ON HILBERT SETS AND
$\mathcal{C}_A(G)$-SPACES WITH NO SUBSPACE ISOMORPHIC TO $c_0$

BY

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Introduction. Arithmetical properties of lacunary sets, in particular Sidon sets, in discrete abelian groups received a great attention during the sixties and seventies (see [9] and [30]). We are interested here in Hilbert subsets of $\mathbb{Z}$, and give a new proof of the fact that a subset of $\mathbb{Z}$ whose uniform density is strictly positive must contain a Hilbert set $\Lambda$; since then $\mathcal{C}_A(T)$ has a subspace isomorphic to $c_0$ (Th. 2), we recover a result of F. Lust-Piquard saying that the uniform density of $\Lambda$ is null if $\mathcal{C}_A$ has no subspace isomorphic to $c_0$.

In the second part, we consider, for a subset $\Lambda$ of a discrete abelian group $\Gamma = \hat{G}$, the property that $\mathcal{C}_A(G)$ does not contain an isomorphic copy of $c_0$. This property appears in the study of Rosenthal sets. Let us recall that a Rosenthal set $\Lambda$ is a subset of $\Gamma$ for which every element of $L^\infty(\Gamma)$ has a continuous representative. This name was given by R. Dressler and L. Pigno after H. P. Rosenthal had constructed such sets which are not Sidon sets ([47]), and they showed that Rosenthal sets are Riesz sets ([13], [14]). By the Bessaga–Pełczyński Theorem, if $\Lambda$ is a Rosenthal set, then $\mathcal{C}_A(G)$ does not have any subspace isomorphic to $c_0$; moreover, this latter property implies that $\Lambda$ is a Riesz set ([29], [31]), but not conversely. F. Lust-Piquard conjectured that $\Lambda$ is a Rosenthal set as soon as $\mathcal{C}_A$ has no subspace isomorphic to $c_0$. We give in the second part of this paper some partial answers, and ask some other connected questions.

The notations will be classical and can be found for instance in [9], [11] or [29].

Hilbert sets. In this section, we give some Banach space and arithmetical properties of Hilbert sets. Let us first recall the definition.

Definition 1. ([25], Def. 7 and Rem. 8.2.1, p. 242). A subset $\Lambda \subseteq \mathbb{Z}$ is said to be a Hilbert set if there are two sequences $(p_n)_{n \geq 1}$ and $(q_n)_{n \geq 1}$ in $\mathbb{Z}$ such
that \( p_n \neq 0 \) for all \( n \geq 1 \) and

\[
A = \bigcup_{n \geq 1} A_n
\]

with \( A_n = \{ q_n + \sum_{k=1}^{n} \varepsilon_k p_k : \varepsilon_k = 0 \text{ or } 1 \} \).

**Theorem 2.** If \( C_A(T) \) has no subspace isomorphic to \( c_0 \), then \( A \subseteq \mathbb{Z} \) does not contain any Hilbert set.

**Proof.** Substituting, if necessary, \( A \) by \( -A \), and making blocks \((p_k + \ldots + p_{k+1})_{n \geq 1} \), we may and do suppose that \((\forall n \geq 1)\) \( p_n \geq 2 \), \( p_{n+1} \geq 2p_n \)

and

\[
\prod_{k \geq 1} \left( 1 - 2\pi \frac{p_k}{p_{k+1}} \right) \geq \frac{9}{10}.
\]

Setting

\[
f_n(x) = e^{2\pi i q_n x} \prod_{k=1}^{2^{n-2}} \left( 1 + e^{2\pi i p_k x} \right) \frac{1 - e^{2\pi i p_{n-1} x}}{2} \cdot \frac{1 - e^{2\pi i p_n x}}{2},
\]

we have \( f_n \in C_A, \|f_n\|_{\infty} \leq 1 \) and \( \|f_n\|_{\infty} \geq 9/10 \), by virtue of

**Lemma 3** ([39], lemme 2). Let \((p_k)_{k \geq 1}\) be a sequence of real positive numbers such that \( p_{k+1} \geq 2p_k \) for all \( k \). On every interval of length \( 2/p_1 \), we can find, for each sequence \((r_k)_{k \geq 1}\) of real numbers, an \( x \in \mathbb{R} \) such that

\[
|e^{2\pi ir_k} - e^{2\pi ip_k}| \leq 4\pi \frac{p_k}{p_{k+1}}, \quad \forall k \geq 1.
\]

