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# ON HILBERT SETS AND $\mathcal{C}_\Lambda(G)$-SPACES WITH NO SUBSPACE ISOMORPHIC TO $c_0$

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**Introduction.** Arithmetical properties of lacunary sets, in particular Sidon sets, in discrete abelian groups received a great attention during the sixties and seventies (see [9] and [30]). We are interested here in Hilbert subsets of $\mathbb{Z}$, and give a new proof of the fact that a subset of $\mathbb{Z}$ whose uniform density is strictly positive must contain a Hilbert set $\Lambda$; since then $\mathcal{C}_\Lambda(\mathbb{T})$ has a subspace isomorphic to $c_0$ (Th. 2), we recover a result of F. Lust-Piquard saying that the uniform density of $\Lambda$ is null if $\mathcal{C}_\Lambda$ has no subspace isomorphic to $c_0$.

In the second part, we consider, for a subset $\Lambda$ of a discrete abelian group $\Gamma = \widehat{G}$, the property that $\mathcal{C}_\Lambda(G)$ does not contain an isomorphic copy of $c_0$. This property appears in the study of Rosenthal sets. Let us recall that a *Rosenthal set* $\Lambda$ is a subset of $\Gamma$ for which every element of $L^\infty_\Lambda(G)$ has a continuous representative. This name was given by R. Dressler and L. Pigno after H. P. Rosenthal had constructed such sets which are not Sidon sets ([47]), and they showed that Rosenthal sets are Riesz sets ([13], [14]). By the Bessaga–Pełczyński Theorem, if $\Lambda$ is a Rosenthal set, then $\mathcal{C}_\Lambda(G)$ does not have any subspace isomorphic to $c_0$; moreover, this latter property implies that $\Lambda$ is a Riesz set ([29], [31]), but not conversely. F. Lust-Piquard conjectured that $\Lambda$ is a Rosenthal set as soon as $\mathcal{C}_\Lambda$ has no subspace isomorphic to $c_0$. We give in the second part of this paper some partial answers, and ask some other connected questions.

The notations will be classical and can be found for instance in [9], [11] or [29].

**Hilbert sets.** In this section, we give some Banach space and arithmetical properties of Hilbert sets. Let us first recall the definition.

DEFINITION 1 ([25], Def. 7 and Rem. 8.2.1, p. 242). $\Lambda \subseteq \mathbb{Z}$ is said to be a *Hilbert set* if there are two sequences $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ in $\mathbb{Z}$ such

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that $p_n \neq 0$ for all $n \geq 1$ and

$$\Lambda = \bigcup_{n\geq 1} \Lambda_n$$

with $\Lambda_n = \{q_n + \sum_{k=1}^n \varepsilon_k p_k : \varepsilon_k = 0 \text{ or } 1\}$.

THEOREM 2. *If $\mathcal{C}_\Lambda(\mathbb{T})$ has no subspace isomorphic to $c_0$, then $\Lambda \subseteq \mathbb{Z}$ does not contain any Hilbert set.*

Proof. Substituting, if necessary, $\Lambda$ by $-\Lambda$, and making blocks $(p_{k_n} + \ldots + p_{k_{n+1}})_{n\geq 1}$, we may and do suppose that

$$(\forall n \geq 1) \quad p_n \geq 2, \quad p_{n+1} \geq 2p_n$$

and

$$\prod_{k\geq 1} \left(1 - 2\pi \frac{p_k}{p_{k+1}}\right) \geq \frac{9}{10}.$$

Setting

$$f_n(x) = e^{2\pi i q_{2n} x} \prod_{k=1}^{2n-2} \left(\frac{1 + e^{2\pi i p_k x}}{2}\right) \frac{1 - e^{2\pi i p_{2n-1} x}}{2} \cdot \frac{1 - e^{2\pi i p_{2n} x}}{2},$$

we have $f_n \in \mathcal{C}_\Lambda$, $\|f_n\|_\infty \leq 1$ and $\|f_n\|_\infty \geq 9/10$, by virtue of

