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ALMOST EVERYWHERE CONVERGENCE OF RIESZ–RAIKOV SERIES

BY

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Let T be a $d \times d$ matrix with integer entries and with eigenvalues > 1 in modulus. Let f be a lipschitzian function of positive order. We prove that the series $\sum_{n=1}^{\infty} c_n f(T^n x)$ converges almost everywhere with respect to Lebesgue measure provided that $\sum_{n=1}^{\infty} |c_n|^2 \log^2 n < \infty$.

1. Introduction. Given an arbitrary nonatomic dynamical system (X, T, μ). Suppose

$$0 \le c_n \downarrow, \quad c_n = O(n^{-1}), \quad \sum_{n=1}^{\infty} c_n = \infty.$$

Then there exists a function $f \in L^\infty(X)$ with $\int f\, d\mu = 0$ such that

$$\sum_{n=1}^{\infty} c_n f(T^n x) \tag{1}$$

diverges μ-a.e. ([3], [4], [8]). On the other hand, it is easy to exhibit some specific functions like $f = g - Tg$ with $g \in L^\infty(X)$ for which the series (1) converges μ-a.e. It is then natural to ask whether there are other classes of functions such that the series (1) converges μ-a.e.

In this paper, we consider a special system $(\mathbb{T}^d, T, dx)$ where T is an endomorphism of $\mathbb{T}^d$ and dx is Haar measure on $\mathbb{T}^d$. We prove that (1) converges a.e. for any lipschitzian continuous function.

In reality, more can be proved. For $f \in \mathcal{C}(\mathbb{T}^d)$, we denote by $\omega_f(\cdot)$ the modulus of continuity of f. For $T \in M_d(\mathbb{Z})$, we denote by $\|T\|$ the operator norm of T corresponding to a given norm on $\mathbb{R}^d$. Our main result is

THEOREM 1. *Let $T_n \in M_d(\mathbb{Z})$ with* $\det T_n \ne 0$ *and $f_n \in \mathcal{C}(\mathbb{T}^d)$ with zero*

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mean value and $\int |f_n|\,dx = O(1)$ $(n \ge 1)$. *Suppose*

$$\sup_n \omega_{f_n}(\tau_{n,p}) = O(p^{-\sigma}) \qquad (\sigma > 0), \tag{2}$$

where

$$\tau_{n,p} = \sum_{k=0}^{\infty} \|T_{n+1}^{-1} \dots T_{n+p+k}^{-1}\| \qquad (n \ge 1,\ p \ge 1).$$

Then the series

$$\sum_{n=1}^{\infty} c_n f_n(T_n T_{n-1} \dots T_1 x) \tag{3}$$

converges a.e. if one of the following conditions is satisfied:

$$\sigma > 1, \qquad \sum_{n=1}^{\infty} |c_n|^2 \log^2 n < \infty, \tag{4}$$

$$\sigma = 1, \qquad \sum_{n=1}^{\infty} |c_n|^2 n^{\varepsilon} \log^2 n < \infty \quad (\textit{for some } \varepsilon > 0), \tag{5}$$

$$\sigma < 1, \qquad \sum_{n=1}^{\infty} |c_n|^2 n^{1-\sigma} \log^2 n < \infty. \tag{6}$$

COROLLARY 1. *Let* $T \in M_d(\mathbb{Z})$ *with all eigenvalues* > 1 *in modulus. For any contiuous function* f *such that* $\omega_f(r) = O(\log(1/r))^{-\sigma}$ (*for some* $\sigma > 0$), *the series* (1) *converges a.e. provided one of the conditions* (4)–(6) *is satisfied.*

The proof of Theorem 1 is based on the following Theorem 2. For $n \ge 1$, let X_n be a finite group equipped with the discrete topology and let μ_n be a probability measure on X_n. Consider then the infinite space $X = \prod X_n$ and the infinite product measure $\mu = \bigotimes \mu_n$. The topology of X can be defined by the usual ultrametric. We denote by $I_n(x)$ the n-cylinder containing x. For $f \in \mathcal{C}(X)$, define

$$\omega_n(f) = \sup_{I_n(x) = I_n(y)} |f(x) - f(y)|.$$

THEOREM 2. *Let* $\{f_n\}_{n \ge 1}$ *be a sequence of continuous functions. Suppose that* f_n *does not depend upon the* $n-1$ *first coordinates and* $\mathbb{E}_\mu f_n = 0$. *Then*

$$|\mathbb{E}_\mu f_n f_{n+p}| \le \omega_{n+p-1}(f_n)\, \mathbb{E}_\mu |f_{n+p}|$$

for $n \ge 1$ *and* $p \ge 1$.