On the other hand, since

\[
|\sin a_1 \sin a_2| + |\cos a_1 \cos a_2 \sin a_3 \sin a_4| + \ldots \\
+ |\cos a_1 \cos a_2 \ldots \cos a_{2n-2} \sin a_{2n-1} \sin a_{2n}| \leq 1,
\]

as remarked in [22], lemme 2, and [40], p. 250, we have

\[
\sum_{n=1}^{\infty} |f_n(x)| \leq 1, \quad \forall x \in T.
\]

The Bessaga–Pełczyński Theorem ([1], Th. 5; see also [11], Th. V.8, or [29], Prop. 2.e.4) ensures then that \( C_A \) has a subspace isomorphic to \( c_0 \).

**Remark.** The proof is taken from arguments Y. Meyer used for ultra-thin symmetric sets ([40], pp. 245–250).

The next result is an arithmetical property of Hilbert sets:

**Theorem 4.** If \( A \subseteq \mathbb{Z} \) does not contain any Hilbert set, the uniform density \( d^*(A) \) of \( A \) is null.
We found two proofs of this theorem. We are only going to detail the second one since we learned afterwards that N. Hindman had already published the first one (see [23], Th. 11.11); let us only mention that one iterates the following result of D. Kazhdan ([43], p. 152; [24], Th. 3.1): if \( A \subseteq \mathbb{N} \) and \( d^*(A) > 0 \), then for each \( n \in \mathbb{N} \) there is \( m \geq n \) such that \( d^*(A \cap (A - m)) > 0 \).

The second proof uses the following notion:

**Definition 5** (Bourgain–Mikheev classes; [2], p. 40; [4], Def. 4.23; [41], Def. 2). Let \( S_0 \) be the class of one-element subsets of \( \mathbb{Z} \). For each ordinal \( \alpha \geq 1 \), \( S_\alpha \) is defined by transfinite induction:

\[
S_\alpha = \{ \Lambda \subseteq \mathbb{Z} : \forall N \geq 1, \exists A_1, \ldots, A_n \in S_\beta, \beta < \alpha, \text{ such that } \lambda, \lambda' \in A \setminus (A_1, \ldots, A_n), \lambda \neq \lambda' \Rightarrow |\lambda - \lambda'| \geq N \}.
\]

Let \( S = S_{\omega_1} = \bigcup_{\alpha < \omega_1} S_\alpha \).

\( \omega_0 \) and \( \omega_1 \) are respectively the first infinite ordinal and the first uncountable ordinal. The class \( S_1 \) is the class of subsets of \( \mathbb{Z} \) whose pace tends to infinity ([9], Def. 8, p. 16, and [7], Def. 5.1). Every Sidon set is in \( S_2 \) ([7], p. 72, Corol.).

The proof of Theorem 4 follows immediately from the next two results:

**Theorem 6** (J. Bourgain, [4], Prop. 4.26). If \( \Lambda \) contains no Hilbert set, then \( \Lambda \in S \).

**Theorem 7**. \( d^*(A) = 0 \) for every \( A \in S \).

Let us recall that

\[
d^*(A) = \lim_{h \to \infty} \left[ \sup_{a \in \mathbb{Z}} \frac{\text{card}(A \cap [a, a + h])}{h} \right].
\]

Actually, I. M. Mikheev shows ([41], Th. 2) that \( d^*(A) = 0 \) for \( A \in S_{<\omega_0} = \bigcup_{\alpha < \omega_0} S_\alpha \); his argument can be pursued transfinitely: of course \( d^*(A) = 0 \) for \( A \in S_0 \) and we assume that \( d^*(A) = 0 \) for all \( A \in S_\beta \) with \( \beta < \alpha \); then if \( A \in S_\alpha \) and \( N \geq 1 \), we can write

\[
A = A_1 \cup \ldots \cup A_K \cup A^*
\]

with \( A_1, \ldots, A_K \in S_\beta, \beta < \alpha \) and \( \sigma(A^*) = \inf \{ |\lambda - \lambda'| : \lambda, \lambda' \in A^*, \lambda \neq \lambda' \} \geq N \); then

\[
d^*(A) \leq d^*(A_1) + \ldots + d^*(A_K) + d^*(A^*) \leq 1/N,
\]

so that \( d^*(A) = 0 \). \( \blacksquare \)