LEMMA 3 ([39], lemme 2). *Let $(p_k)_{k\geq 1}$ be a sequence of real positive numbers such that $p_{k+1} \geq 2p_k$ for all $k$. On every interval of length $2/p_1$, we can find, for each sequence $(r_k)_{k\geq 1}$ of real numbers, an $x \in \mathbb{R}$ such that*

$$|e^{2\pi i r_k} - e^{2\pi i p_k x}| \leq 4\pi \frac{p_k}{p_{k+1}}, \qquad \forall k \geq 1.$$

On the other hand, since

$$|\sin a_1 \sin a_2| + |\cos a_1 \cos a_2 \sin a_3 \sin a_4| + \ldots + |\cos a_1 \cos a_2 \ldots \cos a_{2n-2} \sin a_{2n-1} \sin a_{2n}| \leq 1,$$

as remarked in [22], lemme 2, and [40], p. 250, we have

$$\sum_{n=1}^{\infty} |f_n(x)| \leq 1, \qquad \forall x \in \mathbb{T}.$$

The Bessaga–Pełczyński Theorem ([1], Th. 5; see also [11], Th. V.8, or [29], Prop. 2.e.4) ensures then that $\mathcal{C}_\Lambda$ has a subspace isomorphic to $c_0$. ■

Remark. The proof is taken from arguments Y. Meyer used for ultra-thin symmetric sets ([40], pp. 245–250).

The next result is an arithmetical property of Hilbert sets:

THEOREM 4. *If $\Lambda \subseteq \mathbb{Z}$ does not contain any Hilbert set, the uniform density $d^*(\Lambda)$ of $\Lambda$ is null.*

We found two proofs of this theorem. We are only going to detail the second one since we learned afterwards that N. Hindman had already published the first one (see [23], Th. 11.11); let us only mention that one iterates the following result of D. Kazhdan ([43], p. 152; [24], Th. 3.1): if $\Lambda \subseteq \mathbb{N}$ and $d^*(\Lambda) > 0$, then for each $n \in \mathbb{N}$ there is $m \ge n$ such that $d^*(\Lambda \cap (\Lambda - m)) > 0$. The second proof uses the following notion:

Definition 5 (Bourgain–Mikheev classes; [2], p. 40; [4], Def. 4.23; [41], Def. 2). Let $\mathcal{S}_0$ be the class of one-element subsets of $\mathbb{Z}$. For each ordinal $\alpha \ge 1$, $\mathcal{S}_\alpha$ is defined by transfinite induction:

$$\mathcal{S}_\alpha = \{\Lambda \subseteq \mathbb{Z} : \forall N \ge 1,\ \exists \Lambda_1, \ldots, \Lambda_n \in \mathcal{S}_\beta,\ \beta < \alpha,\ \text{such that}\\ \lambda, \lambda' \in \Lambda \backslash (\Lambda_1, \ldots, \Lambda_n),\ \lambda \ne \lambda' \Rightarrow |\lambda - \lambda'| \ge N\}.$$

Let $\mathcal{S} = \mathcal{S}_{\omega_1} = \bigcup_{\alpha < \omega_1} \mathcal{S}_\alpha$.

$\omega_0$ and $\omega_1$ are respectively the first infinite ordinal and the first uncountable ordinal. The class $\mathcal{S}_1$ is the class of subsets of $\mathbb{Z}$ whose pace tends to infinity ([9], Def. 8, p. 16, and [7], Def. 5.1). Every Sidon set is in $\mathcal{S}_2$ ([7], p. 72, Corol.).

The proof of Theorem 4 follows immediately from the next two results:

Theorem 6 (J. Bourgain, [4], Prop. 4.26). *If $\Lambda$ contains no Hilbert set, then $\Lambda \in \mathcal{S}$.*

Theorem 7. $d^*(\Lambda) = 0$ *for every* $\Lambda \in \mathcal{S}$.