We shall follow the idea of [2], showing quasi-orthogonality. But our techniques are different from those of [2]. In our case, we observe that the general term of (3) is invariant under the action of some finite group which becomes more and more dense when n increases and the group $\mathbb{T}^d$ can be represented by a suitable infinite product of finite groups (see §3). The problem then becomes one on an infinite product space which is treated in §2. The deduction of Theorem 1 from Theorem 2 is given in §4.

We call (1) and (3) *Riesz–Raikov series* because of the first works of D. A. Raikov ([5]) and of F. Riesz ([6]) in the case of one dimension. A similar one-dimensional result is contained in [7].

2. Proof of Theorem 2. We will consider the infinite product measure μ as a G-measure in the sense of [1]. Here is the description.

Let $n \geq 1$. Define, for $x = (x_1, x_2, \ldots) \in X$,

$$F_n(x) = \prod_{j=1}^{n} \mu_j(x_j).$$

We then have

$$\sum_{\gamma \in \Gamma_n} F_n(\gamma) = 1 \qquad (\forall x \in X),$$

where $\Gamma_n = \prod_{j=1}^{n} X_n$. Γ_n will be viewed as a subgroup of X. So, for $x \in X$ and $\gamma \in \Gamma_n$, the group product γx will mean $(x_1 + \gamma_1, \ldots, x_n + \gamma_n, x_{n+1}, \ldots)$. Denote by $\mathcal{F}^n$ the σ-field generated by all but the first n coordinates of X. We have the following three facts:

FACT 1. *The measure μ is actually the unique measure such that for any $n \geq 1$,*

$$\frac{d\mu}{d\mu_n} = F_n \quad \mu\text{-a.e.} \quad \text{where} \quad \mu_n = \sum_{\gamma \in \Gamma_n} \mu \circ \gamma,$$

$\mu \circ \gamma$ being the image of μ under the action of γ.

FACT 2. *For $f \in L^1(\mu)$ we have*

$$\mathbb{E}_\mu(f \mid \mathcal{F}^n) = \sum_{\gamma \in \Gamma_n} f(\gamma x) F_n(\gamma x).$$

FACT 3. *For $f \in \mathcal{C}(X)$, the reverse martingale $\mathbb{E}_\mu(f \mid \mathcal{F}^n)$ converges everywhere* (*even uniformly*) *to $\mathbb{E}_\mu f$.*

Facts 1 and 2 are easily verified and Fact 3 is a consequence of Fact 1 ([1]).

Let us now prove the estimate in Theorem 2. By Facts 3 and 2, we have

$$\mathbb{E}_\mu f_n f_{n+p} = \lim_{N\to\infty} \mathbb{E}_\mu(f_n f_{n+p} \mid \mathcal{F}^N) = \lim_{N\to\infty} \sum_{\gamma \in \Gamma_N} f_n(\gamma x) f_{n+p}(\gamma x) F_N(\gamma x).$$

Let $\widetilde{f}_n(x) = f_n(x_1, \ldots, x_{n+p-1}, 0, \ldots)$. As $f_n = f_n - \widetilde{f}_n + \widetilde{f}_n$, the sum under the limit is bounded by

$$\omega_{n+p-1}(f_n) \sum_{\gamma \in \Gamma_N} |f_{n+p}(\gamma x)| F_N(\gamma x) + \Big| \sum_{\gamma \in \Gamma_N} \widetilde{f}_n(\gamma x) f_{n+p}(\gamma x) F_N(\gamma x) \Big|.$$

Again by Facts 2 and 3, the first sum in the preceding expression has the limit

$$\lim_{N\to\infty}\sum_{\gamma\in\Gamma_N}|f_{n+p}(\gamma x)|F_N(\gamma x)=\mathbb{E}_\mu|f_{n+p}|.$$

Since the function $\widetilde{f}_n(x)$ depends only upon the first $n+p-1$ coordinates and the function f_{n+p} does not depend upon the first $n+p-1$ coordinates, the second sum can be written as

$$\Big(\sum_{\gamma'\in\Gamma_{n+p-1}}\widetilde{f}_n(\gamma' x)F_{n+p-1}(\gamma' x)\Big)\Big(\sum_{\gamma''\in X_{n+p}\times\ldots\times X_N}f_{n+p}(\gamma'' x)\prod_{j=n+p}^{N}\mu_j(\gamma'' x)\Big).$$

The first factor in the preceding product is independent of N and the second one equals $\mathbb{E}_\mu(f_{n+p}\mid\mathcal{F}^N)$ and thus tends to $\mathbb{E}_\mu f_{n+p}=0$ as $N\to\infty$. This completes the proof of Theorem 2. ■

3. Some lemmas. Suppose the conditions of Theorem 1 are satisfied. Before giving the proof of Theorem 1 in the next section, we give here some lemmas.