**Remarks.** 1. By Theorem 4 every \( A \subseteq \mathbb{Z} \) such that \( d^*(A) > 0 \) contains a Hilbert set. An example of Strauss shows that \( A \) does not need to contain a translate of an IP-set (i.e. a Hilbert set with \( q_n = q \) for all \( n \)) ([43], p. 151; [23], Th. 11.6).
2. In [42], Th. 3, I. M. Mikheev showed that if $A \not\subseteq S_{<\omega_0}$, then $A$ contains parallelepipeds of arbitrarily large dimensions, i.e.

$$\bigcup_{n \geq 1} \left\{ q_n + \sum_{k=1}^{n} \varepsilon_k p_{n,k} : \varepsilon_k = 0 \text{ or } 1 \right\}$$

([15], Def., p. 129). The converse is not true: there are easy examples of sets whose pace tends to infinity, i.e. in $S_1$, but which contain parallelepipeds of arbitrarily large dimensions. I. M. Mikheev also showed that $A(p)$-sets ($p > 0$) cannot contain any parallelepiped of arbitrarily large dimensions ([42], Th. 3; see also [15], Th. 4 for $p \geq 1$), hence belong to $S_{<\omega_0}$; however, the set of primes $\mathbb{P}$ contains parallelepipeds of arbitrarily large dimensions ([42], Corol. 3). It is not known if $\mathbb{P} \not\subseteq S_{<\omega_0}$. It is worth pointing out that $\mathcal{C}_\mathbb{P}(\mathbb{T})$ contains subspaces isomorphic to $c_0$ ([36], Th. 2). Nevertheless, it is not known whether $\mathbb{P} \in S$ or $\mathbb{P} \not\subseteq S$. At first glance, $\mathbb{P} \not\subseteq S$ seems to be very strong since it would imply the existence of an infinite sequence $(q_n)_{n \geq 1}$ of prime numbers and an infinite sequence $(p_n)_{n \geq 1}$ of integers such that for every $n \geq 1$ and every $(\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$ the numbers $q_l + \sum_{k=1}^{n} \varepsilon_k p_k$ with $l \geq n$ are all prime; this is connected with the $k$-tuples conjecture (see [10], p. 583). However, if $a\mathbb{Z} + b$ contains neither $-1$ nor $+1$, then $\mathbb{P} \cap (a\mathbb{Z} + b)$ cannot contain a translate of an IP-set, since all points of such a set are accumulation points for the Bohr topology (after [16], Th. 2.19, Lemma 9.4 and Prop. 9.6), whereas all points of $\mathbb{P} \cap (a\mathbb{Z} + b)$ are isolated for this topology.

As a corollary of Theorems 2 and 4, we recover the following result of F. Lust-Piquard ([33], second Th. 3.1; [35], Th. 3), which she showed by using the notion of invariant mean on $\ell_\infty(\mathbb{Z})$:

**Corollary 8.** If $\mathcal{C}_A(\mathbb{T})$ has no subspace isomorphic to $c_0$, then $d^*(A) = 0$.

We finish this section by noting that Bourgain’s Theorem 6 has a converse; this converse, for the dual of the Cantor group instead of $\mathbb{Z}$, is proved in [4], Prop. 4.20, Prop. 4.25 and Corol. 4.24 (see the proof of Corol. 4.28), but Proposition 4.20 is not true for $\mathbb{T}$ (see [4], Remark 2, p. 78).

**Theorem 9.** $A \in S$ implies that $A$ contains no Hilbert set.

**Proof.** The following version of the Hindman–Miliken Theorem is needed:

**Theorem 10 (J. Bourgain, [3], Prop. 2).** If a finite union of sets $A = A_1 \cup \ldots \cup A_n$ contains a Hilbert set, then one of the $A_k$’s also contains a Hilbert set.
Bourgain–Mikheev classes, we define of every set of first kind tends to infinity ([38], p. 31). Analogously to the \( \Lambda \) have the following question: If \( P \) is a Hilbert set and \( \alpha \) is a finite set and \( \beta < \alpha \), we may assume that \( 1 \leq p_1 \leq p_2 \leq \ldots \). For \( N > p_1 \), we can write, since \( A \in S_\alpha \),

\[
A = A_1 \cup \ldots \cup A_K \cup A^*
\]

with \( A_1, \ldots, A_K \in S_\beta, \, \beta < \alpha \) and with the pace

\[
\pi(A^*) = \inf\{ |\lambda - \lambda'| : \lambda, \lambda' \in A^*, \, \lambda \neq \lambda' \} \geq N.
\]