Let us recall that

$$d^*(\Lambda) = \lim_{h\to\infty} \left[ \sup_{a \in \mathbb{Z}} \frac{\operatorname{card}(\Lambda \cap [a, a+h])}{h} \right].$$

Actually, I. M. Mikheev shows ([41], Th. 2) that $d^*(\Lambda) = 0$ for $\Lambda \in \mathcal{S}_{<\omega_0} = \bigcup_{\alpha<\omega_0} \mathcal{S}_\alpha$; his argument can be pursued transfinitely: of course $d^*(\Lambda) = 0$ for $\Lambda \in \mathcal{S}_0$ and we assume that $d^*(\Lambda) = 0$ for all $\Lambda \in \mathcal{S}_\beta$ with $\beta < \alpha$; then if $\Lambda \in \mathcal{S}_\alpha$ and $N \ge 1$, we can write

$$\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_K \cup \Lambda^*$$

with $\Lambda_1, \ldots, \Lambda_K \in \mathcal{S}_\beta$, $\beta < \alpha$ and $\pi(\Lambda^*) = \inf\{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda^*, \lambda \ne \lambda'\} \ge N$; then

$$d^*(\Lambda) \le d^*(\Lambda_1) + \ldots + d^*(\Lambda_K) + d^*(\Lambda^*) \le 1/N,$$

so that $d^*(\Lambda) = 0$. ∎

Remarks. 1. By Theorem 4 every $\Lambda \subseteq \mathbb{Z}$ such that $d^*(\Lambda) > 0$ contains a Hilbert set. An example of Strauss shows that $\Lambda$ does not need to contain a translate of an IP-set (i.e. a Hilbert set with $q_n = q$ for all $n$) ([43], p. 151; [23], Th. 11.6).

2. In [42], Th. 3, I. M. Mikheev showed that if $\Lambda \notin \mathcal{S}_{<\omega_0}$, then $\Lambda$ contains parallelepipeds of arbitrarily large dimensions, i.e.

$$\bigcup_{n\geq 1} \Big\{ q_n + \sum_{k=1}^{n} \varepsilon_k p_{n,k} : \varepsilon_k = 0 \text{ or } 1 \Big\}$$

([15], Def., p. 129). The converse is not true: there are easy examples of sets whose pace tends to infinity, i.e. in $\mathcal{S}_1$, but which contain parallelepipeds of arbitrarily large dimensions. I. M. Mikheev also showed that $\Lambda(p)$-sets ($p > 0$) cannot contain any parallelepiped of arbitrarily large dimensions ([42], Th. 3; see also [15], Th. 4 for $p \geq 1$), hence belong to $\mathcal{S}_{<\omega_0}$; however, the set of primes $\mathbb{P}$ contains parallelepipeds of arbitrarily large dimensions ([42], Corol. 3). It is not known if $\mathbb{P} \notin \mathcal{S}_{<\omega_0}$. It is worth pointing out that $\mathcal{C}_{\mathbb{P}}(\mathbb{T})$ contains subspaces isomorphic to $c_0$ ([36], Th. 2). Nevertheless, it is not known whether $\mathbb{P} \in \mathcal{S}$ or $\mathbb{P} \notin \mathcal{S}$. At first glance, $\mathbb{P} \notin \mathcal{S}$ seems to be very strong since it would imply the existence of an infinite sequence $(q_n)_{n\geq 1}$ of prime numbers and an infinite sequence $(p_n)_{n\geq 1}$ of integers such that for every $n \geq 1$ and every $(\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n$ the numbers $q_l + \sum_{k=1}^{n} \varepsilon_k p_k$ with $l \geq n$ are all prime; this is connected with the *k-tuples conjecture* (see [10], p. 583). However, if $a\mathbb{Z} + b$ contains neither $-1$ nor $+1$, then $\mathbb{P} \cap (a\mathbb{Z} + b)$ cannot contain a translate of an IP-set, since all points of such a set are accumulation points for the Bohr topology (after [16], Th. 2.19, Lemma 9.4 and Prop. 9.6), whereas all points of $\mathbb{P} \cap (a\mathbb{Z} + b)$ are isolated for this topology.

