

$$\sum_{\gamma \in G_n} \lambda(\gamma + G_n^\infty) = 1 \quad \text{and} \quad \lambda(\gamma + G_n^\infty) = \lambda(G_n^\infty).$$

Now given $f \in C(T^d)$, we have by Fact 2,

$$\int_X f \circ q d\mu = \lim_{n \to \infty} \mathbb{E}_\mu(f \circ q \mid F_n^u) = \lim_{n \to \infty} \sum_{h_1, \ldots, h_n} f \circ q(h_1, \ldots, h_n, x_{n+1}, \ldots).$$

Given $\varepsilon > 0$, $f$ being uniformly continuous, there is a $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ if $||x-y||_T < \delta$. Choose an $N > 0$ such that $\sum_{n \geq N} d(0,G_n^u) < \delta$. Then for $n \geq N$ we have

$$\frac{1}{|G_n|} \sum_{h_1, \ldots, h_n} f \circ q(h_1, \ldots, h_n, x_{n+1}, \ldots) = \sum_{\gamma \in G_n} f(\gamma) \lambda(\gamma + G_n^u) + O(\varepsilon).$$

The last sum tends to $\int f d\lambda$ as $N \to \infty$.  ■
4. **Proof of Theorem 1.** Recall that a sequence \( \{h_n\}_{n \geq 1} \) of elements in a Hilbert space is said to be quasi-orthogonal if the bilinear form
\[
\sum_{n,m} \langle h_n, h_m \rangle a_n b_m
\]
on \ell^2(\mathbb{N}^*) \times \ell^2(\mathbb{N}^*) is bounded. Suppose that the Hilbert space is \( L^2(X, \mu) \) for some measure space \((X, \mu)\). For a quasi-orthonormal sequence \( \{h_n\} \subset L^2(\mu) \), we may apply Men'shov’s theorem ([9]), which says that the series
\[
\sum c_n h_n(x)
\]
converges \( \mu \)-a.e. provided the numerical series \( \sum |c_n|^2 \log^2 n \) converges. So, in order to prove Theorem 1, it suffices ([2], p. 237) to show the following estimate, uniform in \( n \):
\[
(9) \quad \int_T f_n(T_n T_{n-1} \ldots T_1 x) f_{n+p}(T_{n+p} T_{n+p-1} \ldots T_1 x) dx = O(p^{-\sigma}).
\]
In fact, let \( h_n(x) = f_n(T_n T_{n-1} \ldots T_1 x) \). If \( \sigma > 1 \), the sequence \( \{h_n\} \) is quasi-orthogonal. If \( \sigma = 1 \), the sequence \( \{n^{-\varepsilon/2} h_n\} \) (\( \forall \varepsilon > 0 \)) is quasi-orthogonal. If \( \sigma < 1 \), the sequence \( \{n^{-(1-\sigma)/2} h_n\} \) is quasi-orthogonal.

Now we deduce (9) from Theorem 2.

According to Lemma 5, we consider the sequence \( f_n \circ \Phi_n \circ q \) (\( n \geq 1 \)) defined on \( X \) and apply Theorem 2 to it. Then to prove (9), it suffices to show
\[
(10) \quad \omega_{n+p-1}(f_n \circ \Phi_n \circ q) = O(p^{-\sigma}).
\]
Suppose \( x = (x_j) \) and \( y = (y_j) \) belong to \( X \) and satisfy \( x_j = y_j \) for \( 1 \leq j \leq n + p - 1 \). We have
\[
q(x) - q(y) = \sum_{j=n+p}^\infty (x_j - y_j).
\]
As \( x_j, y_j \in H_j \subset G^{j-1} \), Lemma 2 implies that there exist \( \xi'_j, \xi''_j \in D \) such that
\[
x_j = \psi_{j-1} \xi'_j, \quad y_j = \psi_{j-1} \xi''_j \pmod{\mathbb{Z}^d}.
\]
Then
\[
\|\Phi_n \circ q(x) - \Phi_n \circ q(y)\|_{T^d} \leq \left\| \Phi_n \left( \sum_{j=n+p}^\infty \psi_j (\xi'_j - \xi''_j) \right) \right\|
\]
\[
= \left\| \sum_{j=n+p}^\infty \Phi_n \psi_j (\xi'_j - \xi''_j) \right\| = \sum_{j=n+p}^\infty \|T_{n+1}^{-1} \ldots T_j^{-1}\|.
\]
With this in mind, we can deduce (10) from (2). ■

**Proof of Corollary.** Let \( \varrho \) be the spectral radius of \( T^{-1} \). By the hypothesis, \( \varrho < 1 \). Let \( \varrho < \varrho_1 < 1 \). For \( n \) sufficiently large we have
\[ \|T^{-n}\| < \varrho_1^n. \]
Consequently,
\[ \tau_{n,p} = O\left( \sum_{k=0}^{\infty} \varrho_1^{p+k} \right) = O(\varrho_1^p). \]
This estimate and the hypothesis on \( f \) allow us to verify condition (2) of Theorem 1 with \( \sigma > 0 \).

REFERENCES