There is an injective map \( j : A^* \to A_1 \cup \ldots \cup A_K \) which associates with every \( \lambda = q_n + \varepsilon_1 p_1 + \sum_{k=2}^n \varepsilon_k p_k \in \Lambda^* \) the number \( \tilde{\lambda} = q_n + \varepsilon_1 p_1 + \sum_{k=2}^n \varepsilon_k p_k \in A \), where \( \varepsilon_1 = 1 \) if \( \varepsilon_1 = 0 \) and \( \tilde{\varepsilon}_1 = 0 \) if \( \varepsilon_1 = 1 \); we have \( \lambda \in A_1 \cup \ldots \cup A_K = A \setminus A^* \) since \( |\lambda - \tilde{\lambda}| = p_1 < N \).

By Theorem 10, either \( A^* \) or one of the \( A_k \)'s contains a Hilbert set. If \( A^* \) contains a Hilbert set, then the image of this Hilbert set by \( j \) is also a Hilbert set and is contained in \( A_1 \cup \ldots \cup A_K \); hence from Theorem 10 again it follows that one of the \( A_k \)'s, which is in \( S_\beta, \beta < \alpha \), contains a Hilbert set.

Since \( S_0 \) obviously does not contain any Hilbert set, an inductive argument leads us to a contradiction. \( \blacksquare \)

**Questions.** (a) Does \( A \) contain a Hilbert set if \( A \subseteq \mathbb{Z} \) is not a Riesz set? It would be sufficient to know if \( "A \cap (A + n)" \) is a Riesz set for every \( n \in \mathbb{Z} \setminus \{0\} \) implies that \( A \) itself is a Riesz set. By Theorem 9, a weaker question is: Is every \( A \) whose pace tends to infinity a Riesz set?

(b) \( A \subseteq \mathbb{Z} \) is said to be of first kind ([9], p. 29) if there is a constant \( C > 0 \) such that for every \( a > 0 \) there is \( A^* \subseteq A \) such that \( A \setminus A^* \) is a finite set and

\[
||P||_\infty \leq C \sup_{0 \leq x \leq a} |P(x)|
\]

for every trigonometric polynomial \( P \) with spectrum in \( A^* \). The interval \( [0, a] \) is said to be \( C \)-associated with \( A^* \). It is easy to see that the pace of every set of first kind tends to infinity ([38], p. 31). Analogously to the Bourgain–Mikheev classes, we define \( \mathcal{P}_0 = S_0 \), and for every ordinal \( \alpha \geq 1 \),

\[
\mathcal{P}_\alpha = \{ A \subseteq \mathbb{Z} : \exists C > 0, \forall a > 0, \exists \Lambda^* \subseteq A \text{ such that } [0, a] \text{ is } C\text{-associated with } A^* \text{ and } A \setminus A^* \text{ is a finite union of elements of } \mathcal{P}_\beta, \beta < \alpha \},
\]

and \( \mathcal{P} = \bigcup_{\alpha < \omega} \mathcal{P}_\alpha \). Then \( \mathcal{P}_1 \) is the class of sets of first kind, and every Sidon set is in \( \mathcal{P}_2 \) ([5], Corol. 3.5 and [6], Part 3, Corol. 2; see also [8], Corol. 3.10). Since \( \mathcal{P}_1 \subseteq S_1 \), it is easy to see by induction that \( \mathcal{P}_\alpha \subset S_\alpha \) for every \( \alpha \). We have the following question: If \( A \notin \mathcal{P} \), does \( \mathcal{C}_A(\mathbb{T}) \) have a subspace isomorphic
It is worth pointing out that there exist sets $\Lambda \subseteq \mathbb{Z}$ whose pace tends to infinity, that is, $\Lambda \in S_1$, but for which $C_\Lambda(\mathbb{T}) \supseteq c_0$. For instance, the set of squares is such a set ([36], Th. 7(b)); another unpublished example is due to J.-P. Kahane; it was communicated to me by M. Dèchamps-Gondim.