As a corollary of Theorems 2 and 4, we recover the following result of F. Lust-Piquard ([33], second Th. 3.1; [35], Th. 3), which she showed by using the notion of invariant mean on $\ell_\infty(\mathbb{Z})$:

COROLLARY 8. *If $\mathcal{C}_\Lambda(\mathbb{T})$ has no subspace isomorphic to $c_0$, then $d^*(\Lambda) = 0$.*

We finish this section by noting that Bourgain's Theorem 6 has a converse; this converse, for the dual of the Cantor group instead of $\mathbb{Z}$, is proved in [4], Prop. 4.20, Prop. 4.25 and Corol. 4.24 (see the proof of Corol. 4.28), but Proposition 4.20 is not true for $\mathbb{T}$ (see [4], Remark 2, p. 78).

THEOREM 9. *$\Lambda \in \mathcal{S}$ implies that $\Lambda$ contains no Hilbert set.*

Proof. The following version of the Hindman–Miliken Theorem is needed:

THEOREM 10 (J. Bourgain, [3], Prop. 2). *If a finite union of sets $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_n$ contains a Hilbert set, then one of the $\Lambda_k$'s also contains a Hilbert set.*

Assume then that $\Lambda \in \mathcal{S}_\alpha$ is a Hilbert set:

$$\Lambda = \bigcup_{n\ge 1}\Big\{q_n + \sum_{k=1}^{n}\varepsilon_k p_k : \varepsilon_k = 0 \text{ or } 1\Big\};$$

we may assume that $1 \le p_1 \le p_2 \le \ldots$ For $N > p_1$, we can write, since $\Lambda \in \mathcal{S}_\alpha$,

$$\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_K \cup \Lambda^*$$

with $\Lambda_1, \ldots, \Lambda_K \in \mathcal{S}_\beta,\ \beta < \alpha$ and with the pace

$$\pi(\Lambda^*) = \inf\{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda^*,\ \lambda \ne \lambda'\} \ge N.$$

There is an injective map $j : \Lambda^* \to \Lambda_1 \cup \ldots \cup \Lambda_K$ which associates with every $\lambda = q_n + \varepsilon_1 p_1 + \sum_{k=2}^n \varepsilon_k p_k \in \Lambda^*$ the number $\widetilde{\lambda} = q_n + \widetilde{\varepsilon}_1 p_1 + \sum_{k=2}^n \varepsilon_k p_k \in \Lambda$, where $\widetilde{\varepsilon}_1 = 1$ if $\varepsilon_1 = 0$ and $\widetilde{\varepsilon}_1 = 0$ if $\varepsilon_1 = 1$; we have $\widetilde{\lambda} \in \Lambda_1 \cup \ldots \cup \Lambda_K = \Lambda \setminus \Lambda^*$ since $|\lambda - \widetilde{\lambda}| = p_1 < N$.

By Theorem 10, either $\Lambda^*$ or one of the $\Lambda_k$'s contains a Hilbert set. If $\Lambda^*$ contains a Hilbert set, then the image of this Hilbert set by $j$ is also a Hilbert set and is contained in $\Lambda_1 \cup \ldots \cup \Lambda_K$; hence from Theorem 10 again it follows that one of the $\Lambda_k$'s, which is in $\mathcal{S}_\beta$, $\beta < \alpha$, contains a Hilbert set.

Since $\mathcal{S}_0$ obviously does not contain any Hilbert set, an inductive argument leads us to a contradiction. ∎

Questions. (a) Does $\Lambda$ contain a Hilbert set if $\Lambda \subseteq \mathbb{Z}$ is not a Riesz set? It would be sufficient to know if "$\Lambda \cap (\Lambda + n)$ is a Riesz set for every $n \in \mathbb{Z}\setminus\{0\}$" implies that $\Lambda$ itself is a Riesz set. By Theorem 9, a weaker question is: Is every $\Lambda$ whose pace tends to infinity a Riesz set?