Recall that $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$ is a quotient space. For simplicity, we introduce the following notation. Let π be the natural projection from $\mathbb{R}^d$ onto $\mathbb{R}^d/\mathbb{Z}^d$. For $x\in\mathbb{R}^d$, we write $\dot{x}=\pi(x)$. By extension, if F is a map with values in $\mathbb{R}^d$, we write $\dot{F}=\pi\circ F$. Similarly, if t is a point of $\mathbb{T}^d$ and G is a subgroup of $\mathbb{T}^d$, we define $[t]_G=t+G$ which is the natural projection from $\mathbb{T}^d$ into $\mathbb{T}^d/G$.

Let Φ be an endomorphism of $\mathbb{R}^d$ defined by a nonsingular matrix with integer entries and Ψ be its inverse. We denote by φ the induced homomorphism of Φ on $\mathbb{T}^d$. Then the relation between φ and Φ is $\pi\circ\Phi=\varphi\circ\pi$, i.e.

$$\dot{\Phi}(x)=\varphi(\dot{x}). \tag{7}$$

The first lemma gives a correspondence between $\mathbb{T}^d/\operatorname{Ker}\varphi$ and $\dot{\Psi}(D)$ where D is the hypercube $[0,1)^d$.

LEMMA 1. *The map* $\pi_\varphi:\dot{\Psi}(D)\to\mathbb{T}^d/\operatorname{Ker}\varphi$ *defined by* $\pi_\varphi(t)=[t]_{\operatorname{Ker}\varphi}$ *is one-to-one.*

Proof. As $D+\mathbb{Z}^d=\mathbb{R}^d$ and Ψ is nonsingular, we have the equality

$$\Psi(D)+\Psi(\mathbb{Z}^d)=\mathbb{R}^d.$$

Notice that $\operatorname{Ker}\varphi=\Psi(\mathbb{Z}^d)/\mathbb{Z}^d$. Thus the preceding equality implies that

$$\dot{\Psi}(D)+\operatorname{Ker}\varphi=\mathbb{T}^d.$$

This equality implies the surjectivity of π_φ. Suppose now we are given two points s and t in $\dot{\Psi}(D)$. Suppose that $[s]_{\operatorname{Ker}\varphi} = [t]_{\operatorname{Ker}\varphi}$. We then have

$$\varphi(s) = \varphi(t).$$

But $s = \dot{\Psi}(x)$ for some $x \in D$ and $t = \dot{\Psi}(y)$ for some $y \in D$. These facts, together with (7) and the last equality, imply $\dot{\Phi}\Psi(x) = \dot{\Phi}\Psi(y)$, which means $x = y \pmod{\mathbb{Z}^d}$. Thus $s = t$, so we have proved the injectivity. ∎

For $n \ge 1$, we denote by Φ_n the endomorphism $T_n T_{n-1} \ldots T_1$ and by φ_n the induced homomorphism on $\mathbb{T}^d$. Let

$$G_n = \operatorname{Ker}\varphi_n, \qquad G^n = \mathbb{T}^d/G_n.$$

Obviously, $\{G_n\}_{n\ge 1}$ is an increasing sequence of finite subgroups of $\mathbb{T}^d$. By Lemma 1, G^n is identified with $\dot{\Psi}_n(D)$. Now we introduce

$$H_n = G_n/G_{n-1} \qquad (N \ge 1)$$

$(G_0 = \{0\})$.