$C_\Lambda(G)$-spaces with no subspace isomorphic to $c_0$. In this section, we recall some questions about these spaces, ask some new ones and give some partial answers. The main question is: Is $\Lambda$ a Rosenthal set if $C_\Lambda(G)$ does not contain a subspace isomorphic to $c_0$? Weaker questions are: Is $\Lambda$ a Rosenthal set if:

(a) $C_\Lambda(G)$ is weakly sequentially complete? A partial answer was given by F. Lust-Piquard: it is “yes” if moreover $L^1/L^1_\Lambda$ has no subspace isomorphic to $\ell_1$ ([32], Ch. 4, Th. 6a, p. 67);

(b) $C_\Lambda(G)$ has the Schur property? ([9], p. 30, question 9);

(c) $L^\infty_\Lambda(G)$ has no subspace isomorphic to $c_0$?

A partial answer was given by F. Lust-Piquard again who showed that $\Lambda$ is a Rosenthal set if $L^\infty_\Lambda$ has the Schur property ([34], Prop. 3.b; [32], Chap. IV, Th. 6b, p. 67; see also [9], p. 30, (v)). Other partial answers are given by the following two propositions:

**Proposition 11.** If $L^\infty_\Lambda(G)$ has Property ($V^*$), then $\Lambda$ is a Rosenthal set.

The hypothesis is satisfied by every Sidon set $\Lambda$. The definition of Property ($V^*$) was given in [45], and [21], Prop. III.1, gives equivalent definitions; for instance: every non-relatively weakly compact bounded set contains a basic sequence whose span is isomorphic to $\ell_1$ and is complemented in the whole space. Banach spaces with Property ($V^*$) are weakly sequentially complete ([45], [21]); in particular, $C_\Lambda(G)$ is weakly sequentially complete if $L^\infty_\Lambda(G)$ has Property ($V^*$).

**Proof of Proposition 11.** Let $f \in L^\infty_\Lambda$. By [19], Prop. I.1, the operator $\hat{C}_f : L^1/L^1_\Lambda \to \ell_1 \subseteq L^\infty_\Lambda$ defined by $\hat{C}_f(g) = f \ast \hat{g}$ is weakly compact; hence $\hat{C}_f \circ \pi : L^1 \to \ell_1 \subseteq C_\Lambda$ is also weakly compact, and so is representable ([12], Th. III.2.12). Therefore $f$ is represented by a continuous function ([37], Prop. II.2) or ([32], Chap. IV, lemme 1, p. 71). ■

**Proposition 12.** If $\Lambda' = \Gamma \backslash (-\Lambda)$ is nicely placed, then $L^\infty_\Lambda(G)$ does have subspaces isomorphic to $c_0$.

Let us recall ([18], Def. 1.4) that $\Lambda'$ is nicely placed if the unit ball of $L^1_{\Lambda'}$ is closed in measure.

**Proof of Proposition 12.** From [17], Th. 3, or [26], Prop. 4, $L^1/L^1_\Lambda$ is an $L$-summand in its bidual, and so has Property ($V^*$) ([46],
Proposition 12 strengthens Remark 4, p. 327 in [27], where it was noticed that $A$ is not a Rosenthal set if $A$ is nicely placed.

Another weaker question is:

Is $A$ an ergodic set if $C_A$ or $L_\infty^\Lambda$ has no subspace isomorphic to $c_0$?

Let us recall ([28]) that $A$ is an \textit{ergodic set} if every $f \in L_\infty^\Lambda(G)$ whose spectrum is contained in a translate of $A$ has a unique invariant mean; every Rosenthal set is ergodic, but not conversely ([36], Th. 4).

Recently, G. Godefroy and F. Lust-Piquard introduced the following extension Property ($\rho$) for the Dirac measure $\delta_0$:

\begin{definition} ([20], Def. V.1).

$\Lambda \subset \Gamma$ has \textit{Property ($\rho$)} if there exists a linear functional $\rho \in [L_\infty^\Lambda(G)]^*$ such that

(i) $\rho : L_\infty^\Lambda(G) \to \mathbb{C}$ is a Borel map for the $w^*$-topology $\sigma(L_\infty^\Lambda, L_1)$;

(ii) $\rho(f) = f(0)$ for every $f \in C_\Lambda(G)$.