(b) $\Lambda \subseteq \mathbb{Z}$ is said to be of *first kind* ([9], p. 29) if there is a constant $C > 0$ such that for every $a > 0$ there is $\Lambda^* \subseteq \Lambda$ such that $\Lambda\setminus\Lambda^*$ is a finite set and

$$\|P\|_\infty \le C \sup_{0\le x\le a} |P(x)|$$

for every trigonometric polynomial $P$ with spectrum in $\Lambda^*$. The interval $[0, a]$ is said to be *C-associated* with $\Lambda^*$. It is easy to see that the pace of every set of first kind tends to infinity ([38], p. 31). Analogously to the Bourgain–Mikheev classes, we define $\mathcal{P}_0 = \mathcal{S}_0$, and for every ordinal $\alpha \ge 1$,

$$\mathcal{P}_\alpha = \{\Lambda \subseteq \mathbb{Z} : \exists C > 0, \forall a > 0, \exists \Lambda^* \subseteq \Lambda \text{ such that } [0,a] \text{ is } C\text{-associated with } \Lambda^* \text{ and } \Lambda\setminus\Lambda^* \text{ is a finite union of elements of } \mathcal{P}_\beta,\ \beta < \alpha\},$$

and $\mathcal{P} = \bigcup_{\alpha<\omega_1} \mathcal{P}_\alpha$. Then $\mathcal{P}_1$ is the class of sets of first kind, and every Sidon set is in $\mathcal{P}_2$ ([5], Corol. 3.5 and [6], Part 3, Corol. 2; see also [8], Corol. 3.10). Since $\mathcal{P}_1 \subseteq \mathcal{S}_1$, it is easy to see by induction that $\mathcal{P}_\alpha \subset \mathcal{S}_\alpha$ for every $\alpha$. We have the following question: If $\Lambda \notin \mathcal{P}$, does $\mathcal{C}_\Lambda(\mathbb{T})$ have a subspace isomorphic to $c_0$? It is worth pointing out that there exist sets $\Lambda \subseteq \mathbb{Z}$ whose pace tends to infinity, that is, $\Lambda \in \mathcal{S}_1$, but for which $\mathcal{C}_\Lambda(\mathbb{T}) \supseteq c_0$. For instance, the set of squares is such a set ([36], Th. 7(b)); another unpublished example is due to J.-P. Kahane; it was communicated to me by M. Déchamps-Gondim.

**$\mathcal{C}_\Lambda(G)$-spaces with no subspace isomorphic to $c_0$.** In this section, we recall some questions about these spaces, ask some new ones and give some partial answers. The main question is: Is $\Lambda$ a Rosenthal set if $\mathcal{C}_\Lambda(G)$ does not contain a subspace isomorphic to $c_0$? Weaker questions are: Is $\Lambda$ a Rosenthal set if:

(a) $\mathcal{C}_\Lambda(G)$ is weakly sequentially complete? A partial answer was given by F. Lust-Piquard: it is "yes" if moreover $L^1/L^1_{\Lambda'}$ has no subspace isomorphic to $\ell_1$ ([32], Ch. 4, Th. 6a, p. 67);

(b) $\mathcal{C}_\Lambda(G)$ has the Schur property? ([9], p. 30, question 9);

(c) $L^\infty_\Lambda(G)$ has no subspace isomorphic to $c_0$?

A partial answer was given by F. Lust-Piquard again who showed that $\Lambda$ is a Rosenthal set if $L^\infty_\Lambda$ has the Schur property ([34], Prop. 3.b; [32], Chap. IV, Th. 6b, p. 67; see also [9], p. 30, (v)). Other partial answers are given by the following two propositions:

PROPOSITION 11. *If $L^\infty_\Lambda(G)$ has Property* (V*), *then $\Lambda$ is a Rosenthal set.*

The hypothesis is satisfied by every Sidon set $\Lambda$. The definition of Property (V*) was given in [45], and [21], Prop. III.1, gives equivalent definitions; for instance: every non-relatively weakly compact bounded set contains a basic sequence whose span is isomorphic to $\ell_1$ and is complemented in the whole space. Banach spaces with Property (V*) are weakly sequentially complete ([45], [21]); in particular, $\mathcal{C}_\Lambda(G)$ is weakly sequentially complete if $L^\infty_\Lambda(G)$ has Property (V*).