LEMMA 2. *Given a point $h \in H_n$, there is one and only one point $t \in G_n \cap \dot{\Psi}_{n-1}(D)$ such that $h = [t]_{G_{n-1}}$.*

Proof. Let $t_0 \in G_n$ be a representative of h. As $\dot{\Psi}_{n-1}(D) + G_{n-1} = \mathbb{T}^d$, there exist a $g \in G_{n-1}$ and a $t \in \dot{\Psi}_{n-1}(D)$ such that $t_0 = t + g$. So $h = [t]_{G_{n-1}}$ and $t \in G_n \cap \dot{\Psi}_{n-1}(D)$ since $g \in G_{n-1} \subset G_n$. Such a t is unique since each point of $\dot{\Psi}_{n-1}(D)$ corresponds to a unique coset of G_n. ∎

Let $\|\cdot\|$ be a norm of $\mathbb{R}^d$. We introduce the associated quotient metric on $\mathbb{T}^d$ defined by

$$d(x,y) = \inf_{z\in\mathbb{Z}^d} \|(x-y) - z\|.$$

This metric on $\mathbb{T}^d$ is invariant under translations. We sometimes write $d(x,y) = \|x-y\|_{\mathbb{T}^d}$. For two subsets A and B of $\mathbb{T}^d$, we denote by $d(A,B)$ the distance from A to B. By the two preceding lemmas, G^n and H_n can be identified with subsets of $\mathbb{T}^d$. From now on G^n and H_n will denote their corresponding subsets on $\mathbb{T}^d$. The following fact is evident.

LEMMA 3. *$d(0, G^n) \le \|\Psi_n\|$ and $d(0, H_n) \le \|\Psi_{n-1}\|$.*

We therefore construct the infinite product $X = \prod_{n=1}^\infty H_n$ equipped with the usual ultrametric, and the map $q : X \to \mathbb{T}^d$ defined by

$$q(h_1, h_2, \ldots) = \sum_{n=1}^\infty h_n.$$

LEMMA 4. *The map $q : X \to \mathbb{T}^d$ is continuous and surjective.*

Proof. We have the continuity because of (2) which implies

$$\sum_n \|T_1^{-1} \ldots T_n^{-1}\| < \infty.$$

Let $\Gamma_n = \prod_{j=1}^n H_j$. Then Γ_n can be regarded as a subset of X. Consider the restriction of q to Γ_n. We claim that $q(\Gamma_n) = G_n$. In fact, first we observe that $q(\Gamma_n) \subset G_n$. Suppose $h'_j, h''_j \in H_j$ $(1 \le j \le n)$ and

$$h'_1 + \ldots + h'_n = h''_1 + \ldots + h''_n.$$

Then $h'_n - h''_n \in G_{n-1}$. According to Lemma 1, this is impossible unless $h'_n = h''_n$. By induction, it follows that $h'_j = h''_j$ $(1 \le j \le n)$. This proves the injectivity of the restriction of q to Γ_n. However, the cardinality of Γ_n is the same as that of G_n, so $q(\Gamma_n) = G_n$. By condition (2), the union of G_n $(n \ge 1)$ is dense in G. Thus the closure of the image of q is G. But X is compact and hence q is surjective. ∎

Let $\{\mu_n\}$ be the sequence of probability mesures defined by

$$\mu_n(h) = |H_n|^{-1} \qquad (h \in H_n).$$

Let $\mu = \bigotimes_{n=1}^\infty \mu_n$ and let $q\mu$ be the image by q of μ. That is to say, $q\mu$ is the measure on $\mathbb{T}^d$ characterized by

$$\int_{\mathbb{T}^d} f\, dq\mu = \int_X f \circ q\, d\mu \qquad (f \in \mathcal{C}(\mathbb{T}^d)). \tag{8}$$

Lemma 5. *If μ is defined as above, then $q\mu$ is Haar measure λ on $\mathbb{T}^d$.*

Proof. $\{\gamma + G^n\}_{\gamma \in G_n}$ being a partition of $\mathbb{T}^d$, we have

$$\lambda(\gamma + G^n) = \frac{1}{|G_n|} \qquad (\gamma \in G_n),$$

because

$$\sum_{\gamma \in G_n} \lambda(\gamma + G^n) = 1 \quad \text{and} \quad \lambda(\gamma + G^n) = \lambda(G^n).$$

Now given $f \in \mathcal{C}(\mathbb{T}^d)$, we have by Fact 2,

$$\int_X f \circ q d\,\mu = \lim_{n\to\infty} \mathbb{E}_\mu(f \circ q \mid \mathcal{F}^n) = \lim_{n\to\infty} \sum_{h_1,\ldots,h_n} f \circ q(h_1, \ldots, h_n, x_{n+1}, \ldots).$$