They pointed out that if the predual $L^1/L_1^\Lambda$ of $L_\infty^\Lambda$ contains no subspace isomorphic to $\ell_1$ (in other words: if $L_\infty^\Lambda$ has the weak Radon–Nikodym property [48], 7.3.8), then, by the Odell–Rosenthal Theorem ([44], [11], p. 215; [29], Th. 2.e.7) each cluster point $\rho$ of an approximate identity $(K_n)_{n \geq 1}$ in $L_1^\Lambda(G)$ gives a functional such that

(i) $\rho : L_\infty^\Lambda \to \mathbb{C}$ is $w^*$-first Baire class,

(ii) $\rho(f) = f(0)$ for every $f \in C_\Lambda$,

so that $A$ has Property ($\rho$). Since they also showed ([20], Prop. V.2) that $A$ having Property ($\rho$) implies $A$ is a Riesz set and since we have the following implications ([31], Th. 3; [33], first Th. 3.1):

$L^1/L_1^\Lambda \not\supseteq \ell_1 \Rightarrow L_\infty^\Lambda \not\supseteq c_0 \Rightarrow C_\Lambda \not\supseteq c_0 \Rightarrow A$ is a Riesz set,

the following question is natural:

If $A$ has Property ($\rho$), does that imply that $C_\Lambda \not\supseteq c_0$? What about the converse?

We now give a more explicit construction for proving, in this translation-invariant setting, a theorem of Bessaga and Pełczyński ([1], Th. 4). We shall denote by $J : L^1/L_1^\Lambda \to M/M_\Lambda$ the canonical isometry, and by $J^* : C_\Lambda^* \to L_\infty^\Lambda$ its adjoint mapping.

\begin{proposition}

Let $(f_n)_{n \geq 1}$ be a basic sequence in $C_\Lambda$ which is equivalent to the canonical basis of $c_0$. There is a subsequence $(g_l)_{l \geq 1} = (f_{n_l})_{l \geq 1}$ such that $J^*$ is an isomorphism between $Y_{\perp\perp}^\Lambda \cong \ell_\infty$ and $J^*(Y_{\perp\perp}^\Lambda)$, where $Y = [g_l, l \geq 1]$.
\end{proposition}
Proof. We may suppose that the sequence \((f_n)_{n \geq 1}\) is normalized and that
\[
\frac{1}{M} \sup_{n \geq 1} |a_n| \leq \left\| \sum_{n=1}^{\infty} a_n f_n \right\|_{\ell^\infty} \leq M \sup_{n \geq 1} |a_n|
\]
for every \((a_n)_{n \geq 1} \in \ell^\infty\). The * means that the series converges in the \(w^*\)-topology. We have:

**Lemma 15.** Let \((f_n)_{n \geq 1}\) be a shrinking basic sequence, with constant \(M\), of normalized continuous functions on \(G\). For each \(\varepsilon > 0\), there are a subsequence \((g_l)_{l \geq 1} = (f_{n_l})_{l \geq 1}\) and a sequence \((\varphi_l)_{l \geq 1}\) of elements with norm \(\leq 2M\) in \(L^1(G)\) such that:

(a) \(|\langle \varphi_l, g_l \rangle| \geq 1 - \varepsilon/2^l\),
(b) \(\|\varphi_l\|_{[g_{l+1}, g_{l+2}, \ldots]} \| \leq \varepsilon/(2^{l+1} \cdot M)\),
(c) \(|\langle \varphi_k, g_l \rangle| \leq \varepsilon/2^l\), \(k \geq l + 1\).

Condition (c) does not appear in the classical proof of Bessaga–Pełczyński’s Theorem.

Proof. Let \((\mu_n)_{n \geq 1}\) be a biorthogonal sequence associated with \((f_n)_{n \geq 1}\) in \(M(G)\):

\[\|\mu_n\|_1 \leq 2M; \quad \langle \mu_n, f_n \rangle = 1; \quad \langle \mu_k, f_n \rangle = 0, \; k \neq n.\]

Let \((K_j)_{j \geq 1}\) be an approximate identity in \(L^1(G)\).