Proof of Proposition 11. Let $f \in L^\infty_\Lambda$. By [19], Prop. I.1, the operator $\check{C}_f : L^1/L^1_{\Lambda'} \to \mathcal{C}_\Lambda \subseteq L^\infty_\Lambda$ defined by $\check{C}_f(g) = f * \check{g}$ is weakly compact; hence $\check{C}_f \circ \pi : L^1 \to \mathcal{C}_\Lambda$ is also weakly compact, and so is representable ([12], Th. III.2.12). Therefore $f$ is represented by a continuous function ([37], Prop. II.2) or ([32], Chap. IV, lemme 1, p. 71). ■

PROPOSITION 12. *If $\Lambda' = \Gamma \backslash (-\Lambda)$ is nicely placed, then $L^\infty_\Lambda(G)$ does have subspaces isomorphic to $c_0$.*

Let us recall ([18], Def. 1.4) that $\Lambda'$ is *nicely placed* if the unit ball of $L^1_{\Lambda'}$ is closed in measure.

Proof of Proposition 12. From [17], Th. 3, or [26], Prop. 4, $L^1/L^1_{\Lambda'}$ is an $L$-summand in its bidual, and so has Property (V*) ([46], Th. 3); therefore it contains complemented subspaces isomorphic to $\ell_1$ ([21], Prop. III.1); hence $L^\infty_\Lambda$ contains subspaces isomorphic to $c_0$. ∎

Proposition 12 strengthens Remark 4, p. 327 in [27], where it was noticed that $\Lambda$ is not a Rosenthal set if $\Lambda'$ is nicely placed.

Another weaker question is:

Is $\Lambda$ an ergodic set if $\mathcal{C}_\Lambda$ or $L^\infty_\Lambda$ has no subspace isomorphic to $c_0$?

Let us recall ([28]) that $\Lambda$ is an *ergodic set* if every $f \in L^\infty(G)$ whose spectrum is contained in a translate of $\Lambda$ has a unique invariant mean; every Rosenthal set is ergodic, but not conversely ([36], Th. 4).

Recently, G. Godefroy and F. Lust-Piquard introduced the following extension Property $(\rho)$ for the Dirac measure $\delta_0$:

Definition 13 ([20], Def. V.1). $\Lambda \subset \Gamma$ has *Property* $(\varrho)$ if there exists a linear functional $\varrho \in [L^\infty_\Lambda(G)]^*$ such that

(i) $\varrho : L^\infty_\Lambda(G) \to \mathbb{C}$ is a Borel map for the $w^*$-topology $\sigma(L^\infty, L^1)$;
(ii) $\varrho(f) = f(0)$ for every $f \in \mathcal{C}_\Lambda(G)$.

They pointed out that if the predual $L^1/L^1_{\Lambda'}$ of $L^\infty_\Lambda$ contains no subspace isomorphic to $\ell_1$ (in other words: if $L^\infty_\Lambda$ has the weak Radon–Nikodym property [48], 7.3.8), then, by the Odell–Rosenthal Theorem ([44]; [11], p. 215; [29], Th. 2.e.7) each cluster point $\varrho$ of an approximate identity $(K_n)_{n\ge1}$ in $L^1(G)$ gives a functional such that

(i) $\varrho : L^\infty_\Lambda \to \mathbb{C}$ is $w^*$-first Baire class,
(ii) $\varrho(f) = f(0)$ for every $f \in \mathcal{C}_\Lambda$,

so that $\Lambda$ has Property $(\varrho)$. Since they also showed ([20], Prop. V.2) that $\Lambda$ having Property $(\varrho)$ implies $\Lambda$ is a Riesz set and since we have the following implications ([31], Th. 3; [33], first Th. 3.1):

$$L^1/L^1_{\Lambda'} \not\supseteq \ell_1 \Rightarrow L^\infty_\Lambda \not\supseteq c_0 \Rightarrow \mathcal{C}_\Lambda \not\supseteq c_0 \Rightarrow \Lambda \text{ is a Riesz set},$$

the following question is natural:

If $\Lambda$ has Property $(\varrho)$, does that imply that $\mathcal{C}_\Lambda \not\supseteq c_0$? What about the converse?