Given $\varepsilon > 0$, f being uniformly continuous, there is a $\delta > 0$ such that $|f(x) - f(y)| \le \varepsilon$ if $\|x - y\|_{\mathbb{T}^d} < \delta$. Choose an $N > 0$ such that $\sum_{n \ge N} d(0, G^n) < \delta$. Then for $n \ge N$ we have

$$\frac{1}{|G_n|} \sum_{h_1,\ldots,h_n} f \circ q(h_1, \ldots, h_n, x_{n+1}, \ldots) = \sum_{\gamma \in G_n} f(\gamma)\lambda(\gamma + G^n) + O(\varepsilon).$$

The last sum tends to $\int f d\lambda$ as $N \to \infty$. ∎

4. Proof of Theorem 1. Recall that a sequence $\{h_n\}_{n\geq 1}$ of elements in a Hilbert space is said to be *quasi-orthogonal* if the bilinear form

$$\sum_{n,m}\langle h_n, h_m\rangle a_n b_m$$

on $\ell^2(\mathbb{N}^*)\times\ell^2(\mathbb{N}^*)$ is bounded. Suppose that the Hilbert space is $L^2(X,\mu)$ for some measure space (X,μ). For a quasi-orthonormal sequence $\{h_n\}\subset L^2(\mu)$, we may apply Men'shov's theorem ([9]), which says that the series $\sum c_n h_n(x)$ converges μ-a.e. provided the numerical series $\sum |c_n|^2\log^2 n$ converges. So, in order to prove Theorem 1, it suffices ([2], p. 237) to show the following estimate, uniform in n:

$$\int\limits_{\mathbb{T}^d} f_n(T_nT_{n-1}\dots T_1x)f_{n+p}(T_{n+p}T_{n+p-1}\dots T_1x)\,dx = O(p^{-\sigma}). \tag{9}$$

In fact, let $h_n(x)=f_n(T_nT_{n-1}\dots T_1x)$. If $\sigma>1$, the sequence $\{h_n\}$ is quasi-orthogonal. If $\sigma=1$, the sequence $\{n^{-\varepsilon/2}h_n\}$ $(\forall\varepsilon>0)$ is quasi-orthogonal. If $\sigma<1$, the sequence $\{n^{-(1-\sigma)/2}h_n\}$ is quasi-orthogonal.

Now we deduce (9) from Theorem 2.

According to Lemma 5, we consider the sequence $f_n\circ\Phi_n\circ q$ $(n\geq 1)$ defined on X and apply Theorem 2 to it. Then to prove (9), it suffices to show

$$\omega_{n+p-1}(f_n\circ\Phi_n\circ q)=O(p^{-\sigma}). \tag{10}$$

Suppose $x=(x_j)$ and $y=(y_j)$ belong to X and satisfy $x_j=y_j$ for $1\leq j\leq n+p-1$. We have

$$q(x)-q(y)=\sum_{j=n+p}^{\infty}(x_j-y_j).$$

As $x_j,y_j\in H_j\subset G^{j-1}$, Lemma 2 implies that there exist $\xi_j',\xi_j''\in D$ such that

$$x_j=\Psi_{j-1}\xi_j',\quad y_j=\Psi_{j-1}\xi_j''\pmod{\mathbb{Z}^d}.$$

Then

$$\begin{aligned}\|\Phi_n\circ q(x)-\Phi_n\circ q(y)\|_{\mathbb{T}^d}&\leq\Big\|\Phi_n\Big(\sum_{j=n+p}^{\infty}\Psi_j(\xi_j'-\xi_j'')\Big)\Big\|\\&=\Big\|\sum_{j=n+p}^{\infty}\Phi_n\Psi_j(\xi_j'-\xi_j'')\Big\|=\sum_{j=n+p}^{\infty}\|T_{n+1}^{-1}\dots T_j^{-1}\|.\end{aligned}$$

With this in mind, we can deduce (10) from (2). ∎

Proof of Corollary. Let ϱ be the spectral radius of T^{-1}. By the hypothesis, $\varrho<1$. Let $\varrho<\varrho_1<1$. For n sufficiently large we have

$\|T^{-n}\| < \varrho_1^n$. Consequently,

$$\tau_{n,p} = O\Big(\sum_{k=0}^{\infty} \varrho_1^{p+k}\Big) = O(\varrho_1^p).$$

This estimate and the hypothesis on f allow us to verify condition (2) of Theorem 1 with $\sigma > 0$. ■

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