Since \(f_1\) is a continuous function,

\[(\exists j_1) \quad j \geq j_1 \Rightarrow \|f_1 * K_j - f_1\|_{\infty} \leq \varepsilon/2;\]

then

\[|\langle \mu_1 * K_j, f_1 \rangle| \geq |\langle \mu_1, f_1 \rangle| - |\langle \mu_1, f_1 * K_j - f_1 \rangle| \geq 1 - \varepsilon/2.\]

We set \(\varphi_1 = \mu_1 * K_j_1\).

Since the basic sequence \((f_n)_{n \geq 1}\) is shrinking, we have

\[\|\varphi_1\|_{[f_n, f_{n+1}, \ldots]} \| \rightarrow 0 \quad \text{as} \; n \rightarrow \infty\]

([29], Prop. 1.b.1); hence

\[(\exists n_2 > n_1 = 1) \quad \|\varphi_1\|_{[f_{n_2}, f_{n_2+1}, \ldots]} \| \leq \varepsilon/(4M).\]

Now, since \(f_{n_2}\) is a continuous function,

\[(\exists j_2 \geq j_1) \quad j \geq j_2 \Rightarrow \|f_{n_2} * K_j - f_{n_2}\|_{\infty} \leq \varepsilon/4.\]

Then, setting \(\varphi_2 = \mu_{n_2} * K_{j_2}\), we have

\[|\langle \varphi_2, f_{n_2} \rangle| \geq |\langle \mu_{n_2}, f_{n_2} \rangle| - |\langle \mu_{n_2}, f_{n_2} * K_{j_2} - f_{n_2} \rangle| \geq 1 - \varepsilon/4,\]

and also

\[|\langle \varphi_2, f_{n_1} \rangle| = |\langle \mu_{n_2}, f_1 * K_{j_2} - f_1 \rangle| \leq \varepsilon/2.\]

Moreover, since \(\|\varphi_2\|_{[f_n, f_{n+1}, \ldots]} \| \rightarrow 0 \quad \text{as} \; n \rightarrow \infty\), we have
The construction will go on by induction.

Now we have
\[ Y^⊥⊥ = \left\{ \sum_{l=1}^{∞} a_l g_l : (a_l)_{l \geq 1} \in \ell_∞ \right\} \]
and
\[ \left\| J^* \left( \sum_{l=1}^{∞} a_l g_l \right) \right\|_{L_A^∞} \leq \left\| \sum_{l=1}^{∞} a_l g_l \right\|_{C_A^{σσ}} \leq M \sup_{l \geq 1} |a_l|. \]

Conversely, let \( \varepsilon > 0 \) and let \( j \) be such that \( |a_j| \geq (1 - \varepsilon) \sup_{k \geq 1} |a_k| \).

With the notations of Lemma 15, we have
\[
2M \left\| J^* \left( \sum_{l=1}^{∞} a_l g_l \right) \right\|_{L_A^∞} \geq \left| \left( \varphi_j, \sum_{l=1}^{∞} a_l g_l \right) \right| \\
\geq |a_j \langle \varphi_j, g_j \rangle| - \sum_{l=1}^{j-1} |a_l \langle \varphi_j, g_l \rangle| - \left| \left( \varphi_j, \sum_{l=j+1}^{∞} a_l g_l \right) \right| \\
\geq |a_j| \left( 1 - \frac{\varepsilon}{2j} \right) - \sup_{l \geq 1} |a_l| \cdot \sum_{l=1}^{j-1} \frac{\varepsilon}{2l} - M \sup_{l \geq j+1} |a_l| \cdot \left\| \varphi_j[\langle g_{j+1}, g_{j+2}, \ldots \rangle] \right\| \\
\geq \sup_{k \geq 1} |a_k| \cdot \left( 1 - \varepsilon \right) \left( 1 - \frac{\varepsilon}{2j} \right) - \sum_{l=1}^{j-1} \frac{\varepsilon}{2l} - \frac{\varepsilon}{2j+1} \\
\geq \sup_{k \geq 1} |a_k| \cdot (1 - 2\varepsilon). \]

Remark. By dominated convergence, if \( g = \sum_{j \geq 1} a_j g_j \in C_A^{σσ} \), a representative of \( J^*(g) \in L_A^∞ \) is given by the function \( \tilde{g} : x \mapsto \langle g, δ_x \rangle = \sum_{j \geq 1} a_j g_j(x) \) where the series is pointwise absolutely convergent.

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