We now give a more explicit construction for proving, in this translation-invariant setting, a theorem of Bessaga and Pełczyński ([1], Th. 4). We shall denote by $J : L^1/L^1_{\Lambda'} \to \mathcal{M}/\mathcal{M}_{\Lambda'}$ the canonical isometry, and by $J^* : \mathcal{C}^{**}_\Lambda \to L^\infty_\Lambda$ its adjoint mapping.

Proposition 14. *Let $(f_n)_{n\ge1}$ be a basic sequence in $\mathcal{C}_\Lambda$ which is equivalent to the canonical basis of $c_0$. There is a subsequence $(g_l)_{l\ge1} = (f_{n_l})_{l\ge1}$ such that $J^*$ is an isomorphism between $Y^{\perp\perp} \cong \ell_\infty$ and $J^*(Y^{\perp\perp})$, where $Y = [g_l, l \ge 1]$.*

Proof. We may suppose that the sequence $(f_n)_{n\ge1}$ is normalized and that
$$\frac{1}{M}\sup_{n\ge1}|a_n| \le \Big\| \sum_{n=1}^{\infty}{}^{*} a_n f_n \Big\|_{\mathcal{C}_\Lambda^{**}} \le M \sup_{n\ge1}|a_n|$$
for every $(a_n)_{n\ge1}\in\ell_\infty$. The $*$ means that the series converges in the $w^*$-topology. We have:

LEMMA 15. *Let $(f_n)_{n\ge1}$ be a shrinking basic sequence, with constant $M$, of normalized continuous functions on $G$. For each $\varepsilon>0$, there are a subsequence $(g_l)_{l\ge1}=(f_{n_l})_{l\ge1}$ and a sequence $(\varphi_l)_{l\ge1}$ of elements with norm $\le 2M$ in $L^1(G)$ such that*:

(a) $|\langle\varphi_l,g_l\rangle|\ge 1-\varepsilon/2^l$,
(b) $\|\varphi_{l|[g_{l+1},g_{l+2},\ldots]}\|\le\varepsilon/(2^{l+1}\cdot M)$,
(c) $|\langle\varphi_k,g_l\rangle|\le\varepsilon/2^l$, $k\ge l+1$.

Condition (c) does not appear in the classical proof of Bessaga–Pełczyński's Theorem.

Proof. Let $(\mu_n)_{n\ge1}$ be a biorthogonal sequence associated with $(f_n)_{n\ge1}$ in $\mathcal{M}(G)$:
$$\|\mu_n\|_1\le 2M;\quad \langle\mu_n,f_n\rangle=1;\quad \langle\mu_k,f_n\rangle=0,\ k\ne n.$$
Let $(K_j)_{j\ge1}$ be an approximate identity in $L^1(G)$.

Since $f_1$ is a continuous function,
$$(\exists j_1)\quad j\ge j_1\Rightarrow\|f_1*K_j-f_1\|_\infty\le\varepsilon/2;$$
then
$$|\langle\mu_1*\check{K}_{j_1},f_1\rangle|\ge|\langle\mu_1,f_1\rangle|-|\langle\mu_1,f_1*K_{j_1}-f_1\rangle|\ge1-\varepsilon/2.$$
We set $\varphi_1=\mu_1*\check{K}_{j_1}$.

Since the basic sequence $(f_n)_{n\ge1}$ is shrinking, we have
$$\|\varphi_{1|[f_n,f_{n+1},\ldots]}\|\to0\qquad\text{as } n\to\infty$$
([29], Prop. 1.b.1); hence
$$(\exists n_2>n_1=1)\quad\|\varphi_{1|[f_{n_2},f_{n_2+1},\ldots]}\|_\infty\le\varepsilon/(4M).$$
Now, since $f_{n_2}$ is a continuous function,
$$(\exists j_2\ge j_1)\quad j\ge j_2\Rightarrow\|f_{n_2}*K_j-f_{n_2}\|_\infty\le\varepsilon/4.$$
Then, setting $\varphi_2=\mu_{n_2}*\check{K}_{j_2}$, we have
$$|\langle\varphi_2,f_{n_2}\rangle|\ge|\langle\mu_{n_2},f_{n_2}\rangle|-|\langle\mu_{n_2},f_{n_2}*K_{j_2}-f_{n_2}\rangle|\ge1-\varepsilon/4,$$
and also
$$|\langle\varphi_2,f_{n_1}\rangle|=|\langle\mu_{n_2},f_1*K_{j_2}-f_1\rangle|\le\varepsilon/2.$$
Moreover, since $\|\varphi_{2|[f_n,f_{n+1},\ldots]}\|\to0$ as $n\to\infty$, we have

$$(\exists n_3 > n_2) \quad \|\varphi_{2|[f_{n_3}, f_{n_3+1},\ldots]}\| \le \varepsilon/(8M).$$

The construction will go on by induction. ■

Now we have

$$Y^{\perp\perp} = \Big\{ \sum_{l=1}^{\infty}{}^{*} a_l g_l : (a_l)_{l\ge 1} \in \ell_\infty \Big\}$$

and

$$\Big\| J^*\Big( \sum_{l=1}^{\infty}{}^{*} a_l g_l \Big) \Big\|_{L^\infty_\Lambda} \le \Big\| \sum_{l=1}^{\infty}{}^{*} a_l g_l \Big\|_{\mathcal{C}^{**}_\Lambda} \le M \sup_{l\ge 1} |a_l|.$$

Conversely, let $\varepsilon > 0$ and let $j$ be such that

$$|a_j| \ge (1-\varepsilon) \sup_{k\ge 1} |a_k|.$$

With the notations of Lemma 15, we have

$$\begin{aligned}
2M\Big\| J^*\Big( \sum_{l=1}^{\infty}{}^{*} a_l g_l \Big) \Big\|_{L^\infty_\Lambda} & \\
&\ge \Big| \Big\langle \varphi_j, \sum_{l=1}^{\infty}{}^{*} a_l g_l \Big\rangle \Big| \\
&\ge |a_j \langle \varphi_j, g_j\rangle| - \sum_{l=1}^{j-1} |a_l \langle \varphi_j, g_l\rangle| - \Big| \Big\langle \varphi_j, \sum_{l=j+1}^{\infty}{}^{*} a_l g_l \Big\rangle \Big| \\
&\ge |a_j| \Big( 1 - \frac{\varepsilon}{2^j} \Big) - \sup_{l\ge 1} |a_l| \cdot \sum_{l=1}^{j-1} \frac{\varepsilon}{2^l} - M \sup_{l\ge j+1} |a_l| \cdot \|\varphi_{j|[g_{j+1}, g_{j+2},\ldots]}\| \\
&\ge \sup_{k\ge 1} |a_k| \cdot \Big[ (1-\varepsilon)\Big( 1 - \frac{\varepsilon}{2^j} \Big) - \sum_{l=1}^{j-1} \frac{\varepsilon}{2^l} - \frac{\varepsilon}{2^{j+1}} \Big] \\
&\ge \sup_{k\ge 1} |a_k| \cdot (1-2\varepsilon). \ ■
\end{aligned}$$

Remark. By dominated convergence, if $g = \sum_{j\ge 1}^{*} a_j g_j \in \mathcal{C}^{**}_\Lambda$, a representative of $J^*(g) \in L^\infty_\Lambda$ is given by the function $\widetilde{g} : x \mapsto \langle g, \delta_x \rangle = \sum_{j\ge 1} a_j g_j(x)$ where the series is pointwise absolutely convergent.

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